Stony Brook University

NAME: Midterm Solutions

RECITATION #: ____

MAT 126 – Spring 2016 – Midterm 2

April 6, 2016

INSTRUCTIONS – PLEASE READ

- Please turn off your cell phone and put it away.
- Please write your name and your section number right now.
- This is a closed book exam. You are NOT allowed to use a calculator or any other electronic device or aid.
- The midterm has 6 problems worth a total of 100 points. Look over your test packet as soon as the exam begins. If you find any missing pages or problems please ask a proctor for another test booklet.
- Show your work. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly. You do not need to simplify your answers unless explicitly instructed to do so.
- Academic integrity is expected of all Stony Brook University students at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1.	
2.	
3.	
4.	
5.	
6.	
Total	

LEC 01	MWF	10:00-10:53am	Joseph Adams
R01	F	1:00-1:53pm	Jaroslaw Jaracz
R02	Tu	4:00-4:53pm	Charles Cifarelli
R03	Tu	1:00-1:53pm	Jaroslaw Jaracz
R04	Th	8:30-9:23am	Alaa Abd-El-Hafez
R05	M	1:00-1:53pm	Thomas Rico
R06	M	9:00-9:53am	Zhuang Tao
R07	W	11:00-11:53am	Dyi-Shing Ou
LEC 02	TuTh	2:30-3:50pm	Raluca Tanase*
R08	Tu	4:00-4:53pm	Gaurish Telang
R09	Tu	1:00-1:53pm	Yuan Gao
R10	Th	1:00-1:53pm	Alaa Abd-El-Hafez
R11	F	1:00-1:53pm	Ruijie Yang
R12	W	12:00-12:53pm	Christopher Ianzano
R13	M	10:00-10:53am	Zhuang Tao
R14	M	12:00-12:53pm	Thomas Rico
LEC 03	MW	4:00-5:20pm	David Kahn
R15	W	9:00-9:53am	Ruijie Yang
R16	Tu	10:00-10:53am	Ying Chi
R17	W	10:00-10:53am	Ying Chi
R18	Th	4:00-4:53pm	Gaurish Telang
R31	W	5:30-6:23pm	Mariangela Ferraro
R32	M	5:30-6:23pm	Charles Cifarelli
R33	Tu	1:00-1:53pm	Yu Zeng

Some trigonometric formulas that might be useful:	
$\sin^2(x) + \cos^2(x) = 1$	$\sin(2x) = 2\sin(x)\cos(x)$
$\tan^2(x) = \sec^2(x) - 1$	$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

Problem 1. (18 points) Compute the following integrals:

a) $\displaystyle\int_0^{\pi/2} x^2 \cos(2x)\, dx$

Sol: Use integration by parts twice

$$\int_0^{\pi/2} x^2\cos(2x)dx = \int_0^{\pi/2} x^2\left(\frac{\sin(2x)}{2}\right)'dx = x^2\frac{\sin(2x)}{2}\Big|_0^{\pi/2} - \int_0^{\pi/2} 2x\cdot\frac{\sin(2x)}{2}\,dx$$

$$\boxed{\begin{matrix} u = x^2 & dv = \cos(2x)dx \\ du = 2x\,dx & v = \frac{\sin(2x)}{2}\end{matrix}}$$

$$= \underbrace{\left(\frac{\pi}{2}\right)^2\frac{\sin(\pi)}{2} - 0^2\frac{\sin(0)}{2}}_{0} + \int_0^{\pi/2} x\left(-\sin(2x)\right)dx.$$

$$\int_0^{\pi/2} x(-\sin(2x))dx = x\frac{\cos(2x)}{2}\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{\cos(2x)}{2}dx = \left(\frac{\pi}{4}\cos(\pi) - 0\right) - \frac{\sin(2x)}{4}\Big|_0^{\pi/2}.$$

$$\boxed{\begin{matrix} u = x & dv = -\sin(2x)dx \\ du = 1\cdot dx & v = \frac{\cos(2x)}{2}\end{matrix}}$$

$$= -\frac{\pi}{4} - \underbrace{\left(\frac{\sin(\pi) - \sin(0)}{4}\right)}_{0} = \boxed{-\frac{\pi}{4}}$$

b) $\displaystyle\int \tan^3(x)\sec(x)\, dx$

Sol: $\int \tan^3(x)\sec(x)\,dx = \int \tan^2 x\cdot\tan x\sec x\,dx = \int(\sec^2 x - 1)\tan x\sec x\,dx$

Use the substitution $u = \sec x$

$du = \tan x\sec x\,dx$

$$= \int (u^2 - 1)\,du = \frac{u^3}{3} - u + c = \frac{\sec^3(x)}{3} - \sec(x) + c$$

