Contents

Chapter 1

Fundamental Group

1.1 Homotopies

Definition 1.1.1. If X, Y are topological spaces and $f, f' : X \to Y$ are two continuous maps then we say that f is *homotopic* to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x), \qquad F(x, 1) = f'(x) \tag{1.1}$$

for all $x \in X$ where $I = [0, 1]$ is the unit interval. We say that F is a *homotopy*.

If f is homotopic to a constant map, we will say that it is *null homotopic*.

Hence a homotopy is a continuous, one-parameter family of maps.

Definition 1.1.2. If X is a topological space, a *path* is a continuous map $f : [0, 1] \to X$. A *path homotopy* is a homotopy that fixes endpoints (see Figure 1.1).

Lemma 1.1.3. *The relation of being homotopic (denoted $\simeq$) or path homotopic (denoted $\simeq_p$) are equivalence relations.*

Proof. We need to check that the relation is reflexive, transitive, and symmetric. Let f, f' :

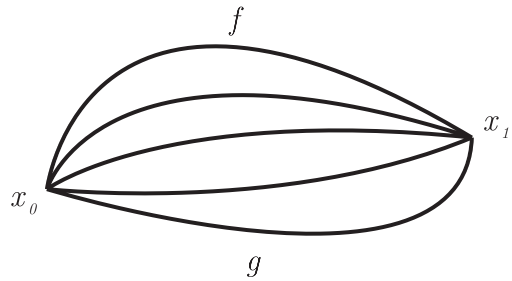

Figure 1.1: An example of a homotopy between the paths f and g with endpoints x_0 and x_1. A few intermediate trajectories have been included.

$X \to Y$ be two continuous functions.

1. Reflexivity: Clearly $f \simeq f$ by taking the identity map to be the desired homotopy.

2. Symmetry: If $f \simeq f'$ let F be a homotopy from f to f' so that $F(0, x) = f(x)$ and $F(1, x) = f'(x)$. Define $H(x, t) = F(x, 1 - t)$ which is clearly a homotopy between f' and f.

3. Transitivity If $f \overset{F}{\simeq} f'$ and $f' \overset{F'}{\simeq} f''$ we define

$$H(x,t) = \begin{cases} F(x, 2t) & t \in \left[0, \frac{1}{2}\right] \\ F'(x, 2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{1.2}$$

By the Pasting Lemma, this is a continuous map and hence is the desired homotopy.

□

Example 1.1.4. 1. Let $f, g : X \to \mathbb{R}^2$ (or in fact $\mathbb{R}^n$). Then f and g are homotopic with the desired homotopy given by

$$F(x, t) = (1 - t)f(x) + tg(x) \tag{1.3}$$

which is the *straight line homotopy.* This example generalizes to any convex space, in which we can see that any two paths are homotopic.

2. Consider the two-sphere. Note that any two paths must again be homotopic, but this is difficult to show directly. Instead of showing this directly, we instead defer to a later section wherein we show that any two paths of a simply connected space are homotopic (see Theorem 1.2.8 and Figure 1.2).

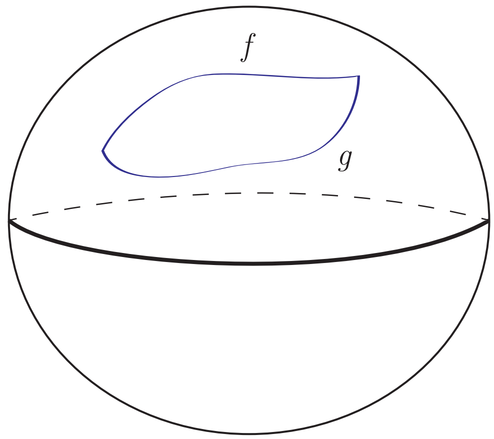

Figure 1.2: Any two paths on the sphere are homotopic. This follows because the sphere is simply connected (see Definition 1.2.7).

3. Consider the punctured Euclidean plane $\mathbb{R}^2 \setminus \{0\}$ and the paths (Figure 1.3)

$$f(s) = (\cos \pi s, \sin \pi s), \qquad g(s) = (\cos \pi s, 2 \sin \pi s), \qquad h(s) = (\cos \pi s, -\sin \pi s). \tag{1.4}$$

   We can easily visualize the necessary homotopy to see that $f \simeq g$; however, the same visualization process will not permit us to pass the punctured origin in the case of f and h. In fact, it turns out that $f \not\simeq h$.

Definition 1.1.5. If $f : [0,1] \to X$ is a path in X from x_0 to x_1 and $g : [0,1] \to X$ is a path from x_1 to x_2 then we can define the star product $f \star g$ defined as

$$h(s) = f \star g = \begin{cases} f(2s) & s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & s \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{1.5}$$

The star product is simply the concatenation of two paths, wherein we adjust the parameters to ensure that the domain of our product still lies on the unit interval.

Proposition 1.1.6. *Let $[f]$ be a path homotopy class of a path f and $[g]$ be a path homotopy class of $[g]$. Then $[f] \star [g]$ is a well defined map.*

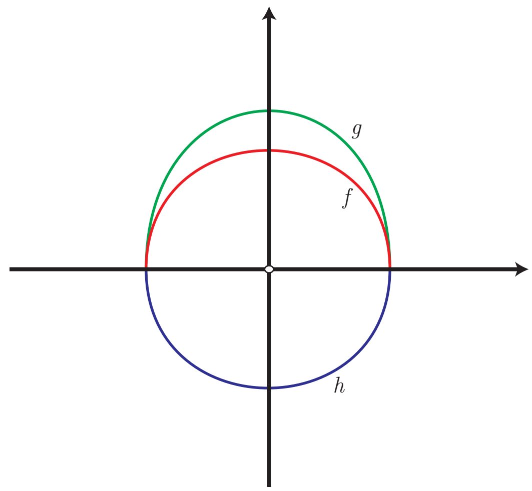

Figure 1.3: Not all paths are homotopic in the punctured Euclidean plane. Intuitively, we are unable to continuously deformed the function h into f because of the hole at the origin.

Proof. Assume that $[f] = [f']$ and $[g] = [g']$. Our goal is to show that $[f] \star [g] = [f'] \star [g']$. The fact that the path classes are equivalent means that we can find homotopies F and G such that $f \overset{F}{\simeq} f'$ and $g \overset{G}{\simeq} g'$. Define the homotopy

$$H(s,t) = \begin{cases} F(2s,t) & s \in \left[0, \frac{1}{2}\right] \\ G(2s-1,t) & s \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{1.6}$$

which is easily seen to be a homotopy from $f \star g$ to $f' \star g'$ and hence $[f] \star [g] = [f'] \star [g']$ as required. □

Our goal will be to show that we can define a group on the set of path homotopy classes. In order to do this, we will need the following results.

Proposition 1.1.7.

1. *Suppose that $K : Y \to Z$ is a continuous map and $F : X \times I \to Y$ is a path homotopy between f and f'. Then $K \circ f$ is path homotopic to $K \circ f'$ via $K \circ F$ (see Figure 1.4).*

2. *If $K : Y \to Z$ is continuous map and f, g are paths in X with $f(1) = g(0)$ then (see Figure 1.5)*
$$K \circ (f \star g) = (K \circ f) \star (K \circ g). \tag{1.7}$$

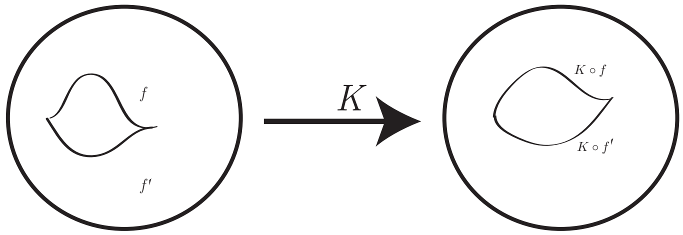

Figure 1.4: The continuous image of a homotopy is still a homotopy.

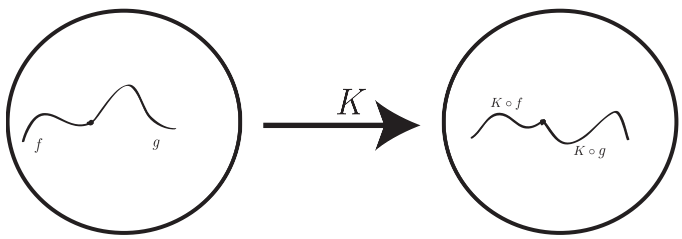

Figure 1.5: Continuous maps between topological spaces preserve the star product.

Proof.

1. Since K and F are both continuous, their composition is continuous and $K \circ F : X \times I \to Z$. We thus need only check the homotopy conditions, which follows since

$$(K \circ F)(x, 0) = K(F(x, 0)) = K(f(x)) = (K \circ f)(x)$$
$$(K \circ F)(x, 1) = K(F(x, 1)) = K(f'(x)) = (K \circ f')(x)$$

2. This fact is trivial so long as we ensure that the endpoints are preserved. We see that this is true since

$$(K \circ f)(1) = K(f(1)) = K(g(0)) = (K \circ g)(0) \tag{1.8}$$

so the star-product will be consistently defined.

□

Theorem 1.1.8. *The operation $\star$ satisfies all the properties for a group binary operation; that is, it is associative, there exists a right and left identity, and there is a two-sided inverse corresponding to the aforementioned identity.*

Proof. 1. ASSOCIATIVITY: We recall that for any two connected intervals $[a, b]$ and $[c, d]$ in $\mathbb{R}$ there exists a linear map $p : [a, b] \to [c, d]$ such that $p(a) = c$ and $p(b) = d$. Such a map is called a positive linear map because its slope is always positive.

