MAT324: Real Analysis – Fall 2016

Assignment 3 – Solutions

Problem 1: Suppose that $E_1, E_2, \ldots$ are measurable subsets of $\mathbb{R}$. Show that if $E_k \supset E_{k+1}$ for all $k \geq 1$ and $m(E_{2016}) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k).$$

Solution. Let $A_1 = \bigcap_{k=1}^{2015} E_k$, and $A_n = E_n$, if $n \geq 2016$. Then the collection A_n satisfies the hypothesis of Theorem 2.13 (ii) (please see the proof there). $\square$

Problem 2: Let $\mathcal{N} \subset [0,1]$ be a non-measurable set. Determine whether the function

$$f(x) = \begin{cases} -x & \text{if} \quad x \in \mathcal{N} \\ x & \text{if} \quad x \notin \mathcal{N} \end{cases}$$

is measurable. Explain.

Solution. The function f is not measurable. Indeed,

$$f^{-1}((-1,0)) \cap (0,1) = \mathcal{N}$$

is a nonmeasurable set, so $f^{-1}((-1,0))$ is nonmeasurable. $\square$

Problem 3: Suppose that, for each rational number q, the set $\{x \mid f(x) > q\}$ is measurable. Can we conclude that f is measurable?

Solution. Yes, it is true. Use the density of $\mathbb{Q}$ in $\mathbb{R}$ to show that for any $a \in \mathbb{R}$,

$$f^{-1}((a,+\infty)) = \bigcup_{\{r\in\mathbb{Q} | r>a\}} f^{-1}((r,+\infty))$$

$\square$

Problem 4: Suppose $f, g : E \to \mathbb{R}$ are measurable functions on $E \in \mathcal{M}$. Show that $h : E \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \dfrac{f(x)}{g(x)} & \text{if} \quad g(x) \neq 0 \\ 0 & \text{if} \quad g(x) = 0 \end{cases}$$

is measurable.

Solution. It suffices to show that if g is a measurable function, then so is the function

$$\left(\frac{1}{g}\right)(x) = \begin{cases} \dfrac{1}{g(x)} & \text{if} \quad g(x) \neq 0 \\ 0 & \text{if} \quad g(x) = 0 \end{cases}$$

If this is proven, then the result follows from the fact that the measurable functions are closed under products. Notice that

1. If $a > 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a,+\infty)) = \left\{x|\frac{1}{g(x)} > a\right\} = \left\{x|g(x) < \frac{1}{a}\right\}$$
2. If $a = 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a,+\infty)) = \{x|g(x) > 0\}$$
3. If $a < 0$, then
$$\left(\frac{1}{g}\right)^{-1}((a,+\infty)) = \left\{x|\frac{1}{g(x)} < a\right\} \cup \{x|g(x) \geq 0\}$$

In each case, the sets are measurable, hence the result follows. □

Problem 5: Let $f : (a,b) \to \mathbb{R}$. If f has a finite derivative at all points then show that f' is measurable.

Solution. For each $n \in \mathbb{N}$, define $f_n : (a,b) \to \mathbb{R}$

$$f_n(x) = \begin{cases} \left(f\left(x + \dfrac{1}{n}\right) - f(x)\right)n, & \text{if } \; x + \frac{1}{n} \in (a,b) \\ 0, & \text{otherwise.} \end{cases}$$

For any $x \in (a,b)$, there exists $N = N(x)$ such that $n > N$ implies $x + \frac{1}{n} \in (a,b)$, hence for every $x \in (a,b)$, $f_n(x) \to f'(x)$. Furthermore, each $f_n(x)$ is measurable (check!). Now use Corollary 3.8 from the textbook. □