Problem 2. (18 points) Compute the following integrals:

a) $\displaystyle\int \frac{5x^2 - x + 2}{(x^2+1)(x-1)}\,dx$

Sol: $\dfrac{5x^2-x+2}{(x^2+1)(x-1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} \Rightarrow 5x^2-x+2 = (x-1)(Bx+C) + A(x^2+1)$

$x=1$: $5\cdot 1^2 - 1 + 2 = 0 + A(1^2+1) \Rightarrow 6 = 2A \Rightarrow A = 3$

$5x^2-x+2 = x^2(A+B) + x(C-B) + (A-C) \Rightarrow \begin{cases} 5 = A+B \\ -1 = C-B \\ 2 = A-C \end{cases} \Rightarrow \begin{cases} B = 2 \\ C = 1 \end{cases}$

Therefore $\displaystyle\int \frac{5x^2-x+2}{(x^2+1)(x-1)}\,dx = 3\int \frac{1}{x-1}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$

$= 3\ln|x-1| + \ln(x^2+1) + \tan^{-1}(x^2+1) + c$

$\displaystyle\int \frac{2x}{x^2+1}\,dx \underset{\boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}}{=} \int \frac{du}{u} = \ln|u| + c = \ln(x^2+1) + c$

b) $\displaystyle\int e^x \sin x\,dx$

Use Integration by parts twice:

Sol 1: $\int e^x \sin x\,dx = \int (e^x)' \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$

$\int e^x \cos x\,dx = \int (e^x)' \cos x\,dx = e^x \cos x - \int e^x(-\sin x)\,dx$ $\Rightarrow$

$\Rightarrow \int e^x \sin dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx \Rightarrow \int e^x \sin x\,dx = \dfrac{e^x \sin x - e^x \cos x}{2} + C$

Sol 2: $\int e^x \sin x\,dx = \int e^x(-\cos x)'\,dx = e^x(-\cos x) - \int e^x(-\cos x)\,dx = -e^x\cos x + \int e^x \cos x\,dx$

$\int e^x \cos x\,dx = \int e^x (\sin x)'\,dx = e^x \sin x - \int e^x \sin x\,dx$

$\Rightarrow \int e^x \sin x\,dx = -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$

$\Rightarrow \int e^x \sin x\,dx = \dfrac{-e^x\cos x + e^x \sin x}{2} + C$

Problem 3. (15 points) Evaluate the integral $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$, for $-3 \le x \le 3$. Simplify your final answer.

Sol: Use the trigonometric substitution $x = 3\sin\theta$

$$dx = 3\cos\theta\, d\theta$$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta|$$

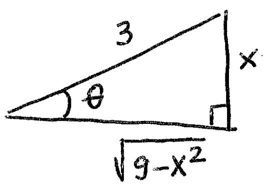
$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{3\cos\theta}\cdot 3\cos\theta\, d\theta = 9\int \sin^2\theta\, d\theta$$

To compute the integral, use the fact that $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{9}{2}\theta - \frac{9}{2}\cdot\frac{\sin(2\theta)}{2} + C = \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C$$

$$= \frac{9}{2}\theta - \frac{9}{4}\, 2\sin\theta\cos\theta + C = \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + C$$

$$x = 3\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$$



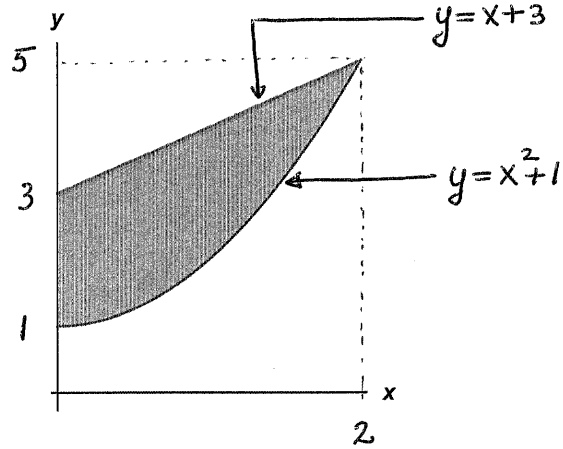
We build a right triangle with an angle θ with $\sin\theta = \frac{x}{3}$ by labeling the opposite side x and hypotenuse 3

$$\left.\begin{aligned}\sin\theta &= \frac{x}{3}\\ \cos\theta &= \frac{\sqrt{9-x^2}}{3}\end{aligned}\right| \Rightarrow \sin(2\theta) = 2\sin\theta\cos(\theta) = \frac{2\sqrt{9-x^2}\cdot x}{9}$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{9-x^2}\cdot x}{2} + C$$