We can use positive linear maps to redefine the star product as follows: Define the positive linear functions

$$p_1 : \left[0, \frac{1}{2}\right] \to [0, 1], \qquad p_2 : \left[\frac{1}{2}, 1\right] \to [0, 1]. \tag{1.9}$$

It is then easy to see that for any two paths γ, λ we have that

$$\gamma \star \lambda = \begin{cases} \gamma \circ p_1 & \text{on } \left[0, \frac{1}{2}\right] \\ \lambda \circ p_2 & \text{on } \left[\frac{1}{2}, 1\right] \end{cases}. \tag{1.10}$$

These maps allow us to go a step further and define a triple product (and in fact, this can be done as often as necessary to create and n-product concatenation operator) in the following sense. For $0 < a < b < 1$ define the positive linear maps

$$p_1 : [0, a] \to [0, 1], \qquad p_2 : [a, b] \to [0, 1], \qquad p_3 : [b, 1] \to [0, 1]. \tag{1.11}$$

For any three continuous maps γ, λ, μ such that $\gamma(1) = \lambda(0)$ and $\lambda(1) = \mu(0)$ we can define the triple product

$$K_{a,b} = \begin{cases} f \circ p_1 & \text{on } [0, a] \\ g \circ p_2 & \text{on } [a, b] \\ h \circ p_3 & \text{on } [b, 1] \end{cases}. \tag{1.12}$$

Hence if $0 < a < b < 1$ and $0 < c < d < 1$ then, by our previous comment, we know there is a positive linear map $p : [a, b] \to [c, d]$ which implies that $[K_{a,b}] = [K_{c,d}]$. Indeed, since $p : I \to I$ and $\mathrm{id} : I \to I$ and all paths on I are homotopic, there must exist a path homotopy P so that $p \overset{P}{\simeq} \mathrm{id}$, in which case $K_{c,d} \circ P$ is a path homotopy between $K_{c,d}$ and $K_{a,b}$.

To show associativity, let $[f], [g], [h]$ denote path classes and notice that

$$(f \star g) \star h = \begin{cases} f(4s) & s \in \left[0, \frac{1}{4}\right] \\ g(4s - 1) & s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ h(2s - 1) & s \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{1.13}$$

which is just the triple product described by $K_{\frac{1}{4},\frac{1}{2}}$. On the other hand

$$f \star (g \star h) = \begin{cases} f(2s) & s \in \left[0, \frac{1}{2}\right] \\ g(4s - 2) & s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ h(4s - 3) & s \in \left[\frac{3}{4}, 1\right] \end{cases} \tag{1.14}$$

which is the triple product $K_{\frac{1}{2},\frac{3}{4}}$. But we know then that $[K_{\frac{1}{4},\frac{1}{2}}] = [K_{\frac{1}{2},\frac{3}{4}}]$ and so the star product is associative as required.

2. Identities: Let $[e]$ denote the path class of null homotopies so that for any path f we have

$$f \star e = \begin{cases} f(2s) & s \in \left[0, \frac{1}{2}\right] \\ f(1) & s \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{1.15}$$

   This function is clearly homotopic to f via the straight line homotopy

$$H(x,t) = (1-t)f(x) + t(f \star e)(x) \tag{1.16}$$

   which is clearly continuous since it is the composition of continuous functions. Hence $[f] \star [e] = [f]$ so that $[e]$ is the identity element.

3. Inverse: For any path $f : [0,1] \to X$ let $\bar{f} : [0,1] \to X$ denote the same path but with reverse orientation; that is, $\bar{f}(t) = f(1-t)$. We claim that $[f] \star [\bar{f}] = [e]$. Indeed, notice that $f \star \bar{f}$ is null homotopic, which is what we wanted to show.

□

1.2 The Fundamental Group

Definition 1.2.1. Pick $x_0 \in X$ which we call the *basepoint*. Paths that both begin and end at x_0 are called *loops*. The set of homotopy classes of loops forms a group under $\star$ relative to x_0, which we denote $\pi_1(X, x_0)$

Example 1.2.2. Consider $\mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$ be arbitrary. Via our discussion before, since $\mathbb{R}^n$ is contractible all paths (and hence all loops) are homotopic to each other. In particular, this means that all loops are nulhomotopic and so $\pi_1(\mathbb{R}^n, x_0) = \{0\}$.

In defining the fundamental group, it was essential that we chose a basepoint for our loops. A natural question then arises as to how much the fundamental group of topological space depends on the basepoint that we choose. It turns out that the answer is actually fairly intuitive. If our space is path connected, then by choosing a different point x_1, we can artificially consider loops about x_1 by concatenating them with a path γ connecting x_0 to x_1 and the loops about x_0. Since this idea is completely symmetric in x_0 and x_1, the converse will also hold true and the fundamental groups should be isomorphic. The details are done below.

Definition 1.2.3. Let α be a path in X from x_0 to x_1. Define

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1), \qquad \hat{\alpha}([f]) = [\bar{\alpha}] \star [f] \star [\alpha] \tag{1.17}$$

where $\bar{\alpha}$ denotes the path traversed in the opposite direction.

Theorem 1.2.4. *The map $\hat{\alpha}$ is a group isomorphism.*

Proof. We begin by showing that the map is well-defined. Let $[f], [g] \in \pi_1(X, x_0)$ with $[f] = [g]$, for which we want to show that $\hat{\alpha}([f]) = \hat{\alpha}([g])$. However, this follows quickly since

$$\begin{aligned}\hat{\alpha}([f]) &= [\bar{\alpha}] \star [f] \star [\alpha] \\ &= [\bar{\alpha}] \star [g] \star [\alpha] \\ &= \hat{\alpha}([g])\end{aligned}$$

Showing that this is a homomorphism is just a simple calculation

$$\begin{aligned}\hat{\alpha}([f]) \star \hat{\alpha}([g]) &= \Big([\bar{\alpha}] \star [f] \star [\alpha]\Big) \star \Big([\bar{\alpha}] \star [g] \star [\alpha]\Big) \\ &= [\bar{\alpha}] \star [f] \star [\alpha] \star [\bar{\alpha}] \star [g] \star [\alpha] \\ &= [\alpha] \star [f] \star [g] \star [\alpha] \\ &= \hat{\alpha}([f] \star [g])\end{aligned}$$

To show that $\hat{\alpha}$ is an isomorphism it is sufficient to find an inverse. By composing with $\hat{\bar{\alpha}}$ we find that

$$\begin{aligned}\hat{\alpha}(\hat{\bar{\alpha}})([h]) &= \hat{\alpha}\Big([\alpha] \star [h] \star [\bar{\alpha}]\Big) \\ &= [\bar{\alpha}] \star [\alpha] \star [h] \star [\bar{\alpha}] \star [\alpha] \\ &= [h]\end{aligned}$$

□

Corollary 1.2.5. *If X is path connected and $x_0, x_1 \in X$ then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Proof. Since X is path connected there exists a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then $\hat{\gamma}$ is an isomorphism by our previous theorem, which is precisely what we wanted to show. □

We have shown that in path connected spaces, all fundamental groups are isomorphic regardless of our choice of basepoint. However, the isomorphisms constructed did depend on our choice of path. Our next result tells us when we can throw away this dependency as well.

Theorem 1.2.6. *The isomorphism of $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ is independent of path if and only if the fundamental group is abelian.*

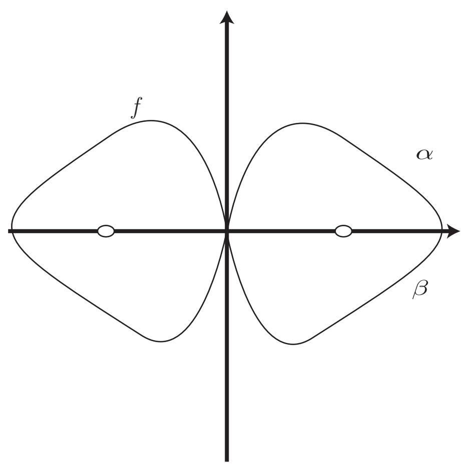

Figure 1.6: There may be ambiguity in the choice of isomorphism between the fundamental groups. Namely, by tracing an isomorphism around a path, what choice we make at the origin could effect the form of the isomorphism in question.

Proof. Suppose that $\pi_1(X, x_0)$ is abelian and define

$$\hat{\alpha}[f] = [\bar{\alpha}] \star [f] \star [\alpha], \qquad \hat{\beta}[f] = [\bar{\beta}] \star [f] \star [\beta]. \tag{1.18}$$

We would like to show that these are equal. Note that $[\bar{\alpha}] \star [\beta]$ is a loop at x_0 and since $\pi_1(X, x_0)$ is abelian, they can be interchanged

$$\begin{aligned}
[\bar{\alpha}] \star [f] \star [\alpha] &= [\bar{\alpha}] \star [\beta] \star [\bar{\beta}] \star [f] \star [\beta] \star [\bar{\beta}] \star [\alpha] \\
&= [\bar{\beta}] \star [f] \star [\beta] \star [\bar{\alpha}] \star [\beta] \star [\bar{\beta}] \star [\alpha] \\
&= [\bar{\beta}] \star [f] \star [\beta]
\end{aligned}$$

On the other hand, suppose the isomorphism is independent of a path. We would like to show that $\pi_1(X, x_0)$ is abelian, so that $[f] \star [g] = [g] \star [f]$. Hence for any two paths α and β connecting x_0 and x_1 we have that

$$[\bar{\alpha}] \star [f] \star [\alpha] = [\bar{\beta}] \star [f] \star [\beta]. \tag{1.19}$$

Applying $[\beta]$ to the left and $[\bar{\alpha}]$ to the right we get

$$([\beta] \star [\bar{\alpha}]) \star [f] = [f] \star ([\beta] \star [\bar{\alpha}]). \tag{1.20}$$

Since α and β have the same endpoints, $\alpha \star \bar{\beta}$ is a loop about x_0 and so the fundamental group is abelian as necessary. □

Definition 1.2.7. If $\pi_1(X, x_0) = \{0\}$ then we say that X is *simply connected.*

Theorem 1.2.8. *In a simply connected space, any two paths having the same initial and final points are path homotopic.*

Proof. For a topological space X let $f : [0, 1] \to X$ and $g : [0, 1] \to X$ be two paths with identical endpoints, say $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. The path $\bar{f} \star g$ is a loop about f, and since X is simply connected it follows that $\bar{f} \star g$ is homotopic to the null homotopy, or equivalently $[\bar{f} \star g] = [e]$. Since $[e]$ is the identity element of the fundamental group, we then have that $[g] = [f] \star [\bar{f} \star g] = [f] \star [e] = [f]$ and so $[f] = [g]$. □

1.3 Covering Spaces

Definition 1.3.1. Let $p : E \to B$ be a continuous surjective map. An open set $U \subseteq B$ is said to be *evenly covered by* p if we can write $p^{-1}(U)$ as the disjoint union of open sets whose restriction to each open set is a homeomorphism onto U. Mathematically,

$$p^{-1}(U) = \bigsqcup_{\alpha} V_\alpha \tag{1.21}$$

where V_α is open and the map $p\Big|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism, for each α.

