1 Exercises 1.1

1.1.1 Transitive: Let $x, y, z \in \mathfrak{R}$ such that $x \leq y$ and $y \leq z$. This means by definition that $y - x \in \mathfrak{R}$ and $z - y \in \mathfrak{R}$. Hence $z - x = (z - y) + (y - x) \in \mathfrak{R}$ by definition of cone.
Reflexive: We note that $x - x = 0$ and we know $0 \in \mathfrak{R}$ as $\mathfrak{R} \cap -\mathfrak{R} = \{0\}$. Hence $x \leq x$.
Antisymmetric: Let $x, y \in \mathfrak{R}$ such that $x \leq y$ and $y \leq x$. By definition, this means that $y - x \in \mathfrak{R}$ and $x - y \in \mathfrak{R}$. Let $z = y - x$. Then $z \in \mathfrak{R}$ and $z \in -\mathfrak{R}$, so by definition of generation cone, this means that $z = 0$. Hence $x = y$.

If we wish to make the order total, then we require that for every pair $x, y \in \mathfrak{R}$ we have that either $x - y \in \mathfrak{R}$ or $y - x \in \mathfrak{R}$. Hence if we had that $\mathfrak{R} \cup -\mathfrak{R} = \mathfrak{X}$ (along with the conditions of generating cone) we could have a total order.

1.1.2 Transitive: Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $x \leq y$ and $y \leq z$. Then $\alpha(y_1 - x_1) \leq y_2 - x_2$ and $\alpha(z_1 - y_1) \leq z_2 - y_2$. Hence

$$\alpha(z_1 - x_1) = \alpha(z_1 - y_1) + \alpha(y_1 - x_1) \leq (z_2 - y_2) + (y_2 - x_2) = z_2 - x_2$$

Reflexive: Obvious because $\alpha(x_1 - x_1) = 0 \leq 0 = x_2 - x_2$, so $x \leq x$.
Antisymmetric: Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $x \leq y$ and $y \leq x$. Then $\alpha(y_1 - x_1) \leq y_2 - x_2$ and $\alpha(x_1 - y_1) \leq x_2 - y_2$. Multiplying the second inequality by -1 we get that $\alpha(y_1 - x_1) \geq y_2 - x_2$. The combination of these two inequality tells us that $\alpha(y_1 - x_1) = y_2 - x_2$. If $y_1 - x_1$ were non-zero then α could be written as the quotient of two integers which contradicts irrationality. Hence $y_1 = x_1$ which implies that $y_2 - x_2 = 0$ so $y_2 = x_2$. Overall then we get that $x = y$.
Total: Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be arbitrary elements of $\mathbb{Z} \times \mathbb{Z}$. Then by total order on $\mathbb{R}$ we know that either $\alpha(y_1 - x_1) \leq y_2 - x_2$ or $\alpha(y_1 - x_1) \geq y_2 - x_2$. If it's the first case, we can conclude that $x \leq y$. If it's the second case we multiply by -1 to conclude that $y \leq x$.

1.1.3 Let X and Y be well ordered sets. Let $\phi : X_\phi \to Y_\phi$ and $\psi : X_\psi \to Y_\psi$ be two order isomorphisms from segments of X to segments of Y. We define $\phi \leq \psi$ if $X_\phi \subset X_\psi$. In particular, this means that if $\phi \leq \psi$ then

$\psi \mid_{X_\phi} = \phi$ which implies that $Y_\phi \subset Y_\psi$. Let T be some chain of such order isomorphisms. In other words, T is a totally ordered subset of the order isomorphisms. We define

$$\Gamma : \bigcup_{\phi \in T} X_\phi \to \bigcup_{\phi \in T} Y_\phi, \Gamma \mid_{X_\phi} = \phi$$

We can think of Γ as the combination of all the functions. Since for $\phi \leq \psi$, we have that $Y_\phi \subset Y_\psi$ there is no ambiguity in definition. Suppose $x \in X_\phi$ and $x \in X_\psi$ for distinct ϕ, ψ in T. Since T is well ordered, we can assume without loss of generality that $\phi \leq \psi$. Therefore, $X_\phi \subset X_\psi$ and $\psi \mid_{X_\phi} = \phi$ hence both ϕ and ψ must map x to the same value.

It is clear that

$$\Gamma^{-1} : \bigcup_{\phi \in T} Y_\phi \to \bigcup_{\phi \in T} X_\phi, \Gamma^{-1} \mid_{Y_\phi} = \phi^{-1}$$

where we know the final quantity exists by virtue of ϕ being an isomorphism. Hence Γ is a bijection.