Problem 4. (25 points) The region R in the first quadrant bounded by $y = x^2+1$ and $y = x+3$ is shown to the right.

a) Identify the two curves on the figure and label their points of intersection and the y-intercepts.



$$x^2+1 = x+3 \Rightarrow x^2-x-2=0$$

$$\Rightarrow x^2+x-2x-2=0 \Rightarrow x(x+1)-2(x+1)=0$$

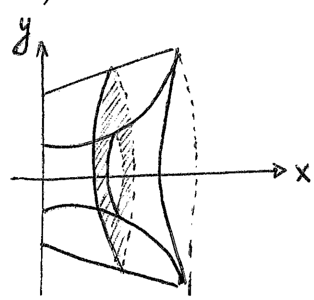
$$\Rightarrow (x-2)(x+1)=0 \Rightarrow x=-1 \text{ or } x=2$$

b) Find the area of the region R.

$$\text{Area} = \int_0^2 \left((x+3)-(x^2+1)\right)dx = \int_0^2 \left(x-x^2+2\right)dx = \frac{x^2}{2}-\frac{x^3}{3}+2x \Big|_0^2$$

$$= \left(\frac{4}{2}-\frac{8}{3}+4\right)-0 = 6-\frac{8}{3} = \frac{10}{3}$$

c) Find the volume of the solid of revolution that results when R is revolved about the x-axis.



A typical cross-section is a washer with outer radius $R = x+3$ and inner radius $r = x^2+1$. The area of the washer is

$$A(x) = \pi(R^2-r^2) = \pi\left((x+3)^2-(x^2+1)^2\right)$$

$$\text{Volume} = \pi\int_0^2 \left((x+3)^2-(x^2+1)^2\right)dx = \pi\int_0^2 (x^2+6x+9)-(x^4+2x^2+1)\,dx$$

$$= \pi\int_0^2 (-x^4-x^2+6x+8)\,dx = \pi\left(-\frac{x^5}{5}-\frac{x^3}{3}+3x^2+8x\right)\Big|_0^2$$

$$= \pi\left(-\frac{32}{5}-\frac{8}{3}+12+16\right) = \pi\left(-\frac{136}{15}+28\right) = \pi\,\frac{284}{15}$$

Problem 5. (16 points) Evaluate the following improper integrals or explain why they diverge. Simply writing "converges" or "diverges" with no explanation or work will result in no credit.

a) $\int_{-1}^{2} \frac{1}{x^{\frac{2}{3}}}\, dx$ The function $\frac{1}{x^{2/3}}$ is continuous on $[-1,0)\cup(0,2]$ and discontinuous at $x=0$.

$$\int_{-1}^{2} \frac{1}{x^{2/3}}\,dx = \int_{-1}^{0} \frac{1}{x^{2/3}}\,dx + \int_{0}^{2} \frac{1}{x^{2/3}}\,dx = \boxed{3+3\sqrt[3]{2}}$$

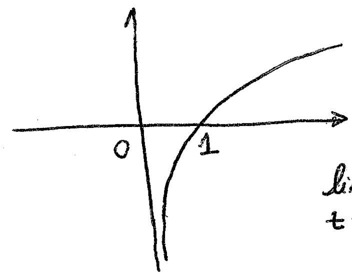
$$\int_{-1}^{0} \frac{1}{x^{2/3}}\,dx = \lim_{t\to0^-}\int_{-1}^{t} \frac{1}{x^{2/3}}\,dx = \lim_{t\to0^-}\int_{-1}^{t} x^{-2/3}\,dx = \lim_{t\to0^-} \frac{x^{-2/3+1}}{-2/3+1}\Bigg|_{-1}^{t}$$

$$= \lim_{t\to0^-} 3x^{1/3}\Big|_{-1}^{t} = \lim_{t\to0^-} 3t^{1/3}-3(-1)^{1/3} = 0-3\sqrt[3]{-1} = 0+3=\boxed{3}$$

$$\int_{0}^{2} \frac{1}{x^{2/3}}\,dx = \lim_{t\to0^+}\int_{t}^{2} \frac{1}{x^{2/3}}\,dx = \lim_{t\to0^+} 3x^{\frac{1}{3}}\Big|_{t}^{2} = \lim_{t\to0^+} 3\cdot2^{\frac{1}{3}}-3\cdot t^{\frac{1}{3}}$$

$$= 3\sqrt[3]{2}-3\cdot\sqrt[3]{0} = \boxed{3\sqrt[3]{2}}$$

b) $\int_{0}^{1} \ln(x)\, dx = \lim_{t\to0^+}\int_{t}^{1} \ln x\, dx = \boxed{-1}$



$y=\ln x$.