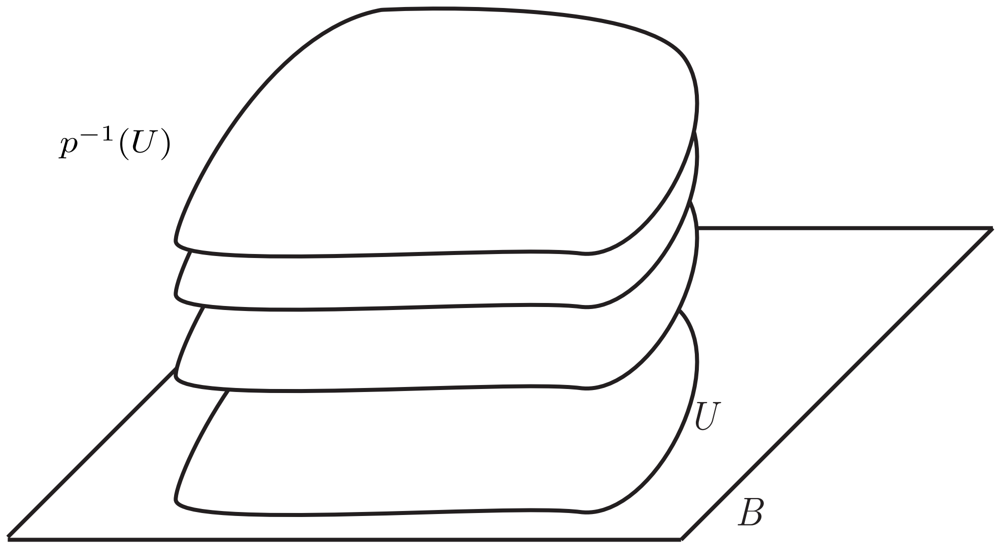

Figure 1.7: Intuitively, we think of the collection of $\{V_\alpha\}$ as a stack of spaces which hover just above $U \subseteq B$. The map p then squishes these V_α onto U.

Definition 1.3.2. Let p be continuous and surjective. If every $b \in B$ has a neighbourhood U that is evenly covered by p then we say that p is a *covering map* and E is a *covering space*.

Note that for a single point $b \in B$, $p^{-1}(b)$ is discrete.

Proposition 1.3.3. *If $p : E \to B$ is a covering map, then p is an open map.*

Proof. If A is open in E we want to show that $p(A)$ is open in B. Let $x \in p(A)$, for which there exists U that is evenly covered by p. If V_α partitions U into slices, we can find a point $y \in A$ such that $p(y) = x$ and choose V_β such that $y \in V_\beta$. Take $V_\beta \cap A$ which is open in E and hence is open in V_β. Since p is a homeomorphism on the restriction to V_β we then have that $p(V_\beta \cap A)$ is open in U and hence open in B. Since x was arbitrary, every point in $p(A)$ can be covered by an open neighbourhood contained entirely in $p(A)$ and so $p(A)$ is open, showing that p is an open map as required. $\square$

Example 1.3.4. 1. The identity map $\mathrm{id} : X \to X$ is a covering map with covering space $X \times \{1, \ldots, n\} \to X$ where $p(x, i) = p(x)$.

2. Consider a covering map $p : \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. Note that the real line is wrapped around the unit circle for each unit interval $[n, n+1]$. To show that the circle can be evenly covered, consider for example the case when the first coordinate is positive and the second coordinate is free to vary. This corresponds to the choice of x such that $\cos 2\pi x > 0$. This is easily seen to be satisfied by $V_n = \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$ for every choice of n.

3. Every covering map is a local homeomorphism, but not every local homeomorphism is a covering map. For example, consider the same map as in Example 2 above, but restricted to only the positive real line; that is, $p : \mathbb{R}^+ \to S^1$. This map *is* a local homeomorphism but is not a covering map since the point $p(0) = (1, 0)$ is not evenly covered. On the other hand $p : S^1 \to S^1$ defined by $z \mapsto z^n$ is a covering map.

As mentioned in Example 1.3.4, all covering maps are local homeomorphisms, essentially by definition. By imposing a few additional properties our base space and covering space, we can deduce when covering maps are precisely homeomorphisms:

Proposition 1.3.5. *Let $p : E \to B$ be a covering map with E path connected. If B is simply connected then p is a homeomorphism.*

Proof. By definition, we know that p is surjective and open and so p^{-1} is necessarily continuous. All that remains to be shown is that p is injective. Since E is path connected we know that for each $b_0 \in B$ the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. Since B is simply connected, $\pi_1(B, b_0) = \{0\}$ for all $b_0 \in B$. The surjectivity of ϕ then implies that the preimage of every point in B is necessarily a singleton. Let $e_0, e_1 \in E$ and assume that $p(e_0) = p(e_1) = b_0$. By our discussion, we then have that $e_0, e_1 \in p^{-1}(b_0) = \{\phi(0)\}$ and hence since this is a singleton set, $e_0 = \phi(0) = e_1$. Thus p is injective as required, and we conclude that p is a homeomorphism. $\square$

Theorem 1.3.6. *Let $p : E \to B$ be a covering map. If B_0 is a subspace of B and $E_0 = p^{-1}(B_0)$ then $p_0 : E_0 \to B_0$ is a covering map and $p_0 = p\Big|_{E_0}$.*

Proof. Let $b_0 \in B_0$ and choose a neighbourhood U of b_0 evenly covered by p. Let $\{V_\alpha\}$ partition U into slices and take $\tilde{V}_\alpha = V_\alpha \cap E_0$. The $\tilde{V}_\alpha$ are then a disjoint collection of sets whose union is $p^{-1}(U \cap B_0)$ and each $\tilde{V}_\alpha$ is homeomorphically mapped to $B_0 \cap U$. □

Theorem 1.3.7. *If $p : E \to B$ and $p' : E' \to B'$ are covering map then $p \times p' : E \times E' \to B \times B'$ is a covering map.*

Proof. The proof of this theorem is fairly rudimentary and is done precisely how one might naively think it is done. Consider $b \in B, b' \in B'$ and take $U \subseteq B, U' \subseteq B'$ to be open neighbourhoods of b and b' that are evenly covered by p and p' respectively. Let $\{V_\alpha\}$ partition U into slices and $\{V'_\alpha\}$ partition U' into slices. Then the sets $\{V_\alpha \times V'_\alpha\}$ are disjoint open sets of $E \times E'$ and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$; hence they constitute a partition of $U \times U'$ into slices, so $p \times p'$ is a covering map. □

Example 1.3.8.

1. The map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ where $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map of the torus T.

2. The map $p \times i : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$ is a covering map for a cylinder. There is a map from $S^1 \times \mathbb{R}_+ \to \mathbb{R}^2 \setminus \{0\}$ which is a homeomorphism via $(x, t) \mapsto tx$.

1.3.1 Liftings

Our goal will be to describe the fundamental groups of our base-space via the (images of) the fundamental group of our covering space. In order to do this, we need to be able to associate paths in one space with those of another. This leads to the notion of a lifting.

Definition 1.3.9. Let $p : E \to B$ be a covering map. If f is a continuous mapping of X into B, a *lifting of f* is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

This is best described as follows: A lifting is a function $\tilde{f} : X \to E$ such that the diagram given in Figure 1.8 commutes.

Example 1.3.10. Consider our standard covering map $p : \mathbb{R} \to S^1$ and $f : [0, 1] \to S^1$ by $f(s) = (\cos \pi s, \sin \pi s)$. Then $\tilde{f}(s) = \frac{s}{2}$ since

$$p \circ \tilde{f}(s) = p(\tilde{f}(s)) = p\left(\frac{s}{2}\right) = \left(\cos 2\pi \frac{s}{2}, \sin 2\pi \frac{s}{2}\right) = (\cos \pi s, \sin \pi s) = f(s). \tag{1.22}$$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Figure 1.8: Commutative diagram describing how lifting maps work. See Definition 1.3.9.

On the other hand, if $g(s) = (\cos \pi s, -\sin \pi s)$ then the lifting is $\tilde{g}(s) = -\frac{s}{2}$, where we have explicitly taken advantage of the even/odd nature of sine and cosine. Notice that despite the fact that f and g have the same endpoints on S^1, their liftings do not share endpoints. They have become "unhooked." Furthermore, notice that while the path f winds around the curve, since a lifting of f manifests as a curve in $\mathbb{R}$ this periodicity is lost.

Lemma 1.3.11. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.*

Proof. Cover B by open sets that are each evenly covered by p. Find a subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ of $[0,1]$ such that for each i the set $f([s_i, s_{i+1}])$ lies in such an open set U (this follows via the Lebesgue numbering lemma). Define $\tilde{f}(0) = e_0$ and assume that we have defined $\tilde{f}$ on $[0, s_i]$, for which we would like to define $\tilde{f}$ on $[s_i, s_{i+1}]$. Note that $f([s_i, s_{i+1}]) \subseteq U$ is evenly covered by p. Take V_α to be a partition of $p^{-1}(U)$ into slices and let us look at $\tilde{f}(s_i)$. Select V_0 such that $\tilde{f}(s_i) \in V_0$ and define $\tilde{f}$ on $[s_i, s_{i+1}]$ as $\left(p\Big|_{V_0}\right)^{-1} \circ f(s)$ which is continuous by the pasting lemma.