Finally, we note that Γ is order preserving. Let $x \in X_\phi$ and $y \in X_\psi$ with $x \leq y$ for some $\phi, \psi \in T$. Suppose that $\phi \leq \psi$. Then $X_\phi \subset X_\psi$ therefore $x, y \in X_\psi$. Since $\Gamma \mid_{X_\psi} = \psi$, it follows that $\Gamma(x) = \psi(x) \leq \psi(y) = \Gamma(y)$. This shows that every chain of order isomorphisms is bounded hence there is a maximal element ϕ by Zorn's lemma.

Suppose the maximal element $\phi : X_\phi \to Y_\phi$ is such that $X_\phi \neq X$ and $Y_\phi \neq Y$. Then $X \backslash X_\phi$ and $Y \backslash Y_\phi$ are non-empty subsets of X and Y and hence by the well-ordering they both contain a smallest element x_0 and y_0. Note that x_0 and y_0 must be in $Upper(X)$ and $Upper(Y)$ respectively. This is because X_ϕ and Y_ϕ are segments hence are of the form $lower(x')$ and $lower(y')$ for some $x' \in X$ and $y' \in Y$ hence have no lower bound (or if they do, they are contained in the respective sets). Hence by defining $\tilde{\phi}$ which agrees with ϕ everywhere but additionally has $\tilde{\phi}(x_0) = y_0$, we get a contradiction since $\phi < \tilde{\phi}$.

1.1.4 Let $\{\alpha_j\}_{j \in J}$ be a collection of ordinal numbers indexed by some set J. Let $\{X_j\}_{j \in J}$ such that the ordinal number for each X_j is α_j. We choose some particular X_j. If X_j is order isomorphic to segments of all other sets then we are done as α_j is the smallest. If X_j is not the smallest then the smaller X_i are isomorphic to (proper) segments of X_j. However these segments are well ordered (their intersection being the smallest element). Thus there is a smallest X_i and we are done.

2 Exercises 1.2

1.2.7 Let us equip $\mathbb{R}$ with the topology $\mathcal{T}$ generated by a basis consisting of half-open intervals of the form $[y, z)$ for $y, z \in \mathbb{R}$ with $y < z$. In order to show that all the basis elements are closed we simply show their complement is open. Let $[y, z)$ be some basis element. It's complement is $(-\infty, y) \cup [z, \infty)$. We can write $(-\infty, y)$ as the union of all $[x, y)$ where $x < y$. Similarly we write $[z, \infty)$ as the union of all $[z, x)$ where $x > z$. Since we can write the complement as some union of the basis elements it follows that the complement is open and hence $[y, z)$ is closed.

I claim that the rationals are dense in $(X, \mathcal{T})$. Let $x \in X$ arbitrary and U some arbitrary neighbourhood of x. We wish to show that U contains some rationals. It is sufficient to show this holds if U is a basis element. Suppose $U = [y, z)$ once again for some $y, z \in \mathbb{R}$ where $y < z$. If $y \in \mathbb{Q}$ we are done. If $y \notin Q$, then we consider the interval (y, x) which is a subset of U. We know that all open intervals intersect with the rationals so in particular this one does. Since the rational numbers are countable we are done.

This topology satisfies the first axiom of countability. Let $x \in \mathbb{R}$ be arbitrary and consider $A_n(x) = [x, x + \frac{1}{n})$ for $n \in \mathbb{N}$. Clearly every neighbourhood of x will contain some $A_n(x)$.

Let $\mathscr{B}$ be some basis for $\mathcal{T}$ and let $x \in \mathbb{R}$. Then we claim that there must exist $B \in \mathscr{B}$ such that $x = \inf(B)$. Suppose this were not the case. Consider $U = [x, x + \delta)$ for some $\delta > 0$ which is open in $\mathcal{T}$. Then U can be written as the union of some $B \in \mathscr{B}$ such that for each B, we have $B \subset U$. Since the infimum of U is x which is contained in U, at least one of the B must have this be the case as well. Thus we can construct a surjection from $\mathscr{B}$ (at least the ones that are bounded below) to $\mathbb{R}$ where we map each basic set to its infimum. However this means that $\mathscr{B}$ cannot be countable.

