MAT 341: Applied Real Analysis – Fall 2015

Practice problems for Midterm 2

The following problems are meant for extra practice only.

Problem 1: Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad u(a,t) = T_0, \quad t > 0;$$
$$u(x,0) = T_0 + T_1 \cos\left(\frac{\pi x}{2a}\right), \quad 0 < x < a.$$

Solution: After you find the steady-state solution, this is Exercise 9 from Ch 2.5; solution at the end of the book.

Problem 2: Find the steady-state solution, the associated eigenvalue problem, and the complete solution of the following problem:

$$\frac{\partial^2 u}{\partial x^2} - \gamma^2 u = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(a,t) = 0, \quad t > 0;$$
$$u(x,0) = \frac{T_1 x}{a}, \quad 0 < x < a.$$

Solution: This is Exercise 5 from Ch 2: Miscellaneous Exercises (page 206); solution at the end of the book.

Problem 3: Find the steady-state solution, the associated eigenvalue problem, and the complete solution of the following problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \quad t > 0;$$
$$u(0,t) = T_0, \quad t > 0;$$
$$u(x,t) \text{ bounded as } x \to \infty;$$
$$u(x,0) = T_0(1 - e^{-2x}), \quad 0 < x.$$

Solution: This is Exercise 11 from Ch 2: Miscellaneous Exercises (page 207) with $\alpha = 2$; solution at the end of the book.

Problem 4: Find the eigenvalues and eigenfunctions of the problem

$$\phi'' + \lambda^2 \phi = 0, \quad 0 < x < 2$$
$$\phi(0) - \phi'(0) = 0$$
$$\phi(2) + \phi'(2) = 0$$

Solution: This is Exercise 3(e) from Ch 2.7 with $a = 2$; solution at the end of the book.

Problem 5: Verify that the eigenvalues and eigenfunctions of the problem

$$(e^x \phi')' + e^x \gamma^2 \phi = 0, \quad 0 < x < a$$
$$\phi(0) = 0 \quad \phi(a) = 0$$

are

$$\gamma_n^2 = \left(\frac{n\pi}{a}\right)^2 + \frac{1}{4}, \quad \phi_n(x) = e^{-\frac{x}{2}} \sin\left(\frac{n\pi x}{a}\right).$$

Is this a regular Sturm-Liouville problem? Find the coefficients for the expansion of the function $f(x) = 1$, $0 < x < a$, in terms of the ϕ_n. To what values does the series converge at $x = 0$ and $x = a$?

Solution: This is Exercise 3 from Ch 2.8; solution at the end of the book.

Problem 6: Solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a, \quad t > 0;$$
$$u(0,t) = 0, \quad u(a,t) = 0, \quad t > 0;$$
$$u(x,0) = 0, \quad 0 < x < a;$$
$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{a}\right), \quad 0 < x < a.$$

Solution: This is similar to homework Exercise 5 from Ch 3.2; solution at the end of the book. Instead of $g(x) = 1$ we use $g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{a}\right)$, which is just a Fourier series that converges uniformly to some function $g(x)$.