To show uniqueness, suppose that $\tilde{g}$ is another lifting of f beginning at e_0, so that $\tilde{g}(0) = e_0 = \tilde{f}(0)$. Suppose that $\tilde{g} = \tilde{f}$ on $[0, s_i]$ and choose V_0 as before. Since $[s_i, s_{i+1}]$ is connected, so too must its image. In particular, since $\tilde{g}(s_i) \in V_0$ we must have that $g([s_i, s_{i+1}]) \subseteq V_0$. Fix an arbitrary point $s \in [s_i, s_{i+1}]$, and define $y = \tilde{g}(s) \in V_0$ which lies in $p^{-1}(f(s))$. This point is unique, so $\tilde{g}(s) = \tilde{f}(s)$ and so by induction, the two liftings are unique. $\square$

Lemma 1.3.12. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $F : I \times I \to B$ be continuous with $F(0,0) = b_0$. Then there exists a unique continuous map $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$ and $p \circ \tilde{F} = F$. Moreover, if F is a path homotopy then $\tilde{F}$ is a path homotopy.*

Proof. Define $\tilde{F}(0,0) = e_0$. Choose a subdivision $0 = s_0 < \cdots < s_m = 1$ and $0 = t_0 < \cdots < t_n = 1$ and define $I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ so that $F(I_i \times J_j) \subsetneq U$, where U is evenly covered by p. Assume that we have extended $\tilde{F}$ and let us define $\tilde{\tilde{F}}$ on I_{i_0} and J_{j_0}. Choose U such that $F(I_{i_0} \times J_{j_0}) \subsetneq U$ and write $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$. Choose V_0 that contains $\tilde{F}(i_0 - 1, j_0 - 1)$ and define $\tilde{F} = \left(p \Big|_{V_0}\right)^{-1} \circ F$ on $I_{i_0} \times J_{j_0}$. Uniqueness follows in a fashion similar to that which we have covered before. $\square$

Suppose that F is a path homotopy so that $0 \times I \mapsto b_0$. If $\tilde{F}$ is a path homotopy of F then $0 \times I \mapsto p^{-1}(b_0)$ but the inverse image of b_0 is discrete and connected, so $0 \times I \mapsto e_0$. Precisely the same argument holds for $1 \times I$ so $1 \times I$ is mapped to the same points by F and $\tilde{F}$.

Theorem 1.3.13. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$, and let f, g be any two paths in B with endpoints b_0 and b_1. Let $\tilde{f}, \tilde{g}$ be their lifting to path in E beginning at e_0. If f and g are path homotopic then $\tilde{f}, \tilde{g}$ end at the same endpoints in E and are path homotopic.*

Proof. Since f and g are path homotopic, let $F : I \times I \to B$ be a path homotopy between them. By Lemma 1.3.12 we know that there exists $\tilde{F} : I \times I \to E$, a lifting of F which is a path homotopy for $\tilde{f}$ and $\tilde{g}$, and $F(0,t) = e_0$ while $F(1,t) = e_1$ for all $t \in I$. The restriction $\tilde{F}|_{I\times\{0\}}$ corresponds to a lift of $F|_{I\times\{0\}}$ and by uniqueness of lifts of path homotopies, we must have $\tilde{F}_{I\times\{0\}} = \tilde{f}(s)$. Similarly, $\tilde{F}_{I\times\{1\}} = \tilde{g}(s)$. Hence both $\tilde{f}$ and $\tilde{g}$ end at the same point. $\square$

Definition 1.3.14. Let $p : E \to B$ be a covering map and let $b_0 \in B, e_0 \in E$ such that $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$ let $\tilde{f}$ be a lifting of f starting at e_0. Define a *lifting correspondence* to be the map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ acting as $\phi[f] = \tilde{f}(1)$.

Intrinsically, we recall that the lifting of a loop need not be a loop. Hence a lifting correspondence simply takes loops in B to the endpoint of the lifting of the homotopy class. However, we also see from Theorem 1.3.13 that lifts of path homotopic functions preserves endpoints, so equivalence classes of $\pi_1(B, b_0)$ all define the same endpoint in $p^{-1}(b_0)$ implies that the lifting correspondence is well defined.

Theorem 1.3.15. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. If E is path connected then $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. If E is simply connected then ϕ is bijective.*

Proof. Suppose that E is path connected. Let $e_1 \in p^{-1}(b_0) \subseteq E$, which we can connect to e_0 via a path since the space is path connected. Let $\tilde{f}$ be such a path and consider $p \circ \tilde{f}$ which is a loop about b_0 in B. The loop $f = p \circ \tilde{f}$ lifts as $\tilde{f}$ starting at e_0 so $\phi[f] = e_1$, and we conclude that ϕ is surjective.

Now suppose that E is simply connected and take $[f], [g] \in \pi_1(B, b_0)$. Suppose that $\phi([f]) = \phi([g])$, so that the corresponding lifts $\tilde{f}$ and $\tilde{g}$ must have the same endpoint (Theorem 1.3.13). Since E is simply connected, we know that all trajectories with the same endpoints are path homotopic, and so $\tilde{f} \simeq \tilde{g}$ with homotopy F. The map $p \circ F$ is a fixed point homotopy between f and g, so $[f] = [g]$ and we conclude that ϕ is injective. Both results together imply that ϕ is bijective. $\square$

1.3.2 The Fundamental Group of the Circle

We are now is a position to calculate the fundamental group of the circle.

Theorem 1.3.16. *The fundamental group of the circle is the integers. Namely,*

$$\pi_1(S^1, b_0) = \mathbb{Z}. \tag{1.23}$$

Proof. Consider the covering map $p : \mathbb{R} \to S^1$ given by $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. Set $b_0 = 0$ so that $p^{-1}(b_0) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, Theorem 1.3.15 tells us that the map $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is surjective. Hence all that remains to be shown is that ϕ is a group homomorphism; that is,

$$\phi([f] \star [g]) = \phi[f] + \phi[g]. \tag{1.24}$$

Fix $[f]$ and $[g]$ in $\pi_1(S_1, b_0)$ with $\phi[f] = n$ and $\phi[g] = m$. Let $\tilde{f}$ and $\tilde{g}$ be their lifts in $\mathbb{R}$ at 0, respectively. Define $\hat{g}(s) = n + \tilde{g}(s)$ on $\mathbb{R}$. Since p is periodic on the integers, we know that $\hat{g}$ is also a lifting of g, but beginning at n. Hence $\tilde{f} \star \hat{g}$ is well defined, is a lifting of $f \star g$, and $\hat{g}(1) = n + m$. Thus

$$\phi([f] \star [g]) = \hat{g}(1) = n + m = \phi[f] + \phi[g]. \tag{1.25}$$

$\square$

Definition 1.3.17. Let $h : X \to Y$ be a continuous map. Then there is a natural map $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $y_0 = h(x_0)$, acting as $h_* f = [h \circ f]$ which is called the *induced homomorphism of* h.

Note that $(k \circ h)_* = k_* \circ h_*$ and that if id is the identity map on topological spaces then id_* is the identity homomorphism. In particular, if h is a homeomorphism, then h_* is an isomorphism. This is another way of saying that the correspondence $h \mapsto h^*$ is a covariant functor between the category of topological spaces with continuous maps and the category of groups with group homomorphisms.

The following is a more general version of Theorem 1.3.15 which will be used in Section 1.11:

Theorem 1.3.18. *Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$.*

1. *The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.*
2. *Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$ of the collection of right cosets of H into $p^{-1}(b_0)$, and this map is bijective if E is path connected.*
3. *If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.*

Proof.

1. We want to show that if $p_*([f])$ is trivial then $[f]$ is trivial; that is, the kernel of this map is trivial. We compute $p_*([f]) = [p \circ f]$, which is trivial, and hence homotopic to the constant map at b_0, say c_{b_0}. Let F be the desired homotopy, so that we may apply the homotopy lifting lemma to get $\tilde{F}$, a lift of F to E. Then $\tilde{F}$ is a homotopy of f to the null map c_{e_0} which is what we wanted to show.

2. We begin by showing that the map Φ is well defined. Suppose $[f] \in H \star [g]$, for which our goal is to show that $\Phi([f]) = \Phi([g])$. The assumption that $[f] \in H \star [g]$ means that f is homotopic to a loop $h = p \circ \tilde{h} \in H$ followed by g; that is, $f \simeq h \star g$. Let $\tilde{h} \star \tilde{g}$ be the lifting of $h \star g$. Since $[f] = [h \star g]$ we then have that $\tilde{f}$ and $\tilde{h} \star \tilde{g}$ must begin at e_0 and end the at same point, so that $\phi[f] = \phi[g]$.

   If E is path connected then by Theorem 1.3.15 we know that ϕ is surjective and so it must follow that Φ is surjective as well.

3. By well-definedness of Φ we know that $\phi[f] = \phi[g]$ if and only if $[f] \in H \star g$. Then

$\phi[f] = e_0$ if and only if $[f] \in H$ which implies that $[g]$ is the constant loop.

□

1.4 Retractions

Definition 1.4.1. Let $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \to A$ such that $r\Big|_A$ is the identity map of A. If such a map exists, we say that A is a *retract* of X.

Lemma 1.4.2. *If $A \subseteq X$ and $r : X \to A$ is a retraction then the induced homomorphism of fundamental groups given by the inclusion map $\iota : A \hookrightarrow X$ is a monomorphism, and the induced homomorphism of the retraction map is an epimorphism.*

Proof. Let $r : X \to A$ be the retraction guaranteed by the hypothesis and consider the homomorphism $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$. If $[f] \in \pi_1(A, a_0)$ we want to show that if $\iota_*[f]$ is trivial then $[f]$ is necessarily nulhomotopic. Suppose that $\iota_*[f]$ is the identity element in $\pi_1(X, a_0)$ so that $\iota \circ f$ is homotopic to the constant map c_{a_0}. If F is the corresponding homotopy, then $r \circ F$ is a homotopy between f and c_{a_0} in A implying that f is nulhomotopic in A.