1.2.8 I claim that $\mathbb{Q} \times \mathbb{Q}$ is dense in $(\mathbb{R}^2, \mathcal{T}^2)$. As before, it suffices to show that $\mathbb{Q} \times \mathbb{Q}$ has non-empty intersections with the basis elements. Let $[y_1, z_1) \times [y_2, z_2)$ be some arbitrary basis element. We know from before that there exists a rational number p in $[y_1, z_1)$ and a rational number q in $[y_2, z_2)$. Then clearly (p, q) in the chosen basis element. As $\mathbb{Q} \times \mathbb{Q}$ is countable, $(\mathbb{R}^2, \mathcal{T}^2)$ is separable by definition.

In order to show that $S = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ is discrete in the relative topology, it suffices to show that the singleton set $\{(x, -x)\}$ is open in the relative topology for all x. Since the arbitrary union of open sets is open, it would follow that every possible subset is open. Then to see that $\{(x, -x)\}$ is open relative to S consider $U = [x, x+\delta) \times [-x, -x+\delta)$ for any $\delta > 0$ and its intersection with S. Intersections occur exactly when there exist $s, t \in \mathbb{R}$ such that $0 \leq s, t < \delta$ with

$$(x + s) + (-x + t) = 0$$

It follows quite immediately that the above equation is satisfied only when $s = -t$. Since s and t are both non-negative this can only occur when they are both 0 and hence $U \cap S = \{(x, -x)\}$ as desired.

In order to see that S is closed we simply show that the complement is closed. Then let $(x, y) \in \mathbb{R}^2 \backslash S$. Suppose $x + y = \delta > 0$. Then consider $B = [x, x + \delta) \times [-x + \delta, -x + 2\delta)$. It is clear that $(x, y) = (x, -x + \delta) \in B$. Additionally we see that B does not intersect S since

$$(x + s) + (-x + t) = 0 \Leftrightarrow s + t = 0$$

where $0 \leq s < \delta$ and $0 < \delta \leq t < 2\delta$. Therefore $B \subset \mathbb{R}^2 \backslash S$.

On the other hand suppose $x + y = -\delta < 0$ (we still assume that $\delta > 0$). Then we consider $B = [x, x + \frac{\delta}{2}) \times [-x - \delta, -x - \frac{\delta}{2})$. Once again we see easily that $(x, y) = (x, -x - \delta) \in B$. Additionally, B does not intersect S since

$$(x + s) + (-x - t) = 0 \Leftrightarrow s - t = 0$$

where $0 \leq s < \frac{\delta}{2}$ and $\frac{\delta}{2} < t \leq \delta$. Hence t is always greater than s. Since the intersection of B with S is empty, we get that $B \subset \mathbb{R}^2 \backslash S$. Since for each point in the complement, we can find a basis that contains the point and is a subset of the complement, it follows that the complement is open. Therefore S is closed.

1.2.9 Let $U(x; \delta) = \{y \in X : d(x, y) < \delta\}$. Then consider $A_n(x) = U\left(x; \frac{1}{n}\right)$. Thus $\mathcal{T}$ satisfies the first axiom of countability.

We know that the second axiom of countability implies separability (we simply choose any element from each of the basis elements). In order to see the reverse, we assume that $(X, \mathcal{T})$ is separable. Let Y be a countable, dense subset of X. Now we consider

$$\mathscr{B} = \bigcup_{y \in Y} \{A_n(y) : n \in \mathbb{N}\}$$

where $A_n(y) = \{x \in X : d(x, y) < \frac{1}{n}\}$. We claim that $\mathscr{B}$ is a basis for $\mathcal{T}$.

Let $x \in U$ for some open U. By definition, there exists some $\epsilon > 0$ such that $U(x;\epsilon) \subset U$. By density of Y, we know there exists a $y \in Y \cap U(x; \frac{\epsilon}{2})$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \frac{\epsilon}{2}$. Then $x \in A_N(y) \subset U$. Therefore we can write U as a union of such elements from $\mathscr{B}$. We note that $\mathscr{B}$ is countable because there is a surjection between $\mathscr{B}$ and $\mathbb{N} \times \mathbb{N}$.

Since the Sorgenfrey line is separable but does not satisfy the second axiom of countability, it cannot be metrizable.

1.2.10 Let $(X, \mathcal{T})$ be a topological space that satisfies the second axiom of countability. Let σ be an open covering of X.

3 Exercises 1.3

1.3.4 Suppose $Y \subset X$ is open. Let $\mathscr{F}$ be a filter that converges to some $y \in Y$. Then $Y \in \mathcal{O}(y)$ and by definition $\mathcal{O}(y) \subset \mathscr{F}$. Therefore $Y \in \mathscr{F}$ for all such $\mathscr{F}$.

