MAT 351 Differential Equations: Dynamics & Chaos
Spring 2016

Assignment 6

Due Thursday, **April 21**, in class.

Problem 1: Consider the oscillator $\ddot{x} - (\mu - x^2)\dot{x} + x = 0$. Show that the system undergoes a Hopf bifurcation at $\mu = 0$. Is this subcritical or supercritical?

Problem 2: Consider the Lorenz system of equations

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - \frac{8}{3}z. \end{aligned}$$

for $r > 0$.

a) Find the linearized system at the origin. This system has the form $\dot{X} = AX$, where A is the Jacobian matrix at $(0, 0, 0)$.

b) Compute the eigenvalues of the matrix A.

c) By studying the eigenvalues from part b), show that the origin is asymptotically stable for $r < 1$ and unstable for $r > 1$.

Problem 3: Consider the system

$$\begin{aligned} \dot{x} &= -\nu x + zy \\ \dot{y} &= -\nu y + (z - a)x \\ \dot{z} &= 1 - xy \end{aligned}$$

where $a, \nu > 0$ are parameters.

a) Show that the system is dissipative.

b) Show that the fixed points may be written in parametric form $x^* = \pm k$, $y^* = \pm\frac{1}{k}$, and $z^* = \nu k^2$, where k verifies the equation $\nu(k^4 - 1) = ak^2$.

c) (Extra Credit - 3p) Classify the fixed points.

Note from Strogatz: These equations were proposed by Rikitake (1958) as a model for the self-generation of the Earths magnetic field by large current-carrying eddies in the core. Computer experiments show that the model exhibits chaotic solutions for some parameter values. These solutions are loosely analogous to the irregular reversals of the Earths magnetic field inferred from geological data.

Problem 4: Consider the following familiar system in polar coordinates: $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$. Let A be the unit circle $x^2 + y^2 = 1$.

a) Is A an invariant set? Does A attract an open set of initial conditions?

b) Is A an attractor? If not, explain why not?