Similarly, let $[f] \in \pi_1(A, a_0)$ and choose a representative f so that $f : [0, 1] \to A$ is a loop in A about the point a_0. Since $A \subseteq X$ then f is also a loop in X so $[f] \in \pi_1(X, a_0)$. Finally, since $r\Big|_A = \mathrm{id}_A$ and the image of f lies entirely within A we get that

$$(r \circ f)(x) = (\mathrm{id} \circ f)(x) = f(x), \qquad \forall x \in [0, 1] \tag{1.26}$$

showing that $[r \circ f] = [f]$. Thus

$$r_*[f] = [r \circ f] = [f] \tag{1.27}$$

and since $[f] \in \pi_1(A, a_0)$ was arbitrary, we conclude that r_* is surjective. Alternatively, consider the commutative diagram given in Figure 1.9

Since the map $r \circ j = \mathrm{id}$ then the functorial properties of the induced homomorphism implies that $(r \circ j)_* = r_* \circ j_* = \mathrm{id}$. This in turn implies that j_* is injective and r_* is surjective. □

$$\begin{tikzcd}
A \arrow[rr, hook, "j"] \arrow[dr, "\mathrm{id}"'] & & X \arrow[dl, two heads, "r"] \\
& A &
\end{tikzcd}$$

Figure 1.9: A commutative diagram describing how the retraction of a space $X \twoheadrightarrow A$ and the inclusion $A \hookrightarrow X$ interact.

Theorem 1.4.3. *There is no retraction of B^2 to S^1.*

Proof. We know that $\pi_1(B^2, b_0) = \{0\}$ since B^2 is simply connected. On the other hand, $\pi_1(S^1, b_0) = \mathbb{Z}$. Clearly there is no injection of $\mathbb{Z} \hookrightarrow \{0\}$ so no such retraction may exist. $\square$

Definition 1.4.4. Let X, Y be topological spaces and $p : X \to Y$ a surjective map. Then p is a *quotient map* provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Note that a sufficient (albeit, not necessary) condition to be a quotient maps it to be continuous, surjective, and open.

Theorem 1.4.5. *Let $p : X \to Y$ be a quotient map, Z a topological space, and $g : X \to Z$ a map that is constant on $p^{-1}(\{y\})$ for each $y \in Y$ (that is, g is constant on fibres of p). Then g induces a map $f : Y \to Z$ such that $f \circ p = g$ and f is continuous if and only if g is continuous.*

The commutative diagram given in Figure 1.10 should elucidate the statement of this theorem, which will be advantageous in proving Lemma 1.4.6.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, "f"'] & \end{tikzcd}$$

Figure 1.10: The universal property of quotient maps written as a commutative diagram.

Lemma 1.4.6. *Let $h : S^1 \to X$ be a continuous map. Then the following are equivalent*

1. *h is nulhomotopic.*
2. *h extends to a continuous map $k : B^2 \to X$.*
3. *h_* is the trivial homomorphism of fundamental groups.*

Proof. Our program will consist of showing that

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1). \tag{1.28}$$

(1) $\Rightarrow$ (2) Consider the cylinder $S^1 \times I$ and the projection map $\pi : S^1 \times I \to B^2$ acting as $\pi(x,t) = (1-t)x$. Let $H : S^1 \times I \to X$ be the homotopy between h and the constant map, in which case k is an extension of h induced by H guaranteed by Theorem 1.4.5.

$$\begin{tikzcd} S^1 \times I \arrow[r, "H"] \arrow[d, "\pi"'] & X \\ B^2 \arrow[ur, "k"'] & \end{tikzcd}$$

(2) $\Rightarrow$ (3) Consider the commutative diagram given in Figure 1.12 where $\iota : S^1 \hookrightarrow B^2$ is the inclusion map. Hence $h = k \circ \iota$ implies that $h_* = k_* \circ \iota_*$ and define $f = h \circ g$. Then $\iota_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0)$ which is a trivial homomorphism since $\pi_1(B^2, b_0)$ is trivial, hence the composition h_* is trivial.

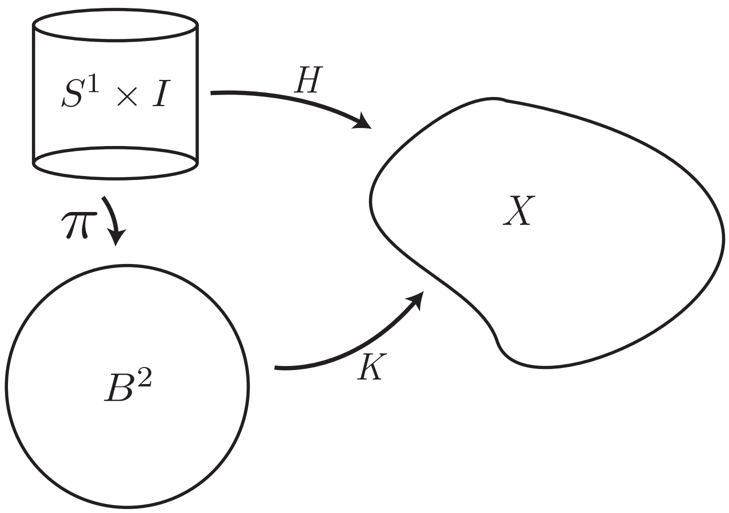

Figure 1.11: A diagram describing how the maps are related.

$$\begin{tikzcd}
I \arrow[r, "g"] & S^1 \arrow[r, "\iota"] \arrow[dr, "h"'] & B^2 \arrow[d, "k"] \\
 & & X
\end{tikzcd}$$

Figure 1.12: Commutative diagram for the implication (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1) Recall that we have a covering map $p : I \to S^1$ by $p(x) = (\cos 2\pi x, \sin 2\pi x)$, and so consider the loop given by this map. Since h_* is trivial we know that $h_*[p]$ is contractible. Let $\pi : I \times I \to I \times S^1$ and $\tilde{H} : S^1 \times I \to X$ corresponding via the lifting lemma to the homotopy $H : I \times I \to X$. Then this is the desired homotopy showing that h is nulhomotopic.

$\square$

Corollary 1.4.7. *The inclusion map $j : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is not nulhomotopic, and* $\mathrm{id} : S^1 \to S^1$ *is not nulhomotopic.*

Proof. Note that S^1 is a retract of $\mathbb{R}^2 \setminus \{0\}$ which implies that j_* is injective. Since trivial maps are necessarily non-injective, Lemma 1.4.6 implies that j^* is not trivial and so not nulhomotopic. lhomotopic.

On the other hand, consider id : $S^1 \to S^1$ so that $\mathrm{id}_* : \pi(S^1, b_0) \to \pi_1(S^1, b_0)$ is the identity homomorphism which implies that it is not trivial and hence not nulhomotopic. □

Theorem 1.4.8. *Given a non-vanishing vector field on B^2 there exists a point of S^1 where the vector points directly inward, and a point that points directly outward.*

Proof. Let $v : B^2 \to \mathbb{R}^2 \setminus \{0\}$ be our non-vanishing vector field, and restrict this to S^1. The map $\omega = v\Big|_{S^1}$ is extendable to a disk and so is nulhomotopic. For the sake of contradiction, assume there is no point x such that $v(x)$ is directed inward. Now $\omega(x)$ must be homotopic to the inclusion map $j : S^1 \to \mathbb{R}^2 \setminus \{0\}$ and so let $F : S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$ be the homotopy between j and $\omega(x)$ described explicitly by $F(x,t) = xt + (1-t)\omega(x)$. This does not pass through the origin since if it did then $\omega(x) = \frac{xt}{1-t}$ which would imply that $\omega(x)$ points directly inward. Thus j_* is homotopic to $\omega(x)$ which is null homotopic, and Corollary 1.4.7 tells us that j_* is not nulhomotopic so we have arrived at a contradiction. To prove the existence of an outward pointing vector field, we apply this exact technique to the negative vector field. □

Theorem 1.4.9 (Brouwer Fixed Point Theorem)**.** *If $f : B^2 \to B^2$ is continuous then there exists $x \in B^2$ such that $f(x) = x$; that is, x is a fixed point of f.*

Proof. For the sake of contradiction, assume that there is no point x such that $f(x) = x$. Let $v(x) = f(x) - x$ which is a nowhere vanishing vector field. By Theorem 1.4.8 we know there is a point $x \in S^1$ such that $v(x)$ points directly outward. Hence $v(x) = ax$ for some positive constant $a > 0$ and so $f(x) = ax + x = (1+a)x$ which lies outside of B^2, and hence is a contradiction. □

Theorem 1.4.10 (Borsuk-Ulam Theorem)**.** *Every continuous function $f : S^n \to \mathbb{R}^n$ maps some pair of antipodal points to the same point.*

The Borsuk-Ulam theorem implies that there are not continuous injective maps from $S^n \to \mathbb{R}^n$. To consider this proof, we will need to introduce some notation and minor results.

Definition 1.4.11. A map $h : S^k \to S^m$ is said to be *antipode-preserving* if $h(-x) = -h(x)$ for all $x \in S^n$.

Theorem 1.4.12. *If $h : S^1 \to S^1$ is continuous and antipode-preserving then h is not nulhomotopic.*

Proof. It is sufficient to prove this for any map $h : S^1 \to S^1$ such that $h(b_0) = b_0$ where $b_0 = (1, 0)$. Indeed, if $h(b_0) = b_1$ then let $\rho : S^1 \to S^1$ be a rotation on S^1 such that $\rho(b_1) = b_0$. Certainly rotations are antipode-preserving, and so the composition $\rho \circ h$ is antipode preserving and $(\rho \circ h)(b_0) = b_0$.

