0.1 20/09/07 MAT1300 Notes

Disclaimer: The following is merely a reproduction of my notes from class and have not been verified other than a quick mental check. I do not attempt to claim complete accuracy.

We begin with defining a functional structure. This will be used to give an alternate definition of manifold which we later show to equivalent to definition given previously.

Definition 1 *A functional structure on a topological space X is an assignment, defined on the open sets of X that takes $U \mapsto F_X(U)$ such that:*
1) $F_X(U)$ is a subalgebra of the algebra of continuous functions on U
2) $F_X(U) \supseteq \{constant functions\}$
3) if $V \subseteq U$ and $f \in F_X(U)$ then $f|_V \in F_X(U)$
4) if $U = \bigcup_\alpha U_\alpha$ and $f : U \to \mathbb{R}$ then for $f |_{U_\alpha} \in F_X(U_\alpha) \Rightarrow f \in F_X(U)$

Example 1:
On arbitrary topological spaces, the set of locally constant function and the set of continuous functions are examples of functional structures. When $X = \mathbb{R}^n$, C^∞, and analytic functions also each induce functional structures.

Definition 2 *Given $Y \subseteq X$ and V open in Y define the set*

$$F_Y(V) = \{f : V \to \mathbb{R} \mid \forall y \in V \; \exists \; a \; nbd \; of \; U \in X \; and \; g \in F_X(U) \; such \; that \; g |_{V \cap U} = f |_{V \cap U}\}$$

Example 2:
$V \mapsto F_y(V)$ is a functional structure on Y

Example 3:
$\mathbb{T}^2 \subset \mathbb{R}^3$ Consider an embedding of the torus into $\mathbb{R}^3$ and a functional structure on $\mathbb{R}^3$. One thus induces a functional structure on the torus. It should be noted the functional structure on the torus will be different not only if the functional structure on $\mathbb{R}^3$ is different but also if a different embedding is used.

Example 4:
We consider the functional structure on $\mathbb{R}^3$, $F^3_{\mathbb{R}}(U)$, to be the C^∞ functions. Consider the following embeddings of $\mathbb{R}$ into $\mathbb{R}^2$:
a)let $\iota_a : \mathbb{R} \to \mathbb{R}^2$ be given by $\iota_a(x) = (x, 0)$
b)let $\iota_b : \mathbb{R} \to \mathbb{R}^2$ be given by $\iota_b(x) = (x, \mid v \mid)$

What are $F_{\mathbb{R}}^{\iota_a}$ and $F_{\mathbb{R}}^{\iota_b}$?
$F_{\mathbb{R}}^{\iota_a}$ are all the C^∞ functions. Indeed, any smooth function on the plane when restricted to the line will also be smooth and any smooth function on the line can be extended to a smooth function on the plane by simply maintaining the value at (x,0) for all (x,y) thus proving both inclusions..
However, there are more than just the smooth functions in $F_{\mathbb{R}}^{\iota_a}$. Indeed, the same vertical extension trick works to show $F_{\mathbb{R}}^{\iota_b} \supseteq F_{\mathbb{R}}^{\iota_a}$. Consider the trivially smooth function that takes $(x, y) \to y$ on the plane. The restriction to the range of ι_b however yields $\mid x \mid$ which is not a smooth function. Thus the containment $F_{\mathbb{R}}^{\iota_b} \supset F_{\mathbb{R}}^{\iota_a}$ is proper.

Definition 3 *We say the pair* (X, F_X) *is isomorphic to* (Y, F_Y) *if* $\exists\ \varphi : X \to Y$*, a homeomorphism, and* $\forall$ *open* U *in* X *and* $\forall f : \varphi(U) \to \mathbb{R}$ *then* $f \in F_Y(\varphi U) \Leftrightarrow f \circ \varphi \in F_X(U)$

We now give the alternate definition of an n dimensional manifold.

Definition 4 *An n-dimensional differentiable manifold is a Hausdorff, second countable functionally structured space which is locally isomorphic to* $(\mathbb{R}^n, C^\infty)$

Theorem 1 *Our two definitions for a manifold are indeed equivalent.*

Proof:
This proof consists of four parts. The first two parts respectively start with a manifold in one of the two senses and construct a manifold in the other sense. It then must be checked that these constructions are compatible in that if one starts with a manifold defined in one sense, constructs the manifold in the other sense and then from this new manifold constructs a new manifold again in the original sense then this should be the same as the original manifold.

Assume M is a manifold in the first sense (the "atlas" definition). We will construct a manifold in the second sense (the "functional structure" definition)
Consider an open $U \subseteq M$. Then $F_M(U) = \{f : U \to \mathbb{R} \mid f \circ \varphi^{-1} \textit{is smooth on } \varphi(U \cap U_\varphi)\ \forall (U, \varphi) \textit{ in the atlas}\}$
Check that all properties are satisfied.

For the other construction, start with M a manifold in the second sense and define the atlas to be the set of all local isomorphisms from neighborhoods of points in M to neighborhoods of points in $\mathbb{R}^n$. The only non trivial property that needs to be checked is the "transition function" one and this, along with the rest of the proof, will be done next class (most likely).