MAT324: Real Analysis – Fall 2016
Assignment 5 – Solutions

Problem 1: Suppose $\int_E f\,dm = \int_E g\,dm$ for every measurable set $E \in \mathcal{M}$. Show that $f = g$ almost everywhere.

Solution. See the proof of Theorem 4.22. $\square$

Problem 2: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions on $E \in \mathcal{M}$. If $\{f_n\}$ decreases to f almost everywhere and $\int_E f_1 dm < \infty$, then show that

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

Hint: Look at the sequence $g_n = f_1 - f_n$.

Solution. Consider the sequence of measurable functions $g_n = f_1 - f_n$. Since $\{f_n\}$ is a decreasing sequence, the sequence $\{g_n\}$ is an increasing sequence of nonnegative measurable functions converging to $g = (f_1 - f)$. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_E g_n dm = \int_E g dm.$$

On the other hand, since $\int_E f_1 dm < \infty$, and the f_n's decrease, monotonicity gives us $\int_E f_n dm < \infty$. Then, for each $n \in \mathbb{N}$, we have

$$\int_E g_n dm = \int_E (f_1 - f_n) dm = \int_E f_1 dm - \int_E f_n dm.$$

Likewise,

$$\int_E g dm = \int_E (f_1 - f) dm = \int_E f_1 dm - \int_E f dm.$$

The result now follows from cancellation (notice that it is necessary to assume $\int_E f_1 dm < \infty$ for this). $\square$

Problem 3: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. Show that

$$\int \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int f_n dm.$$

Solution. Consider the sequence of measurable functions $g_n = \sum_{k=1}^{n} f_k$. This sequence is clearly increasing since $f_k \geq 0$ for all $k \geq 1$, and $g_n(x) \to g(x) = \sum_{k=1}^{\infty} f_k(x)$, pointwise for every x. Convergence follows from the Monotone Convergence Theorem. $\square$

Problem 4: Compute the following limits if they exist and justify the calculations:

a) $\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx$

b) $\lim_{n\to\infty} \int_0^\infty \frac{n^2 x e^{-n^2x^2}}{1+x} dx.$

c) $\lim_{n\to\infty} \int_1^\infty \frac{n^2 x e^{-n^2x^2}}{1+\sqrt[n]{x}} dx.$

Solution.

a) The integrand is dominated by
$$g(x) = \frac{1}{(1+\frac{x}{2})^2},$$
which is integrable (check!). In addition, the integrand goes to zero for every x. The Dominated Convergence Theorem gives that the limit is zero.

b) See the worked out example example online and make the change of variables $y = nx$. The computations are exactly the same. The limit exists and is $\frac{1}{2}$.

c) The limit exists and is 0. Check that the integrands are dominated by the same L^1 function as before. □

Problem 5: Suppose $E \in \mathcal{M}$. Let (g_n) be a sequence of integrable functions which converges a.e. to an integrable function g. Let (f_n) be a sequence of measurable functions which converge a.e. to a measurable function f. Suppose further that $|f_n| \le g_n$ a.e. on E for all $n \ge 1$. Show that if $\int_E g\, dm = \lim_{n\to\infty} \int_E g_n\, dm$, then $\int_E f\, dm = \lim_{n\to\infty} \int_E f_n\, dm$.

Hint: Rework the proof of the Dominated Convergence Theorem.

Solution. Since $\pm f_n \le |f_n| \le g_n$ we get that $g_n \pm f_n \ge 0$. Apply Fatou's Lemma to the sequences $g_n - f_n$ and $g_n + f_n$ and carefully work out the details. □

Problem 6: Let $E \in \mathcal{M}$. Let (f_n) be a sequence of integrable functions which converges a.e. to an integrable function f. Show that $\int_E |f_n - f|\, dm \to 0$ as $n \to \infty$ if and only if $\int_E |f_n|\, dm \to \int_E |f|\, dm$ as $n \to \infty$.

Solution. First, suppose (f_n) is a sequence of integrable functions which converges a.e. to an integrable function f and $\int_E |f_n - f|\, dm \to 0$ as $n \to \infty$. We need to show that $\int_E |f_n|\, dm \to \int_E |f|\, dm$. The reverse triangle inequality tells us that
$$||f_n(x)| - |f(x)|| \le |f_n(x) - f(x)|$$
We integrate both sides and pass to the limit as $n \to \infty$.

For the converse, apply the previous problem to the sequence $|f_n - f|$ which is dominated by the sequence of functions $g_n = |f_n| + |f|$. □

Problem 7: Consider two functions $f, g : [0,1] \to [0,1]$ given by
$$f(x) = \begin{cases} \frac{1}{q} & \text{if} \quad x = \frac{p}{q} \in \mathbb{Q}, \text{ where } p \text{ and } q \text{ are relatively prime} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Show that f is Riemann integrable on $[0,1]$, but g is not Riemann integrable on $[0,1]$.

SOLUTION. For f, see Example 4.6 from the textbook. The function f is continuous at all irrational points, hence almost everywhere. For g, notice that the set of discontinuities is the whole interval $[0,1]$. Then use Theorem 4.23, part (i). □

Problem 8: Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x > 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Show that f has an improper Riemann integral over the interval $[0,\infty)$, but f is not Lebesgue integrable.

SOLUTION. Notice that if $a, A > 0$,

$$\int_a^A \frac{\sin x}{x} dx = \frac{\cos a}{a} - \frac{\cos A}{A} - \int_a^A \frac{\cos x}{x^2}$$

The integral on the right-hand side is convergent, whereas the diference also is, hence the Riemann integral exists. However, f is not Lebesgue integrable. Indeed, $|f|$ is not integrable since

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx = \frac{2}{(k+1)\pi}$$

and therefore

$$\lim_{n\ to \infty} \int_1^n \frac{|\sin x|}{x} dx \geq \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{1}{k} = \infty$$

diverges. □