MAT 351 Differential Equations: Dynamics & Chaos
Spring 2016

Project ideas

Project title due: Thursday, April 14.
The final project is due: Monday, May 16 at 5:30pm.
Note: Project 2 and Project 7 can be done in groups of 1 or 2 students. Project 3+4 (together) can be done in groups of 2 students.

Topic 1: This year we had a particularly warm winter, which was characterized by El Niño phenomenon. El Niño or, more precisely **El Niño-Southern Oscillation (ENSO)**, is a quasi-periodic climate pattern that occurs across the equatorial Pacific Ocean roughly every three to seven years. It is characterized by a change in sea surface temperatures (SSTs) in the eastern Pacific off the coast of Peru and accompanying changes in the air pressure difference between the central and western Pacific Ocean (Tahiti and Darwin, Australia). The following system of equations (with dimensionless variables and parameters) is a good model for studying ENSO:

$$\begin{aligned}\dot{x} &= -x + \frac{\lambda}{b}(bx+y) - \epsilon(bx+y)^3\\ \dot{y} &= -ry - \alpha bx\end{aligned}$$

where $r, \alpha, b, \epsilon, \lambda$ are all positive numbers. We are looking for the oscillatory behavior that characterizes the El Niño phenomenon

1. What is a quasi-periodic solution (or function)?

1. Compute the Jacobian A at the fixed point $(0,0)$ and find its eigenvalues. Find a condition on λ such that A has a pair of complex conjugate eigenvalues $\rho_1 = \beta - i\omega$ and $\rho_2 = \beta + i\omega$. Classify the fixed point at $(0,0)$.

2. Prove that a Hopf bifurcation occurs at a critical value $\lambda = \lambda_c$ and find λ_c. Decide whether the bifurcation is subcritical or supercritical. Find the value ω_c at the critical value λ_c (note that β and ω depend on λ).

3. Find the nontrivial solutions to the linearized system $(\dot{x}, \dot{y}) = A(x,y)$ at the parameter $\lambda = \lambda_c$. Show that $y(t)$ can be written as:

$$y(t) = -\frac{\alpha b}{\sqrt{\alpha(1+r)}}x(t-\eta), \quad \text{where} \quad \eta = \frac{1}{\omega_c}\tan^{-1}\left(\frac{\omega_c}{r}\right),$$

which shows that the trajectories of x and y coincide, but y lags behind x with a lag given by η. Thus, this ENSO model predicts that the negative thermocline depth anomaly follows the same oscillatory pattern as the SST anomaly but with a time lag η.

4. Consider $r = \frac{1}{4}$, $\alpha = \frac{1}{8}$, and $\lambda = \frac{3}{4}b$. Find λ_c, ω_c, and the time lag η. Suppose the time unit is two months, what is the predicted period? (this is the period of the function $y(t)$ from above). What does the factor η predict in this case? Are there better models for studying El Niño?

A comprehensive description of El Niño and deduction of the dimensionless model can be found in Chapter 16 from:

Hans Kaper, Hans Engler, *Mathematics and Climate*, Society for Industrial and Applied Mathematics (SIAM), 2013.

It is useful to read this chapter beforehand for a better understanding of the project (especially the last question).

Topic 2: The Fitzhugh-Nagumo system is a simplified model that describes the electrochemical transmission of neuronal signals along the cell membrane. Although the model is not entirely accurate, it capture the essential behavior of nerve impulses.

The **Fitzhugh-Nagumo system** of equations is given by

$$\begin{aligned} \dot{x} &= y + x - \frac{x^3}{3} + I \\ \dot{y} &= -x + a - by \end{aligned}$$

where a and b are constants satisfying $0 < \frac{3}{2}(1-a) < b < 1$ and I is a parameter. In these equations x is similar to the voltage and represents the excitability of the system; the variable y represents a combination of other forces that tend to return the system to rest. The parameter I is a stimulus parameter that leads to excitation of the system (I is like an applied current).

1. First assume that $I = 0$. Prove that this system has a unique equilibrium point (x^*, y^*). *Hint:* Use the geometry of the nullclines for this rather than explicitly solving the equations. Also remember the restrictions placed on a and b.

2. Prove that this equilibrium point is always a sink.

3. Now suppose that $I \neq 0$. Prove that there is still a unique equilibrium point $(x^*(I), y^*(I))$ and that $x^*(I)$ varies monotonically with I.

