MAT 341 – Applied Real Analysis
Spring 2017

Midterm 1 – February 28, 2017
Solutions

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
TOTAL	

Problem 1: (30 points) Consider the function $f(x) = 3 - x$, $0 < x < 3$.

a) Sketch both the even and odd periodic extensions of f on the interval $[-6, 6]$.

SOLUTION.

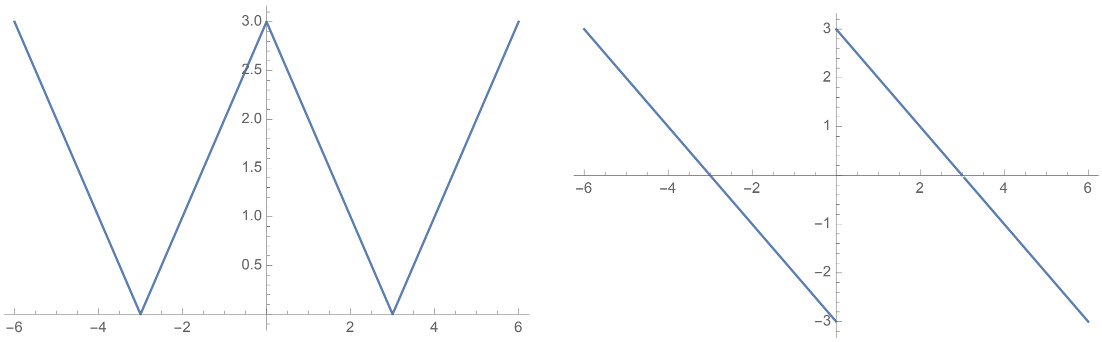

Figure 1: LEFT: even extension. RIGHT: odd extension.

$\square$

b) Find the Fourier cosine series of f.

SOLUTION. The cosine series of f is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right),$$

where

$$a_0 = \frac{1}{3}\int_0^3 (3-x)\,dx = \frac{1}{3}\left(3x - \frac{x^2}{2}\right)\Big|_0^3 = \frac{3}{2}$$

and

$$\begin{aligned} a_n &= \frac{2}{3}\int_0^3 (3-x)\cos\left(\frac{n\pi x}{3}\right)\,dx = 2\int_0^3 \cos\left(\frac{n\pi x}{3}\right)\,dx - \frac{2}{3}\int_0^3 x\cos\left(\frac{n\pi x}{3}\right)\,dx \\ &= -\frac{6}{(n\pi)^2}(\cos(n\pi) - 1) = \frac{6(1-(-1)^n)}{n^2\pi^2}. \end{aligned}$$

This gives

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6(1-(-1)^n)}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right).$$

$\square$

(Problem 1 continued)

c) Find the Fourier sine series of f. The sine series of f is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right),$$

where

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 (3-x)\sin\left(\frac{n\pi x}{3}\right)\,dx = 2\int_0^3 \sin\left(\frac{n\pi x}{3}\right)\,dx - \frac{2}{3}\int_0^3 x\sin\left(\frac{n\pi x}{3}\right)\,dx \\ &= \frac{6}{n\pi}(1-\cos(n\pi)) + \frac{6}{n\pi}\cos(n\pi) = \frac{6}{n\pi}. \end{aligned}$$

This gives

$$f(x) \sim \sum_{n=1}^{\infty} \frac{6}{n\pi}\sin\left(\frac{n\pi x}{3}\right).$$

□

d) To what value does the Fourier cosine series converge at $x = 2$? At $x = 6$? To what value does the Fourier sine series converge at $x = 2$, $x = 6$? Do the series from parts b) and c) converge uniformly?

Solution. At $x = 2$, the Fourier cosine series converges to $\frac{f(2-)+f(2+)}{2} = 1$. At $x = 6$ the cosine series converges to $\frac{f(0-)+f(0+)}{2} = 3$. Similarly, at $x = 2$, the Fourier sine series converges to 1. At $x = 6$, the sine series converges to 0.

The even periodic extension is continuous and piecewise smooth. Therefore the Fourier cosine series converges uniformly. The odd periodic extension is not continuous; it has a jump discontinuity at $x = 0$. Therefore the Fourier sine series does not converge uniformly (and the Gibbs phenomenon occurs). □

Problem 2: (25 points) The Fourier cosine series of $f(x) = x^2 - 2x$, $0 < x < 4$ is

$$x^2 - 2x = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n^2} \cos\left(\frac{n\pi x}{4}\right).$$

a) Does the Fourier cosine series of f converge uniformly in the interval $[0, 4]$? Explain.

SOLUTION. We notice that

$$\sum_{n=1}^{\infty} |a_n| + |b_n| \leq 4\frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

This implies that the Fourier cosine series converges uniformly. □

b) Compute the Fourier sine series of the derivative $f'(x)$ if it exists. Is the convergence uniform? If it doesn't exist, explain why it does not exist.

SOLUTION. The even periodic extension of f is continuous and piecewise differentiable. On a period interval, say $[-4, 4]$, f is not differentiable precisely when $x = -4, 0, 4$. Then the differentiated Fourier series of $f(x)$ converges to $f'(x)$ at each point where $f''(x)$ exists (see Theorem 6 from Section 1.5). Clearly, f is twice differentiable on $0 < x < 4$. It follows that the Fourier sine series of $f'(x)$ exists and

$$f'(x) = -\frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n^2} \cdot \frac{n\pi}{4} \sin\left(\frac{n\pi x}{4}\right) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n} \sin\left(\frac{n\pi x}{4}\right).$$

However, the convergence is not uniform. The even extension of f has the formula

$$f(x) = \begin{cases} x^2 + 2x & \text{if } -4 < x < 0, \\ x^2 - 2x & \text{if } \quad 0 < x < 4 \end{cases}$$

Note that there is a jump discontinuity in the graph of $f'(x)$ at $x = 0$, so the Fourier sine series of f' does not converge uniformly. This could also be observed from the coefficients of the series, which grow only like $1/n$. □

(Problem 2 continued)

c) Determine the Fourier sine series of $g(x) = x^3 - 3x^2$, $0 < x < 4$.