Consider the map $q : S^1 \to S^1$ be given by $q(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta)$ which is both a covering and quotient map. We want to know if $q \circ h$ preserves fibres, for which we examine the diagram below:

$$\begin{tikzcd}
S^1 \arrow[r, "h"] \arrow[d, "q"'] \arrow[dr, "q \circ h" description] & S^1 \arrow[d, "q"] \\
S^1 \arrow[r, "k"] & S^1
\end{tikzcd}$$

Note that $q(h(-z)) = q(-h(z)) = q(h(z))$ and so fibres are indeed preserved. Then $k \circ q = q \circ h$ and we want to show that h_* is non-trivial. To do this, we will show that k_* is non-trivial from which the non-triviality of h_* will follow. Let $\tilde{f}$ be any path in S^1 from b_0 to $-b_0$. Then $f = q \circ \tilde{f}$ is a non-trivial element in $\pi_1(B, b_0)$. But

$$k_*([f]) = k \circ f = k \circ q \circ \tilde{f} = q \circ h \circ \tilde{f} \tag{1.29}$$

and is non-trivial since $h \circ \tilde{f}$ is a path from b_0 to $-b_0$. □

Theorem 1.4.13. *There is no continuous antipode-preserving map $g : S^2 \to S^1$.*

Proof. For the sake of contradiction, assume that $g : S^2 \to S^2$ is antipode preserving and restrict g to the equator of E of S^2. Then $\tilde{g} = g\Big|_E : S^1 \to S^2$ is still an antipode-preserving map. Hence the induced homomorphism $\tilde{g}_*$ is non-trivial, but since $\tilde{g}_*$ can be extended to a disk it must be trivial, a contradiction. □

Proof of Borsuk-Ulam Theorem. Suppose that we have a continuous map $f : S^2 \to \mathbb{R}^2$. For the sake of contradiction, assume that there is no pair of antipodal points that map to the same point; that is, $f(x) \neq f(-x)$ for all $x \in S^2$. Now consider the map $F : S^2 \to S^1$ given by

$$F(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|} \tag{1.30}$$

which is a continuous, well-defined, antipode preserving map. But this is a contradiction to Theorem 1.4.13, so such a pair must exist. □

1.4.1 Applications of Borsuk-Ulam

Theorem 1.4.14 (Bisection Theorem)**.** *Given two polygonal regions in $\mathbb{R}^2$ there exists a line in $\mathbb{R}^2$ that bisects each of them.*

Proof. Denote our polygons by A_1 and A_2 and let $u \in S^2$ and P be a plane in $\mathbb{R}^3$ through the origin which is orthogonal to u. Let $f_i(u)$ be the area of the portion of A_i that lies on the same side of P as u, so that $f_i(u) + f_i(-u) = A_i$. Hence we can define a map $F : S^2 \to \mathbb{R}^2$ given by

$$F(u) = (f_1(u), f_2(u)). \tag{1.31}$$

By Borsuk-Ulam, we know there exists $u \in S^2$ such that $F(u) = F(-u)$ which implies that $f_i(u) = f_i(-u)$ for $i = 1, 2$ or $f_i(u) = \frac{1}{2}A_i$ for $i = 1, 2$. □

Theorem 1.4.15. *If S^2 is written as the union of three closed sets $A_i, i = 1, 2, 3$ then at least one of these sets must contain a pair of antipodal points.*

Proof. Define distance measures $d_i : S^2 \to \mathbb{R}$ by $d_i(x) = \inf_{y \in A_i} |x - y|$, which are continuous functions since they are compositions of continuous functions. Define the map $d : S^2 \to \mathbb{R}^2$ by $d(x) = (d_1(x), d_2(x))$ which must also be continuous. Applying Borsuk-Ulam, we know there exists a pair of antipodal points $\pm a$ such that $d_1(a) = d_1(-a)$ and $d_2(a) = d_2(-a)$. If either of these is zero, we conclude that $\pm a \in A_i$ for $i = 1$ or $i = 2$. If both are strictly positive then $\pm a \in A_3$. In either case, our result follows. □

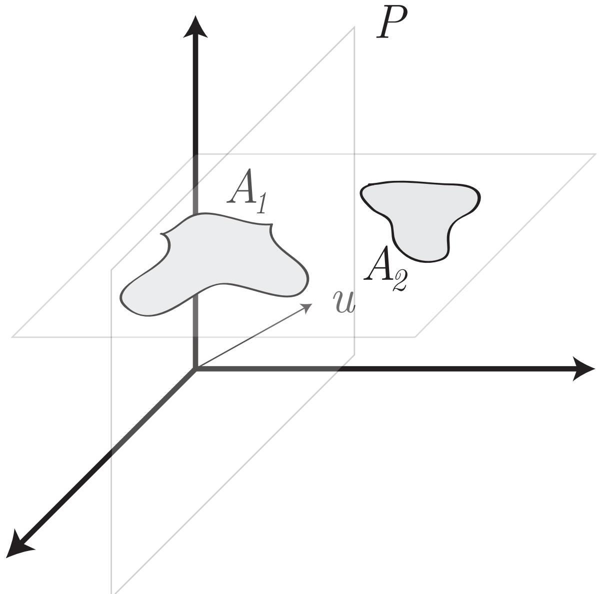

Figure 1.13: The vector $u \in \mathbb{R}^2$ determines the plane P which divides our two regions. An appropriate choice of u will prove the proof for the theorem.

We note that this does not extend to more than three closed sets, as we can inscribe a sphere within a tetrahedron and define our closed sets as the projection of the tetrahedral faces onto the sphere. However, using the generalization of Borsuk-Ulam, we may show that any decomposition of S^n into $n+1$ closed sets guarantees that one such set will contain a pair of antipodal points. It turns out that this statement is actually equivalent to Borsuk-Ulam, and its converse is left as an exercise.

Theorem 1.4.16. *If $f : S^2 \to \mathbb{R}^2$ is a continuous function that satisfies $f(-x) = -f(x)$ for all x then there exists $x_0 \in S^2$ with $f(x_0) = 0$.*

Proof. Since f is continuous, Borsuk-Ulam implies that there exists a point $x_0 \in S^2$ such that $f(x_0) = f(-x_0)$; hence

$$f(x_0) = f(-x_0) = -f(x_0) \tag{1.32}$$

implies that $2f(x_0) = 0$, or rather $f(x_0) = 0$. Hence the point guaranteed by Borsuk-Ulam is the same that satisfies our problem. $\square$

Just like Theorem 1.4.15, this theorem is equivalent to Borsuk-Ulam as we now show:

Theorem 1.4.17. *Suppose that for every continuous map $f : S^n \to \mathbb{R}^n$ satisfying $f(-x) = -f(x)$ there exists some point such that $f(x) = 0$. Then for every continuous map $g : S^n \to \mathbb{R}^n$ there is a pair of antipodal points such that $g(x) = g(-x)$.*

Proof. Let $g : S^n \to \mathbb{R}^n$ be a continuous function and define $f : S^n \to \mathbb{R}^n$ by $f(x) = g(x) - g(-x)$. Clearly $f(-x) = -f(x)$ and so by assumption, there is a point $a \in S^n$ such that $f(a) = 0$ so that $g(a) = g(-a)$ which is precisely what we wanted to show. □

1.5 Deformation Retractions

Lemma 1.5.1. *Suppose that we have two continuous map $h, k : (X, x_0) \to (Y, y_0)$. If h and k are homotopic and the image of $x_0 \in X$ is fixed during the homotopy then $h_* = k_*$.*

Proof. Let H be the homotopy between h and k. If f is any loop in X then $H(f(s), t)$ is a homotopy between $h \circ f$ and $k \circ f$. □

Theorem 1.5.2. *The inclusion map $j : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ induces an isomorphism of fundamental groups.*

Proof. Let $r : \mathbb{R}^n \setminus \{0\} \to S^n$ by $x \mapsto \frac{x}{\|x\|}$. The map $r \circ j : S^n \to S^n$ is the identity and so $(r \circ j)_* = r_* \circ j_* = \mathrm{id}_*$. Furthermore, the map $j \circ r : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ is not identity but is homotopic to the identity via the straight line homotopy $H(x, t) = (1 - t)x + \frac{tx}{\|x\|}$. Now the image of any $b_0 \neq 0$ is fixed under this map, and so by Lemma 1.5.1 we have that $(j \circ r)_* = j_* \circ r_*$ is the identity homomorphism. Both directions implies that j_* is an isomorphism. □

Definition 1.5.3. A subset A of X is a *deformation retract* if there exists a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$, $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$.

Theorem 1.5.4. *If A be a deformation retract of X then the inclusion map $j : A \hookrightarrow X$ induces an isomorphism of fundamental groups.*

Proof. This follows immediately by emulating the proof if Theorem 1.5.2. □

Example 1.5.5.

1. Note that $S = \mathbb{R}^3 \setminus \{(0, 0, z) : z \in \mathbb{R}\}$ can deformation retract onto $\mathbb{R}^2 \setminus \{0\}$, which can deformation retract to the circle, and so $\pi_1(S, x_0) = \mathbb{Z}$.

2. Consider $\mathbb{R}^2 \setminus \{p, q\}$ for distinct points p, q. We can pull everything down to two circles concentric with p, q, so that the fundamental group of $\mathbb{R}^2 \setminus \{p, q\}$ is the fundamental group of the figure 8 (the free group on two generators).

3. Consider the solid torus $S^1 \times D^2$. This can be retracted to an annulus in the plane, which can in turn be deformation retracted to the circle S^1. Hence $\pi(S^1 \times D^2) = \pi(S^1) = \mathbb{Z}$.

4. Consider the punctured torus $\mathbb{T}^2 \setminus \{p\}$. Taking $\mathbb{T}^2$ to be $\mathbb{R}^2/\mathbb{Z}^2$ centred at the puncture p we can retract everything to the boundaries. After applying our quotient, we get the figure eight group again.

5. Consider $\mathbb{R}^3$ with the non-negative x, y, z-axes deleted. Retract everything to a sphere, which will have three punctures. By placing one at the north-pole, stereographic projection implies that the fundamental group will be the free group on two generators.

Definition 1.5.6. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that $g \circ f : X \to X$ is homotopic to the identity map $\mathrm{id}_X : X \to X$, and $f \circ g : Y \to Y$ is homotopic to the identity map on Y. Then we say that f and g are a *homotopy equivalences* and that f and g are *homotopy inverses* of one another.