$\lim_{t\to0^+} \ln x = -\infty$

Use Integration by parts:

$$\int_{t}^{1} \ln x\, dx = \int_{t}^{1} 1\cdot\ln x\, dx$$

$$= x\ln x\Big|_{t}^{1} - \int_{t}^{1} x\cdot\frac{1}{x}\,dx = (1\cdot\ln 1 - t\cdot\ln t) - \int_{t}^{1} 1\,dx$$

$$= -t\ln t - x\Big|_{t}^{1} = -t\ln t - (1-t)$$

$$\lim_{t\to0^+}\int_{t}^{1} \ln x\, dx = \lim_{t\to0^+} -t\ln t - 1 + t = \underbrace{\left(\lim_{t\to0^+} -t\ln t\right)}_{0} + \underbrace{\left(\lim_{t\to0^+} (-1+t)\right)}_{-1} = \boxed{-1}$$

Use l'Hospital to compute $\lim_{t\to0^+} -t\ln t = \lim_{t\to0^+} \frac{\ln t}{-\frac{1}{t}} \overset{\frac{-\infty}{-\infty}}{=}$

$$= \lim_{t\to0^+} \frac{(\ln t)'}{\left(-\frac{1}{t}\right)'} = \lim_{t\to0^+} \frac{\frac{1}{t}}{\frac{1}{t^2}} = \lim_{t\to0^+} t = \boxed{0}$$

Problem 6. (8 points) Determine whether the following statements are true or false. Circle your response and give a brief explanation (a reason why it's true or an example where it fails).

a) TRUE FALSE Suppose that f is a continuous function on $(-\infty, \infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to+\infty} \int_{-R}^{R} f(x)\,dx.$$

Sol: Assume for example that $f(x) = x$. $\Rightarrow \int_{-R}^{R} x\,dx = \frac{x^2}{2}\Big|_{-R}^{R} = \frac{R^2 - (-R)^2}{2} = 0$

$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} x\,dx = 0$.

However $\int_{-\infty}^{\infty} x\,dx$ is divergent, because $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$

and $\int_{-\infty}^{\infty} x\,dx$ converges if and only if both $\int_{-\infty}^{0} x\,dx$ and $\int_{0}^{\infty} x\,dx$ converge.

$\int_{0}^{\infty} x\,dx = \lim_{R\to\infty} \int_{0}^{R} x\,dx = \lim_{R\to\infty} \frac{x^2}{2}\Big|_{0}^{R} = \lim_{R\to\infty} \frac{R^2}{2} = \infty$

$\Rightarrow \int_{0}^{\infty} x\,dx$ diverges, so $\int_{-\infty}^{\infty} x\,dx$ diverges as well.

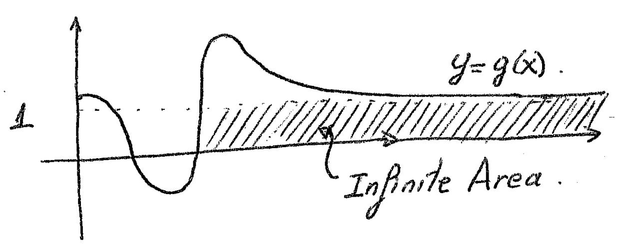
Some Correct Rules: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$ where a is any real number

$= \lim_{R_1\to-\infty} \lim_{R_2\to\infty} \int_{R_1}^{R_2} f(x)\,dx = \lim_{R_1\to-\infty} \int_{R_1}^{\infty} f(x)\,dx = \ldots$

b) TRUE FALSE Let g be a continuous function on $[0, +\infty)$ such that $\lim_{x\to+\infty} g(x) = 1$.

Then the improper integral $\int_0^{+\infty} g(x)\,dx$ is always divergent.

Sol: Since $\lim_{x\to\infty} g(x) = 1$, for x large enough we have $g(x) \simeq 1$.



Assume for the moment that $g(x) = 1$.

$\int_0^{\infty} 1\,dx = \lim_{R\to\infty} \int_0^R 1\,dx = \lim_{R\to\infty} x\Big|_0^R$

$= \lim_{R\to\infty} R - 0 = \infty$

For any general function $g(x)$, since g is continuous and $\lim_{x\to\infty} g(x) = 1$, there exists Some R large such that for all $x > R$ we have $g(x) \simeq 1 > 0.9$

So $\int_0^{\infty} g(x)\,dx = \underbrace{\int_0^R g(x)\,dx}_{\text{this is not improper since } g \text{ is continuous on } [0,R]} + \int_R^{\infty} g(x)\,dx$ $\longrightarrow$ this is divergent, since

$\int_R^{\infty} g(x)\,dx > \int_R^{\infty} 0.9\,dx = \infty$