In order to see the converse, let Y be such that for every filter $\mathscr{F}$ that converges to some point y in Y, we have that $Y \in \mathscr{F}$. For each $y \in Y$, we define

$$\mathscr{F}_y = \{A \subset X : \exists U_y \in \mathcal{T}. y \in U_y \subset A\}$$

It is clear that $\mathcal{O}(y) \subset \mathscr{F}_y$ hence each $\mathscr{F}_y$ converges to y. Thus by assumption, each $\mathscr{F}_y$ contains Y. In particular, this means that for each $y \in Y$ there exists U_y open such that $U_y \subset Y$. Therefore we can write

$$Y = \bigcup_{y \in Y} U_y$$

which implies that Y is open.

Let $\mathscr{G}$ be a filter with $\mathscr{F}$ as a subfilter. Let x be a convergence point of $\mathscr{F}$. Then $\mathcal{O}(x) \subset \mathscr{F}$. Since $\mathscr{F} \subset \mathscr{G}$, it follows that $\mathcal{O}(x) \subset \mathscr{G}$ and hence G converges to x.

1.3.5 Let $\mathscr{U}$ be an ultrafilter. Let Y be some arbitrary subset of X and suppose $\mathscr{U}$ contains neither Y nor $X \backslash Y$. It is clear that $\mathscr{U}$ does not contain any subset of Y (if it did by the second condition Y would be in $\mathscr{U}$ and we would be done). Similarly $\mathscr{U}$ cannot contain subsets of $X \backslash Y$. Let $\mathscr{U}'$ be U along with Y and all the necessary subsets needed to make $\mathscr{U}'$ a filter (in particular the intersections of all members of $\mathscr{U}$ with Y as well as all sets that contain Y) We know that the intersection of any $A \in \mathscr{U}$ with Y cannot be empty as that would imply that A is a subset of $X \backslash Y$. Thus $\mathscr{U}'$ is a filter that contains $\mathscr{U}$ as a proper filter leading to a contradiction. Additionally, we note that both Y and $X \backslash Y$ cannot be in $\mathscr{U}$ as their intersection is the empty set which is definitionally not an element of any filter.

Let $\mathscr{F}$ be a filter $\{\mathscr{F}_j\}_{j \in J}$ be the collection of filters such that $\mathscr{F} \subset \mathscr{F}_j$ for all j. Their union is clearly a filter so by Zorn's lemma there is a maximal filter $\mathscr{U}$ and by maximality it cannot be properly contained in any other filter.

1.3.6 Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X. Let $\mathscr{F}$ be defined as follows

$$\mathscr{F} = \{A \subset X : (x_\lambda) \text{ eventually in } A\}$$

Clearly $\phi \notin \mathscr{F}$. Let A, B be elements of $\mathscr{F}$. Then there exists $\lambda_1, \lambda_2 \in \Lambda$ such that for all $\lambda \geq \lambda_1$ we have $x_\lambda \in A$ and for $\lambda \geq \lambda_2$ we have $x_\lambda \in B$. Since Λ is upward filtering, there exists λ' that is greater than both λ_1 and λ_2. Then for $\lambda \geq \lambda'$ we have $x_\lambda \in A$ and $x_\lambda \in B$ thus $x_\lambda \in A \cap B$. This means that $A \cap B \in \mathscr{F}$. The second condition clearly holds.

Suppose (x_λ) converges to some $x \in X$. This means that (x_λ) is eventually in every $A \in \mathcal{O}(x)$. Thus $\mathcal{O}(x) \subset \mathscr{F}$ implying that $\mathscr{F}$ also converges to x. On the other hand suppose the corresponding filter for the net converges to some $x \in X$ and hence contains $\mathcal{O}(x)$. By definition, this means that the net is eventually in every $A \in \mathcal{O}(x)$. Hence the net converges to x.

1.3.7 Let $(x, A) \in \Lambda$. It is clear that $(x, A) \leq (x, A)$ since $A \subset A$. Additionally let $(x, A), (y, B)$ and (z, C) be such that $(x, A) \leq (y, B)$ and $(y, B) \leq (z, C)$. This means that $B \subset A$ and $C \subset B$. This directly implies that $C \subset A$ and hence $(x, A) \leq (z, C)$. This confirms that the $\leq$ defined is indeed a preorder. In order to see that its upward filtering let (x, A) and (y, B) be arbitrary elements of Λ. Since A, B are elements of a filter, we know that their intersection is non-empty. Let $z \in A \cap B$. Then $(x, A) \leq (z, A \cap B)$ and $(y, B) \leq (z, A \cap B)$ and so we are done.