Lemma 1.5.7. *Let h, k be continuous maps and let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. If $H : X \times I \to Y$ is the homotopy between h and k then α is the path $\alpha(t) = H(x_0, t)$.*

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "h"] \arrow[dr, "k_*"'] & \pi_1(Y,y_0) \arrow[d, "\hat{\alpha}"] \\
& \pi_1(Y,y_0)
\end{tikzcd}$$

Proof. Let $H : X \times I \to X$ be the homotopy between H and K. Let $f : I \to X$ be a loop at x and let $\alpha = H(x_0, t)$ **Complete**. □

Corollary 1.5.8. *Homotopy equivalences yield isomorphisms. That is, if $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi(X, x_0) \to \pi(Y, f(x_0))$ is a group isomorphism.*

1.6 Seifert-Van Kampen Theorem

Let G be a given group, which we write as $G = \langle S|R\rangle$ where S is a set of elements and R is a set of relations.

Theorem 1.6.1 (Baby Seifert-Van Kampen Theorem)**.** *Suppose that $X = U \cup V$ where $U, V \subseteq X$ are open and $U \cap V$ is path connected. Take $x_0 \in U \cap V$, and let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ by inclusion maps. Then the images of the induced homorphisms $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ generate the fundamental group of X.*

Proof. Let f be a path about x_0, which is homotopic to a collection $g_1 \star g_2 \star \cdots \star g_n$ where each g_i is strictly contained in either U of V.

By the Lebesgue Numbering Lemma, there exists a subdivision of I, say, $a_0 < a_1 < \cdots < a_n$ such that $f([a_i, a_{i+1}])$ lies in U or V. We would like to ensure that every point of the subdivision is contained within $U \cap V$. This can be done as follows: Suppose that $f(a_i) \in U$ but not in V, then $f([a_{i-1}, a_i]) \subseteq U$ and $f_([a_i, a_{i+1}]) \subseteq U$ so $f([a_{i-1}, a_{i+1}]) \subseteq U$ and remove a_i from the subdivision. Hence without loss of generality we may assume that for each partition point a_i, $a_i \in U \cap V$.

Since $U \cap V$ is path connected, let σ_i be a path connecting a_i to x_0. Hence each $g_i = \sigma_{i-1} \star f_i \star \overline{\sigma_{i-1}}$ where $f_i = f \Big|_{[a_{i-1}, a_i]}$. □

Corollary 1.6.2. *Suppose that $X = U \cup V$ where $U, V \subseteq X$ are open sets where $U \cap V$ is path connected and U and V are each simply connected. Then X is simply connected.*

Proof. The generators of the fundamental group of X come from the generators of U and V, both of which are trivial. Hence X is itself trivial and we conclude that X is simply connected. □

Corollary 1.6.3. *If $n \geq 2$ then S^n is simply connected.*

Proof. Let $U = S^n \setminus \{N\}$ and $V = S^N \setminus \{S\}$ where N and S are the north and south poles respectively. Then $U \cap V = S^n \setminus \{N, S\}$ which is topologically a cylinder and hence path connected. By Seifert-Van Kampen, we conclude that S^n is simply connected. □

Our next goal is to calculate the fundamental group of $\mathbb{RP}^2$. Consider the antipodal map $a : S^2 \to S^2$ which acts by $a : x \mapsto -x$. The real projective space is then the quotient under the identification of $x \sim a(x)$. If $p : S^2 \to \mathbb{RP}^2$ taking $x \mapsto [x]$ the equivalence class, is a covering map of $\mathbb{RP}^2$ by the sphere. By lifting correspondence, we conclude that $\pi_1(\mathbb{RP}^2, x_0) \cong \mathbb{Z}/2$.

Theorem 1.6.4 (Seifert-Van Kampen Theorem)**.** *Let $X = U_1 \cup U_2$ where $U_1, U_2 \subseteq X$ are open and connected. Suppose that $U_1 \cap U_2 \neq \emptyset$ is path connected. To find a presentation of $\pi_1(X, x_0)$ Let*

$$\begin{aligned} \pi_1(U_1 \cap U_2, x_0) &= \langle S | R \rangle \\ \pi_1(U_1, x_0) &= \langle S_1 | R_1 \rangle \\ \pi_1(U_2, x_0) &= \langle S_2 | R_2 \rangle \end{aligned}$$

and the set R_S is $S_1 \cup S_2$ under the relation $\langle\langle \phi_1 \star S \rangle\rangle \langle\langle \phi_2 \star S \rangle\rangle^{-1} = 1$. Then

$$\pi_1(X, x_0) = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup R_S \rangle. \tag{1.33}$$

Lemma 1.6.5. *The generators $S_1 \cup S_2$ of $\pi_1(X, x_0)$ satisfy R_1, R_2, R_S.*

$$
\begin{tikzcd}
 & U_1, x_0 \arrow[dr, "\psi_1"] & \\
U_1 \cap U_2, x_0 \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & & X_1, x_0 \\
 & U_2, x_0 \arrow[ur, "\psi_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(U_1, x_0) \arrow[dr, "(\psi_1)_*"] & \\
\pi_1(U_1 \cap U_2, x_0) \arrow[ur, "(\phi_1)_*"] \arrow[dr, "(\phi_2)_*"'] & & \pi_1(X_1, x_0) \\
 & \pi_1(U_2, x_0))_* \arrow[ur, "(\psi_2"'] &
\end{tikzcd}
$$

Proof. The map ψ_j^* is a homomorphism so $\psi_j^* S_j$ satisfies the relation that S_j satisfies in $\pi_1(U_j, x_0)$. Since the diagram is commutative we have $(\psi_1)_* \circ (\phi_1)_* S = (\psi_2)_* \circ (\phi_2)_* S$ and so $\langle\langle (\phi_1)_* S \rangle\rangle = \langle\langle (\phi_2)_* S \rangle\rangle$. □

Theorem 1.6.6. *If the elements of $S_1 \cup S_2$ in $\pi_1(X, x_0)$ satisfy some relation then this relation follows R_1, R_2, R_S.*

Proof. Let $\alpha_1^{\epsilon(1)} \alpha_2^{\epsilon(2)} \cdots \alpha_k^{\epsilon(k)} = 1$ for $\epsilon(1) = \pm 1$ be a relation of elements of $S_1 \cup S_2 \subseteq \pi_1(X, x_0)$. For each i select a loop f_i in $U_{\alpha(i)}$ based at x_0. Notice that

$$[f_i] = \alpha_i, \qquad [\bar{f}_i] = \alpha_i^{-1}. \tag{1.34}$$

Let $[f] = [f_1] \star [f_2] \star \cdots \star [f_k]$ where without loss of generality, we have chosen representatives such that all powers are positive. By our original relations on the α_i, $[f]$ must then be homotopic to e_{x_0} the constant map at x_0. Let $F(t, s)$ be such a homotopy; that is, $F(t, 0) = f(t)$ and $F(t, 1) = x_0$. By the Lebesgue Number Lemma, there exists a partition

$$0 = t_0 < t_1 < \cdots < t_m = 1, \qquad 0 = s_0 < s_1 < \cdots < s_n = 1 \tag{1.35}$$

so that $F(R_{ij})$ is either in U_1 or in U_2, where $R_{ij} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$ is a rectangle in $I \times I$. For each i, j let $a_{ij} : I \to X$ be a path from x_0 to $F(t_i, s_j)$ that lies in U_1 if $F(t_i, s_i) \in U_1$ and lies in U_2 if $F(t_i, s_i) \in U_2$ and lies in the intersection $U_1 \cap U_2$ if $F(t_i, s_i) \in U_1 \cap U_2$. Set

$$b_{ij}(t) = F\left((1-t)t_{i-1} + tt_i, s_j\right), \qquad c_{ij}(t) = F\left(t_i, (1-t)s_{j-1} + ts_j\right). \tag{1.36}$$

Thus $b_{i,j-1} \star c_{ij}$ is homotopic to $c_{i-1,j} \star b_{ij}$ so define

$$f_{ij} = a_{i-1,j} \star b_{ij} \star \bar{a}_{ij}, \qquad g_{ij} = a_{i,j-1} \star c_{ij} \star \bar{a}_{ij}. \tag{1.37}$$

Then $f_{i,j-1} \star g_{ij}$ is path homotopic to $g_{i-1,j} \star f_{ij}$ so $[f_{i,j-1}] = [g_{i-1,j}][f_{ij}][\bar{g}_{ij}]$ and we get

$$\langle\langle [f_i, g_{i-1}] \rangle\rangle = \langle\langle [g_{i-1}, j] \rangle\rangle \langle\langle [f_{ij}] \rangle\rangle \langle\langle [\bar{g}_{ij}] \rangle\rangle. \tag{1.38}$$

Write $\alpha = \alpha_1^{\epsilon(1)}\alpha_2^{\epsilon(2)}\cdots\alpha_k^{\epsilon(k)} = 1\ [f_1]\star\cdots\star[f_k] = [e_{x_0}]$ so that

$$\begin{aligned}\alpha = \alpha_1^{\epsilon(1)}\cdots\alpha_k^{\epsilon(k)} &= \langle\langle[f_{10}]\rangle\rangle\cdots\langle\langle[f_{m0}]\rangle\rangle \\ &= \langle\langle[g_{01}]\rangle\rangle\langle\langle[f_{11}]\rangle\rangle\langle\langle[\bar{g}_{11}]\rangle\rangle\langle\langle[g_{11}]\rangle\rangle\langle\langle[f_{21}]\rangle\rangle\langle\langle[\bar{g}_{21}]\rangle\rangle\cdots \\ &\cdots\langle\langle[\bar{g}_{m-1,1}]\rangle\rangle\langle\langle[g_{m-1,1}]\rangle\rangle\langle\langle[f_{m1}]\rangle\rangle \\ &= [e_{x_0}].\end{aligned}$$

□

Example 1.6.7. Let us compute the fundamental group of the torus. Fix arbitrary distinct points $y, x_0 \in \mathbb{T}^2$. Consider the torus as $\mathbb{R}^2/\mathbb{Z}^2$ and label the boundaries as a_1 and a_2. Let $U_1 = \mathbb{T}^2 \setminus \{y\}$ and $U_2 = \mathbb{T}^2 - \{a_1 \cup a_2\}$. Now $U_1 \cap U_2$ is homeomorphic to a punctured disk, which has the circle as a deformation retract and so $\pi(U_1 \cap U_2, x_0) \cong \mathbb{Z}$, with generator say $[\gamma]$. The set U_2 is simply connected so $\pi_1(U_2, x_0) = \{0\}$ and $\pi_1(U_1, x_0) = \langle A_1, A_2 | \emptyset\rangle$ is the free group with two generators (see Figure 1.14).

