Fourier Series with Mathematica

Abstract

The computation of Fourier series is extremely useful as a way to break up an arbitrary periodic function into a set of simple trigonometric terms than can be solved individually, and then recombined to obtain the solution to the original problem or an approximation to it.

1 Introducing the Fourier Coefficients

We say that a function f is piecewise continuous on an interval I if f is bounded and continuous except for finitely many jump discontinuities.

Definition 1 *Let f be a piecewise continuous function on $I = [-\pi, \pi]$. The **Fourier series** of f is*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \tag{1}$$

*The numbers a_n and b_n are called the **Fourier coefficients** of f and are given by the formulas*

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \ \ n = 0, 1, 2, \ldots \tag{2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt, \ \ n = 1, 2, \ldots. \tag{3}$$

Theorem 1 *Let f be a piecewise continuous function on $I = [-\pi, \pi]$ and suppose f has a piecewise continuous derivative. Then the Fourier series is convergent and*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(x-) + f(x+)}{2} \tag{4}$$

In particular, the Fourier series is equal to $f(x)$ at all points x where f is continuous.

Remark 1 *If f is any integrable function then the coefficients a_n and b_n may be computed. However, there is no assurance that the Fourier series will converge to f if f is an arbitrary integrable function. In general, we write*

$$f\left(x\right) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(nx\right) + b_n \sin\left(nx\right) \tag{5}$$

if the series converges almost everywhere to f.

2 Graphical approximation

Figure 1 represents a plot in Mathematica of the function $\sinh\left(x\right)$ against its Fourier series to 10, and 50 terms in different colors in the range $-\pi \leq x \leq \pi$. We also give the plot in Mathematica of the piecewise continuous function

$$h(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ -1 & -\pi \leq x < 0 \end{cases}$$

plotted together with its Fourier series with 1, 5 and 20 terms.

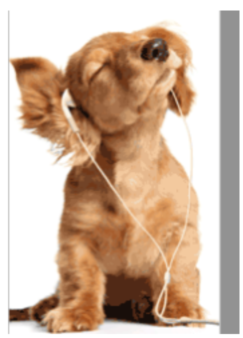

Figure 1: Replace the picture by the plot obtained using Mathematica.