Suppose $\mathscr{F}$ converges to a point $x \in X$. Then $\mathcal{O}(x) \subset \mathscr{F}$. Let $(x_A, A) \in \Lambda$ where $A \in \mathcal{O}(x)$. Let (x_B, B) be any majorant of (x_A, A). Then $B \subset A$ therefore $x_B \in A$ implying that for all elements greater than (x_A, A), the net is always in A. Hence the net converges to x.

In order to see the converse, suppose the net converges to $x \in X$. Let A be some arbitrary set in $\mathcal{O}(x)$. Then there is some x_λ that is in A. In particular this means that there is some $(x_B, B) \in \Lambda$ such that $x_B \in B \subset A$. Since $B \in \mathscr{F}$ this means that $A \in \mathscr{F}$. Hence $\mathcal{O}(x) \subset \mathscr{F}$ allowing us to conclude that the filter converges to x.

Lemma 3.1. *Additionally, let $\mathscr{U}$ an ultrafilter and $A \in \mathscr{U}$. Then for all $B \subset A$ we have that either $B \in \mathscr{U}$ or $A \backslash B \in \mathscr{U}$.*

Proof. By previous results, we know that either $B \in \mathscr{U}$ or $X \backslash B \in \mathscr{U}$. If $B \in \mathscr{U}$, we are done. If $X \backslash B \in \mathscr{U}$, we simply not that $A \backslash B = A \cap (X \backslash B)$ and we are done. □

4 Exercises 1.4

1.4.6 Let X, Y, Z be topological spaces and $f : X \times Y \to Z$ continuous. For every $x \in X$, we define $f_x : Y \to Z$ as $f_x(y) = f(x, y)$. Then we claim that f_x is continuous for all x. First for $x \in X$, we define $i_x : Y \to X \times Y$ given by $i_x(y) = (x, y)$ which is clearly continuous. Then we simply note that $f_x = f \circ i_x$. Since the composition of continuous functions is continuous, it follows that f_x is continuous.

1.4.8 We first wish to show that x is a limit of point of A and y is a limit point of B if and only if (x, y) is a limit point of $A \times B$.

First we assume that x and y are limit points of A and B respectively. Then let W be any neighbourhood of (x, y) in $X \times Y$. There must exist U, V open in X and Y respectively such that $(x, y) \in U \times V \subset W$. By assumption, there exists $x' \neq x$ in $U \cap A$ and $y' \neq y$ in $V \cap B$. This means that $(x', y') \in U \times V \cap A \times B$. Since W was an arbitrary neighbourhood, this holds for all neighbourhoods allowing us to conclude that (x, y) is a limit point of $A \times B$.

Now let us assume that (x, y) is a limit point of $A \times B$. Let U be some open neighbourhood of x and V some open neighbourhood of y. Then $U \times V$ intersects $A \times B$ in a point distinct from (x, y).

1.4.16 Let X be a topological space and E and F closed subsets such that $E \cup F = X$. Then we show that X and $E \cap F$ being connected implies that both E and F are connected. As usual, it is easier to work with disconnectedness so we will instead prove that E or F being disconnected implies that at least one of $E \cup F$ and $E \cap F$ is disconnected.

Without loss of generality, suppose E is disconnected. Then by *E 1.4.13* E contains non-trivial clopen sets. Let A be such a set. If $A \cap F = \phi$, then A is clopen in $E \cup F$ since $F \subset X \backslash A$ (so in particular we can write X as the union of A and $X \backslash A$).

1.4.19 Let $f : \mathbb{R}^n \to Y$ be continuous. We define $F : [0, 1] \times \mathbb{R}^n \to Y$ as $F(t, x) = f((1 - t)x)$. We see that F is continuous because $F = f \circ h$ where $h : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ with $h(s, x) = (1 - s)x$. Since f and g are continuous, it follows that F is continuous.

Similarly for $g : X \to \mathbb{R}^n$ we define $G(t, x) = (1 - t)g(x)$.

1.4.20 Let X, Y, Z be topological spaces. Then it is clear that X is homotopic to itself since we can take f and g to be the identity maps themselves. It is clear from the definition that if X is homotopic to Y then Y is homotopic to X. Thus all that remains to show is that homotopy is transitive. First we show that if $g \sim \mathrm{id}_X$ and f continuous then $f \circ g \sim f$.

