MAT 351 Differential Equations: Dynamics and Chaos
Spring 2016

Midterm – March 31, 2016

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
TOTAL	

Problem 1: (22 points) Consider a two-dimensional system $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ and f is a $\mathcal{C}^1$ function.

a) Give a short definition for the following notions:

hyperbolic fixed point

closed orbit

limit cycle

Hopf bifurcation

(Problem 1 continued)

b) Give an example of a system that undergoes a Hopf bifurcation. No proof is required.

c) Sketch a phase portrait of a system that has a *stable limit cycle*, a *heteroclinic orbit*, and a *nonlinear center*. Sketch some typical trajectories for your system.

Problem 2: (22 points) Consider the differential equation

$$\dot{x} = \lambda - \frac{x^2}{1 + x^2}, \qquad x \in \mathbb{R},\ \lambda \in \mathbb{R}.$$

Find the equilibrium points and discuss their stability. Find the values of λ at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Sketch the bifurcation diagram of fixed points x^* vs. λ.

Problem 3: (28 points) Consider the two-dimensional system:

$$\begin{aligned} \dot{x} &= y - y^2 \\ \dot{y} &= \sin(x) \end{aligned}$$

a) Find an energy function $E(x, y)$ of the form $E(x, y) = \alpha \cos(x) + f(y)$, for some constant α and some function f. Verify that $E(x, y)$ is constant along trajectories.

b) Show that the fixed points $((2n + 1)\pi, 0)$ and $(2n\pi, 1)$ are nonlinear centers.

(Problem 3 continued)

c) Show that the fixed points $(2n\pi, 0)$ and $((2n+1)\pi, 1)$ are saddles.

d) Sketch the phase portrait for this system.

Problem 4: (28 points) Consider the two-dimensional system

$$\begin{aligned}\dot{x} &= y + x(1 - a - x^2 - y^2)\\ \dot{y} &= -x + y(1 - x^2 - y^2)\end{aligned}$$

where a is a constant such that $0 < a < 1$.

a) Determine the equilibrium points and classify them as sinks, sources, or saddles. Draw the phase portrait near the equilibrium points.

b) Using $x = r\cos(\theta)$ and $y = r\sin(\theta)$, rewrite the system in polar coordinates.

(Problem 4 continued)

c) Let $r_1 = \sqrt{1-a} - \epsilon$ and $r_2 = 1 + \epsilon$, for some $\epsilon > 0$ small enough. Show that there is at least one limit cycle in the region $R = \{(r, \theta) : r_1 \leq r \leq r_2\}$.

d) Suppose there are several limit cycles. Explain why they all must have the same period (the period will depend on the parameter a though).