MAT 351 Differential Equations: Dynamics & Chaos

Spring 2016

Assignment 4

Due Tuesday, **March 8**, in class.

Problem 1: For each of the following systems, find the equilibrium points, classify them and sketch the neighboring trajectories.

a) $\dot{x} = x - y$, $\dot{y} = x^2 - 4$

b) $\dot{x} = y$, $\dot{y} = xy + x - 1$

c) $\dot{x} = x(x^2 + y^2)$, $\dot{y} = y(x^2 + y^2)$

Does the linearized system accurately describe the local behavior near the equilibrium points?

Problem 2: Consider the system $\dot{x} = \sin(y)$, $\dot{y} = \cos(x)$ in the rectangle $-3 < x < 5$, $-4 < y < 5$

a) Find all fixed points, determine their stability and sketch the neighboring trajectories.

b) Does the system have homoclinic trajectories, heteroclinic trajectories, closed orbits, or limit cycles? Sketch some of them, if they exist.

Problem 3: Consider the pendulum equation, $\ddot{x} + \sin(x) = 0$, $-8 < x < 8$.

a) Find the equilibrium points and discuss the linearization of the pendulum equation in the neighborhood of each equilibrium point.

b) Multiply both sides of the pendulum equation by $\dot{x}$ and show that the *energy* function $E(\dot{x}, x) = \frac{1}{2}(\dot{x})^2 - \cos(x)$ is constant along trajectories.

c) (Extra Credit - 3p) Sketch the phase portrait of the full nonlinear equation.

Problem 4: Discuss the local and global behavior of solutions of

$$\begin{aligned} \dot{r} &= ar - r^5 \\ \dot{\theta} &= 1 \end{aligned}$$

as the parameter a passes through 0. Does the system have a limit cycle for some value of a?

Problem 5: The relativistic equation for the orbit of a planet around the sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \epsilon u^2$$

where $u = \dfrac{1}{r}$ and r, θ are the polar coordinates of the planet in its plane of motion. The parameter α is positive and can be found explicitly from classical Newtonian mechanics; the term ϵu^2 is Einsteins correction. Here ϵ is a very small positive parameter (so that $\epsilon\alpha \approx 0$).

a) Rewrite the equation as a system in the (u, v) phase plane, where $v = \dfrac{du}{d\theta}$.

b) Find all the equilibrium points of the system.

c) Show that one of the equilibria is a center in the (u, v) phase plane, according to the linearization. Is it a *nonlinear* center?

d) (Extra Credit - 2p) Show that the equilibrium point found in c) corresponds to a circular planetary orbit.

Problem 6: Show that the nonlinearly damped oscillator $\ddot{x} + (\dot{x})^3 + x = 0$ has no periodic solutions. *Hint:* Analyze the energy function $E(\dot{x}, x) = \frac{1}{2}\left((\dot{x})^2 + x^2\right)$.