Let $F : [0,1] \times X \to Y$ be the homotopy between g and id_X. Then let $G : [0,1] \times X \to Y$ with $G(t,x) = f(G(t,x))$. It is clear that $G(0,x) = f(g(x))$ and $G(1,x) = f(x)$. Additionally G is continuous as it is the composition of two continuous functions.

Now suppose $X \sim Y$ and $Y \sim Z$. Let $f_1 : X \to Y$ and $g_1 : Y \to X$ be continuous maps from such that $f_1 \circ g_1 \sim \mathrm{id}_Y$ and $g_1 \circ f_1 \sim id_X$. Let f_2 and g_2 be similar maps between Y and Z. Then $f_2 \circ f_1$ is a continuous map from X to Z and $g_1 \circ g_2$ is a continuous map from Z to X. Additionally

$$(f_2 \circ f_1) \circ (g_1 \circ g_2) \sim f_2 \circ (f_1 \circ g_1) \circ g_2 \sim f_2 \circ \mathrm{id}_Y \circ g_2 \sim f_2 \circ g_2 \sim \mathrm{id}_Z$$

It is clear then that $(g_1 \circ g_2) \circ (f_2 \circ f_1) \sim \mathrm{id}_X$ therefore $X \sim Z$. Hence we conclude that homotopy is an equivalence relation.

It is clear that homeomorphic spaces are homotopic as we can simply take f to be a homeomorphism with f^{-1} as its inverse.

1.4.21 Suppose X is homotopic to a point. Let $x \in X$ be a point with $f : X \to \{x\}$ and $g : \{x\} \to X$ with $g(x) = x'$ such that $f \circ g$ and $g \circ f$ are homotopic to the respective identities. Then note that $g \circ f$ is a constant map that is homotopic to the identity.

Now suppose id_X is homotopic to a constant function. Let $x \in X$ be the constant that they are mapped to. Then we define $f : X \to \{x\}$ and $g : \{x\} \to X$ with $g(x) = x$. Clearly $f \circ g$ is the constant function that is homotopic to id_X. On the other hand, $g \circ f = \mathrm{id}_{\{x\}}$. This means that X is homotopic to a point and is thus contractible.

Let $X \subset \mathbb{R}^n$ be convex. Choose $c \in X$. We define $F : [0,1] \times X \to X$ defined by $F(t,x) = (1-t)x + tc$. Clearly F is continuous and homotopy between id_X and a constant function and hence X is contractible.

1.4.23 Let $f, g : [0,1] \to S^2$ be loops. For convenience, we use spherical/polar coordinates, i.e. $f(t) = (1, \theta_t, \phi_t)$. Then we define $H : [0,1] \times [0,1] \to S^2$ where $H(s,t) = (1, (1-s)\theta_{f(t)} + s\theta_{g(t)}, (1-s)\phi_{f(t)} + s\phi_{g(t)})$. This means that any two loops are homotopic so $\pi(S^2) = \{0\}$.

1.4.24 Let X and Y be homotopic spaces. This means there exists $f : X \to Y$ and $g : Y \to X$ continuous with $f \circ g$ and $g \circ f$ homotopic to the identity. We define $H_f : \pi(X) \to \pi(Y)$ with $H_f([l]) = [f(l)]$. Similarly we have $H_g : \pi(Y) \to \pi(X)$ with $H_g([k]) = [g(k)]$. I claim that H_f and H_g are inverses of one another.

$$H_f(H_g([k])) = H_f([g(k)]) = [f(g(k))]$$

Since $f \circ g$ is homotopic to the identity it follows that $[f(g(k))] = [k]$. The other equation is also easily verified confirming that the above are inverses. All that remains to show is that H_f and H_g preserve the group operation. In other words, what we wish to show is that

$$[f(l_1 l_2)] = [f(l_1)][f(l_2)]$$

We note that

$$l_1 l_2 = \begin{cases} l_1(t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ l_2(t) & \text{for } \frac{1}{2 \leq t \leq 1} \end{cases}$$

Then

$$f(l_1 l_2) = \begin{cases} f(l_1(t)) & \text{for } 0 \leq t \leq \frac{1}{2} \\ f(l_2(t)) & \text{for } \frac{1}{2 \leq t \leq 1} \end{cases}$$

This is of course exactly $f(l_1)f(l_2)$ and hence the desired equality holds.