4. Determine values of $x^*(I)$ for which the equilibrium point is a source and show that there must be a stable limit cycle in this case.

5. When $I \neq 0$, the point (x^*, y^*) is no longer an equilibrium point. Nonetheless we can still consider the solution through this point. Describe the qualitative nature of this solution as I moves away from 0. Explain in mathematical terms why biologists consider this phenomenon the “excitement” of the neuron.

6. Consider the special case where $a = I = 0$. Describe the phase plane for each $b > 0$ (no longer restrict to $b < 1$) as completely as possible. Describe any bifurcations that occur.

7. Now let I vary as well and again describe any bifurcations that occur. Describe in as much detail as possible the phase portraits that occur in the I, b-plane, with $b > 0$.

8. Extend the analysis of the previous problem to the case $b \leq 0$.

9. Now fix $b = 0$ and let a and I vary. Sketch the bifurcation plane (the I, a-plane) in this case.

This project and a brief description on neurodynamics can be found in Chapter 12.5 from:
R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013.

Topic 3: Hamiltonian systems are fundamental to classical mechanics; they provide an equivalent but more geometric version of Newtons laws. They are also central to celestial mechanics and plasma physics, where dissipation can sometimes be neglected on the time scales of interest. We restrict our attention to Hamiltonian systems in $\mathbb{R}^2$, which is a system of the form:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x, y) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x, y), \end{aligned} \tag{1}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function called the Hamiltonian function.

1. Show that H is constant along every solution curve. Check that any system of the form $\ddot{x} + f(x) = 0$ is a Hamiltonian system.

2. Let (x^*, y^*) be a non-degenerate equilibrium point of a Hamiltonian system (that is, the determinant of the Jacobian at (x^*, y^*) is nonzero). Show that (x^*, y^*) is either a saddle or a center. Recall that (x^*, y^*) is a saddle for the system (1) iff it is a saddle of the Hamiltonian function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a center for (1).

3. There is an interesting relationship between the gradient system and the Hamiltonian system. Show that the system given by $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ is a Hamiltonian system if and only if the system *orthogonal* to it, given by $\dot{x} = g(x, y)$, $\dot{y} = -f(x, y)$ is a gradient system. To illustrate the orthogonality, consider the Hamiltonian function $H(x, y) = y \sin(x)$ and sketch the phase portraits of the Hamiltonian system and its gradient system (on the same graph).

4. Consider the equations for a nonlinear pendulum

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -bv - \sin(\theta) + k. \end{aligned} \tag{2}$$

Here θ gives the angular position of the pendulum (assumed to be measured in the counterclockwise direction) and v is its angular velocity. The parameter $b > 0$ measures the damping. The parameter $k \geq 0$ is a constant torque applied to the pendulum in the counterclockwise direction.

a) Find all equilibrium points for this system and determine their stability.

b) Suppose $k > 1$. Prove that there exists a periodic solution for this system in a region R of the form $R = \{(\theta, v) : 0 < v_1 < (k - \sin(\theta))/b < v_2\}$.

c) Find a Hamiltonian function and use it to prove that when $k > 1$ there is a unique periodic solution for this system.

d) Are there any parameter values for which a stable equilibrium and a periodic solution coexist?

Useful references for this project are:

Steven Strogatz, *Nonlinear dynamics and Chaos: with applications to physics, biology, chemistry, and engineering*, 2nd ed., Addison-Wesley Pub. 2014.

R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013

Topic 4: Consider the Hamiltonian systems from Topic 3.

1. Do the first two parts of Topic 3.

2. State the Andronov-Hopf Bifurcation Theorem for a two-dimensional system.

3. Prove the **Lyapunov Center Theorem** as a consequence of the Hopf Bifurcation Theorem.

   Theorem 1 (Lyapunov Center Theorem)**.** *Assume that* $(0,0)$ *is a center equilibrium of the Hamiltonian system* (1) *and that* $\pm\lambda i$ *are simple eigenvalues of the Jacobian* A *of the vector field at* $(0,0)$ *(assume* $\lambda > 0$*). Then each neighborhood of the center contains periodic orbits, whose periods approaches* $2\pi/\lambda$ *as they approach the center*

The Lyapunov Center Theorem (together with a proof) and the Hopf Bifurcation Theorem can be found in:

K. Alligood, T. Sauer, J. Yorke, *Chaos: an introduction to dynamical systems*, Springer, New York, 1996.

The Hopf Bifurcation Theorem can also be found in Chapter 6 of:

Wei-bin Zhang, *Differential equations, bifurcations, and Chaos in economics*, World Scientific 2005.