SOLUTION. Note that $g(x) = 3\displaystyle\int_0^x f(t)\,dt$, so

$$\begin{aligned}
g(x) &= 3\int_0^x \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1+3(-1)^n}{n^2}\cos\left(\frac{n\pi t}{4}\right)\,dt \\
&= 4x + \frac{3\cdot 16}{\pi^2}\sum_{n=1}^{\infty} \frac{1+3(-1)^n}{n^2}\frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right) \\
&= 4x + \frac{3\cdot 64}{\pi^3}\sum_{n=1}^{\infty} \frac{1+3(-1)^n}{n^3}\sin\left(\frac{n\pi x}{4}\right)
\end{aligned}$$

We need to compute the Fourier sine series for x, $0 < x < 4$. Doing similar computations as in Problem 1c) we find

$$x = \sum_{n=1}^{\infty} \frac{8(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{4}\right).$$

The Fourier sine series of $g(x)$ is

$$\begin{aligned}
g(x) &= 4\sum_{n=1}^{\infty} \frac{8(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{4}\right) + \frac{3\cdot 64}{\pi^3}\sum_{n=1}^{\infty} \frac{1+3(-1)^n}{n^3}\sin\left(\frac{n\pi x}{4}\right) \\
&= \sum_{n=1}^{\infty}\left(32\frac{(-1)^n}{n\pi} + 3\cdot 64\frac{1+3(-1)^n}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{4}\right).
\end{aligned}$$

□

Problem 3: (20 points) Consider the partial differential equation

$$20\frac{\partial^2 u}{\partial x^2} - 10\frac{\partial u}{\partial t} + 17\,u = 0.$$

a) Let $u(x,t) = e^{\lambda t} w(x,t)$, where λ is a constant. Find the corresponding partial differential equation for w. You are not asked to solve it.

SOLUTION. We have $u_{xx} = e^{\lambda t} w_{xx}$ and $u_t = e^{\lambda t}(\lambda w + w_t)$. Plugging back into the given equation yields

$$20u_{xx} - 10u_t + 17u = e^{\lambda t}(20w_{xx} - 10\lambda w - 10w_t + 17w) = 0.$$

Thus the PDE for w is $20w_{xx} - (10\lambda - 17)w - 10w_t = 0$. $\square$

b) Find a value for λ so that the partial differential equation for w found in part a) has no term in w. Then write the PDE as $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial w}{\partial t}$ and determine k.

SOLUTION. We want the coefficient of w to be zero, that is $10\lambda - 17 = 0$, which gives $\lambda = \dfrac{17}{10}$. The equation for w can be written as $w_{xx} = \dfrac{10}{20} w_t$, so $k = 2$. $\square$

Problem 4: (25 points) Consider the heat problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{9}\frac{\partial u}{\partial t}, \qquad 0 < x < 3, \qquad t > 0;$$

$$u(0,t) = 20, \quad u(3,t) = 50, \qquad t > 0;$$

$$u(x,0) = 60 - 2x, \qquad 0 < x < 3.$$

a) Find the steady state solution. State the problem satisfied by the transient solution.

Solution. The steady state solution verifies the equation $v''(x) = 0$, with boundary conditions $v(0) = 20$ and $v(3) = 50$. We find $v(x) = 10x + 20$. The transient solution is $w(x,t) = u(x,t) - v(x)$ and verifies the PDE:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{9}\frac{\partial w}{\partial t}, \qquad 0 < x < 3, \qquad t > 0;$$

$$w(0,t) = 0, \quad w(3,t) = 0, \qquad t > 0;$$

$$w(x,0) = 40 - 12x, \qquad 0 < x < 3.$$

□

b) Find the temperature $u(x,t)$.

Solution. The solution to the homogeneous equation is

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x) e^{-k\lambda_n^2 t},$$

where $\lambda_n = \frac{n\pi}{3}$ and $k = 9$. We simplify this and get

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right) e^{-(n\pi)^2 t}.$$

We find the coefficients c_n from the initial condition $w(x,0) = 40 - 12x$:

$$\begin{aligned} c_n &= \frac{2}{3}\int_0^3 (40 - 12x)\sin\left(\frac{n\pi x}{3}\right)\, dx = \frac{80}{3}\int_0^3 \sin\left(\frac{n\pi x}{3}\right)\, dx - 8\int_0^3 x\sin\left(\frac{n\pi x}{3}\right)\, dx \\ &= \frac{80}{n\pi}(1 - \cos(n\pi)) + \frac{72}{n\pi}\cos(n\pi) = \frac{8}{n\pi}\left(10 - (-1)^n\right). \end{aligned}$$

Therefore, the transient solution is

$$w(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{10 - (-1)^n}{n} \sin\left(\frac{n\pi x}{3}\right) e^{-(n\pi)^2 t}$$

and $u(x,t) = w(x,t) + v(x)$.

Note that the computations are similar to Problem 1c. □

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C \qquad \int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1+\cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1-\cos(2a)}{2}$$