MAT324: Real Analysis – Fall 2016
Assignment 2

Due Tuesday, **September 20**, in class.

Problem 1: Suppose $E_1, E_2 \subseteq \mathbb{R}$ are measurable sets. Show that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Problem 2: Construct a *Cantor-like* closed set $\mathcal{C} \subset [0,1]$ so that at the k^{th} stage of the construction one removes 2^{k-1} centrally situated open intervals each of length ℓ_k, with

$$\ell_1 + 2\ell_2 + \ldots + 2^{k-1}\ell_k < 1.$$

Suppose ℓ_k are chosen small enough so that $\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1$.

a) Show that $m(\mathcal{C}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k$ and conclude that $m(\mathcal{C}) > 0$.

b) Give an example of a sequence $(\ell_k)_{k\geq 1}$ that verifies the hypothesis.

Problem 3: Let $E_1, E_2, \ldots, E_{2016} \subset [0,1]$ be measurable sets such that $\sum_{k=1}^{2016} m(E_k) > 2015$. Show that $m\left(\bigcap_{k=1}^{2016} E_k\right) > 0$.

Problem 4: Suppose $A \in \mathcal{M}$ and $m(A\Delta B) = 0$. Show that $B \in \mathcal{M}$ and $m(A) = m(B)$.

Problem 5: Suppose $A \subset E \subset B$ where A and B are measurable sets of finite measure. Show that if $m(A) = m(B)$, then E is measurable.

Problem 6: Suppose $E \in \mathcal{M}$ and $m(E) > 0$. Prove that there exists an open interval I such that

$$m(E \cap I) > 0.99 \cdot m(I).$$

Hint: Argue by contradiction, using the regularity of Lebesgue measure. See Theorems 2.17, 2.29.