Topic 5: For much of the 20th century, chemists believed that all chemical reactions tended monotonically to equilibrium. This belief was shattered in the 1950s when the Russian biochemist Belousov discovered that a certain reaction involving citric acid, bromate ions, and sulfuric acid, when combined with a cerium catalyst, could oscillate for long periods of time before settling to equilibrium. The concoction would turn yellow for a while, then fade, then turn yellow again, then fade, and on and on like this for over an hour. This reaction, now called the Belousov-Zhabotinsky reaction (the **BZ reaction**, for short), was a major turning point in the history of chemical reactions. Now, many systems are known to oscillate. Some have even been shown to behave chaotically.

One particularly simple chemical reaction is given by a chlorine dioxide-iodine-malonic acid interaction. The exact differential equations modeling this reaction are extremely complicated. However, there is a planar nonlinear system that closely approximates the concentrations of two of the reactants. The system is

$$\begin{aligned} \dot{x} &= a - x - \frac{4xy}{1+x^2} \\ \dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right) \end{aligned}$$

where x and y represent the concentrations of I^- and ClO_2^-, respectively, and a and b are positive parameters.

1. Find all equilibrium points for this system. Linearize the system at your equilibria and determine the type of each equilibrium.

2. In the ab-plane, sketch the regions where you find asymptotically stable or unstable equilibria.

3. Identify the a, b-values where the system undergoes bifurcations. What kind of bifurcations are these?

4. Using the nullclines for the system together with the Poincaré-Bendixson theorem, find the a, b-values for which a stable limit cycle exists. Why do these values correspond to oscillating chemical reactions?

The project was taken from Chapter 10 of:
R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013

For more details on this reaction, see the following article: Lengyel, I., Rabai, G., and Epstein, I. *Experimental and modeling study of oscillations in the chlorine dioxide-iodine-malonic acid reaction.* J. Amer. Chem. Soc. 112 (1990), 9104.

The very interesting history of the BZ-reaction is described in: Winfree, A. T. *The prehistory of the Belousov-Zhabotinsky reaction.* J. Chem. Educ. 61 (1984), 661.

Topic 6: This project deals with the existence of periodic points of functions defined on an interval or on the real line. A point x is a periodic point of period p for the function f if $f^p(x) = x$. It is of prime period if there is no smaller number $0 < q < p$ such that $f^q(x) = x$. Here $f^p(x)$ means $f \circ f \circ f \ldots \circ f(x)$. For example $f^2(x) = f(f(x))$.

1. Explain what Sharkovskii's ordering is.

2. Give a proof of **Sharkovskii's Theorem**.

Theorem 2 (Sharkovskii)**.** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous map and has an orbit of prime period p. If $p \rhd q$ in the Sharkovskii's ordering, then f has an orbit of period q.*

3. Explain the meaning of "period 3 implies chaos".

4. Give some applications of Sharkovskii's Theorem. For example, can a continuous function on $\mathbb{R}$ have a periodic point of period 176 but not one of period 96? Why? Or prove that if a continuous function $f : [0, 1] \to [0, 1]$ has a periodic point of period 2014, then f has a periodic point of period 100. Does Sharkovskii's Theorem hold for continuous functions $f : \mathbb{R}^2 \to \mathbb{R}^2$?

Aside from Strogatz, these are also useful references (they include proofs):

Robert Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Westview Press, 2003.

K. Alligood, T. Sauer, J. Yorke, *Chaos: an introduction to dynamical systems*, Springer, New York, 1996.

Topic 7: Another interesting project related to **Quantum Mechanical Systems** and **anisotropic Kepler problem** can be found in Chapter 13 of:

R. Devaney, M. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed. Elsevier Academic Press 2013

Topic 8: The subject of Differential Equations, Dynamical Systems and Chaos is a vast subject and many other topics are possible:

a) A project in **Complex Dynamics** (which requires some knowledge of Complex Analysis). This would include a description of the Julia set, the Mandelbrot set, local behavior around fixed points, a classification of the possible Fatou components, hyperbolicity (and the role of the critical points), Chaos, etc.

b) A study of the **van der Pol equation** and Liénard's Theorem.

c) The analysis of a **Lotka-Volterra** equation model of population dynamics and ecology.

d) A new topic!

Please discuss these additional topics with me to ensure that the level of difficulty is within the framework of the course.