MAT324: Real Analysis – Fall 2016

Assignment 7 – Solutions

Problem 1: Which of the following statements are true and which are false? Explain.

a) $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$

b) $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$

c) $L^1[3,5] \subset L^2[3,5]$

d) $L^2[3,5] \subset L^1[3,5]$

Solution.

a) The statement is false. The function

$$f(x) = \sum_{n=2}^{\infty} n\chi_{[n+\frac{1}{n^3}, n+\frac{2}{n^3}]}$$

belongs to $L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$

b) The statement is false. Indeed, the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\chi_{[n,n+1)}$$

belongs to $L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$.

c) The statement is false. To prove it, modify the function on part (a) in the following way. Let $C = \sum_{n=1}^{\infty} n^{-3}$. Define $s_0 = 3$, $s_n = s_{n-1} + \frac{1}{Cn^3}$, and $E_n = [s_{n-1}, s_n]$, for $n \in \mathbb{N}$ (notice that the E_n intersect only at the endpoints). Then

$$g(x) = \sum_{k=1}^{\infty} k\chi_{E_k} \in L^1[3,5] \setminus L^2[3,5]$$

d) It is true. Check proposition 5.3 on textbook. □

Problem 2: Let f be a positive measurable function defined on a measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$. Prove that

$$\left(\int_E f\,dm\right)\left(\int_E \frac{1}{f}\,dm\right) \geq m(E)^2.$$

Hint: Apply Cauchy-Schwarz inequality.

Solution. Follow the hint, if $f > 0$ then:

$$\left(\int_E (\sqrt{f})^2\,dm\right)^{\frac{1}{2}} \left(\int_E \frac{1}{(\sqrt{f})^2}\,dm\right)^{\frac{1}{2}} \geq \left(\int_E 1\,dm\right) = m(E).$$ □

Problem 3: Let $f_n \in L^1(0,1) \cap L^2(0,1)$ for all $n \geq 1$. Prove or disprove the following:

a) If $\|f_n\|_1 \to 0$ then $\|f_n\|_2 \to 0$.

b) If $\|f_n\|_2 \to 0$ then $\|f_n\|_1 \to 0$.

Solution.

a) Consider the sequence of functions $f_n(x) = n^2 \chi_{[0,\frac{1}{n^3}]}$. Then $\|f_n\|_1 = n^{-1}$, so $\|f_n\|_1 \to 0$. On the other hand, $\|f_n\|_2 = \sqrt{n} \to \infty$.

b) Cauchy-Schwarz inequality gives $\|f_n\|_1 \leq \sqrt{\|f_n\|_2}$. This proves the statement. □

Problem 4: Show that it is impossible to define an inner product on the space $\mathcal{C}([0,1])$ of continuous function $f : [0,1] \to \mathbb{R}$ which will induce the sup norm $\|f\|_{\text{sup}} = \sup\{|f(x) : x \in [0,1]\}$.

Solution. Recall that if a norm $\|\cdot\|$ on a space E is induced by an inner product, then it satisfies the parallelogram law:

$$\|x-y\|^2 + \|x+y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall \ x, y \in E$$

Consider the functions $f, g, \in \mathcal{C}([0,1])$, defined by

$$f(x) = \begin{cases} 1-2x & \text{if} \quad x \in [0, \frac{1}{2}] \\ 0 & \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \frac{1}{2}] \\ 2x-1 & \text{if} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

Check that the parallelogram law is not satisfied. □

Problem 5: Decide whether each of the following is Cauchy as a sequence in $L^2(0,\infty)$.

a) $f_n = \dfrac{1}{n^2}\chi_{[0,\frac{1}{n^3}]}$

b) $f_n = \dfrac{1}{x^2}\chi_{(n,\infty)}$.

Solution. See Exercise 5.3 from the textbook. □