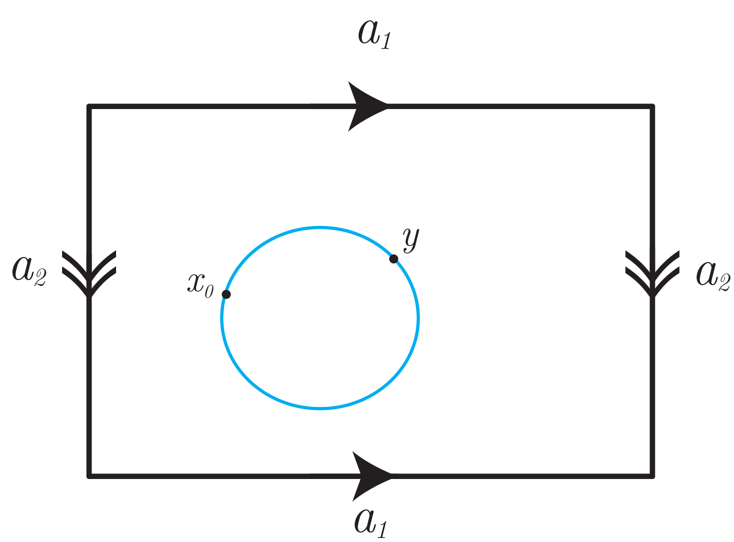

Figure 1.14: Applying the Seifert-van Kampen theorem to the Torus.

Seifert-van Kampen says then says that $\pi_1(\mathbb{T}^2, x_0) = \left\{A_1, A_2 | A_1A_2A_1^{-1}A_2^{-1} = 1\right\} = \mathbb{Z}\oplus\mathbb{Z}$.

Example 1.6.8. Let us calculate the fundamental group of the figure-eight. Consider

$$C_1 = \left\{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = 1\right\}, \qquad C_2 = \left\{(x, y) \in \mathbb{R}^2 : (x+1)^2 + y^2 = 1\right\} \tag{1.39}$$

so that the figure eight is the union $C_1 \cup C_2$. Let x_0 be the origin and define $U = C_1 \cup C_2 \setminus \{(-1, 0)\}$ and $V = C_1 \cup C_2 \setminus \{(0, 1)\}$, so that $\pi_1(U, x_0) = \mathbb{Z}$ which has presentation $\langle \alpha : \emptyset\rangle$. On the other hand, $\pi_1(V, x_0) = \mathbb{Z}$ so that this presentation is given via $\langle \beta : \emptyset\rangle$. The set

$U \cap V = C_1 \cup C_2 \setminus \{(-1,0),(1,0)\}$ is simply connected and the holes allow each path to be deformed to a point, hence $\pi_1(U \cap V, x_0) = \{0\}$. By Seifert-van Kampen, $\pi_1(8, x_0) = \langle \alpha, \beta : \emptyset \rangle$ which is just the free group on two elements.

Example 1.6.9. Let us calculate the fundamental group of $\mathbb{RP}^2$ using Seifert-van Kampen. We recall that we can visualize $\mathbb{RP}^2$ as the quotient of a disk, identifying antipodes. Let a denote one of the edges of the disk and define

$$U = \mathbb{RP}^2 \setminus \{a\}, \qquad V = \mathbb{RP}^2 \setminus y \tag{1.40}$$

for which $\pi_1(\mathbb{RP}^2 \setminus a, x_0) = \{0\}$ since the quotient is a disk. On the other hand, $\pi_1(V, x_0) = \mathbb{Z}$ since removing y will allow everything to deformation retract to a which will quotient to S^1. Let δ be a path that goes from x_0 to x_1 along d, and set $A = \delta \star a \star \bar{\delta}$ which is the generator $\langle A : \emptyset \rangle$. Now $U \cap V$ is a punctured open disk, for which $\pi_1(U \cap V, x_0) = \mathbb{Z}$ since we can deformation retract to S^1 on the boundary. If γ is any loop whose centre contains the puncture, this is a generator, so we write $\mathbb{Z} = \langle \gamma : \emptyset \rangle$. By Seifert-van Kampen, $\pi_1(X, x_0) = \langle S_1 \cup S_2 : R_1 \cup R_2 \cup R_s \rangle$, so we need to consider the relation R_S. Note that γ is homotopic to a constant map in U and is homotopic to $\delta \star a \star a \star \bar{\delta}$ in V so by taking equivalence classes $[\delta \star a \star a \star \bar{\delta}] = A^2$. Since these must be invariant of the choice presentation, we get that $A^2 = e$. Plugging these in we find

$$\pi_1(\mathbb{RP}^2, x_0) = \big\langle A : A^2 = e \big\rangle \cong \mathbb{Z}/2. \tag{1.41}$$

Example 1.6.10. Consider the space whose pre-quotient is given by Figure 1.15. Let $U_1 = X \setminus b$ and $U_2 = X \setminus (a_1 \cup a_2)$. Note that by identifying the a_1 and a_2 we get the figure eight, and so $\pi_1(U_1, x_0) \cong \mathbb{Z} \star \mathbb{Z}$, so this has presentation $\langle A_1, A_2 : \emptyset \rangle$. Let the contraction of the interior be β. After identifying the edges b we have S^1 and so $\pi_1(U_2, x_0) \cong \mathbb{Z}$, denote this presentation by $\langle B : \emptyset \rangle$. Finally, consider $U \cap V$ around which a loop will given us $\pi_1(U \cap V, x_0) = \mathbb{Z} = \langle \Gamma : \emptyset \rangle$. To conclude the overlapping relation, we have that $A_2 A_1^2 A_2^{-1} = B^3$ so

$$\pi_1(X, x_0) = \big\langle A_1, A_2, B : A_2 A_1^2 A_2^{-1} = B^3 \big\rangle. \tag{1.42}$$

Theorem 1.6.11 (Classical Seifert-van Kampen Theorem)**.** *If $j : \pi_1(U, x_0) \star \pi_1(V, x_0) \to \pi_1(x, x_0)$ is a homomorphism of the free product that extends the homomorphism ψ_{1*}, ψ_{2*} then it is surjective and its kernel is the least normal subgroup N of the free product that contains all elements represented by words of the form $\phi_1(g)^{-1}\phi_2(g)$ where $g \in \pi_1(U \cap V, x_0)$.*

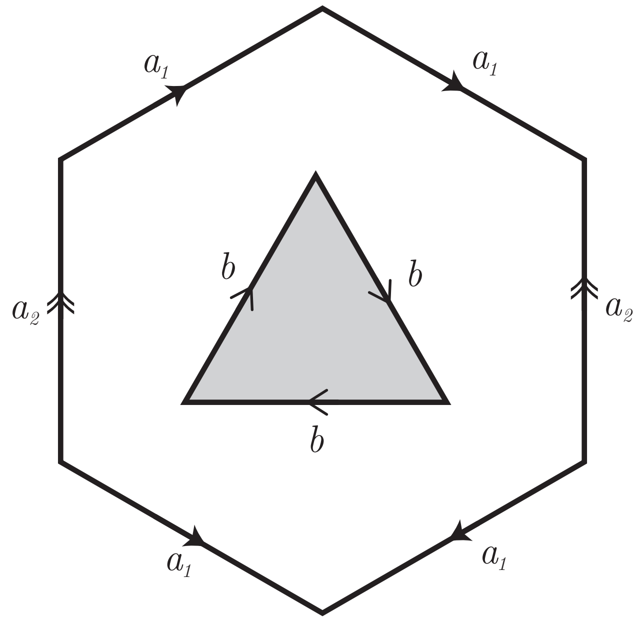

Figure 1.15: An abstract topological space viewed as a pre-quotient.

1.7 Adjoining a Two-Cell

Theorem 1.7.1. *Let X be a Hausdorff space and let A be a closed, path connected subspace of X. Suppose that there exists a continuous map $h : B^2 \to X$ such that h maps $\operatorname{int} B^2$ bijectively to $X \setminus A$ and maps $S1 = \partial B^2$ into A. Let $p \in S^1$ and let $a = h(p)$ and let $k : (S^1, p) \to (A, a)$ be the map obtained by restricting h. Then the homomorphism*

$$i_* : \pi_1(A, a) \to \pi_1(X, a) \tag{1.43}$$

induced by inclusion is surjective, and its kernel is the least normal subgroup $\pi_1(A, a)$ containing the image of $k_* : \pi_1(S^1, p) \to \pi_1(A, a)$ generated by $h \circ f$ where f is a generator of $\pi_1(S^1, p)$.

To see what this means, recall that we have a covering map $p \times p : \mathbb{R}^2 \to \mathbb{T}^2$. By taking a unit square (topologically equivalent to S^1), this maps to a a figure 8 on the torus. In a sense, it is almost as though we have "pasted" a square onto the area defined by the figure-8 space (here called A) allowing us to write

$$\pi_1(\mathbb{T}^2, p) = \langle a, b | bab^{-1}a^{-1} = e \rangle. \tag{1.44}$$

Proof. We want to show that the fundamental group of X is obtained from the fundamental group of A by "killing off" the class $k_*[f]$ where f generated $\pi_1(S, p)$. Let $U = X \setminus \{x_0\}$

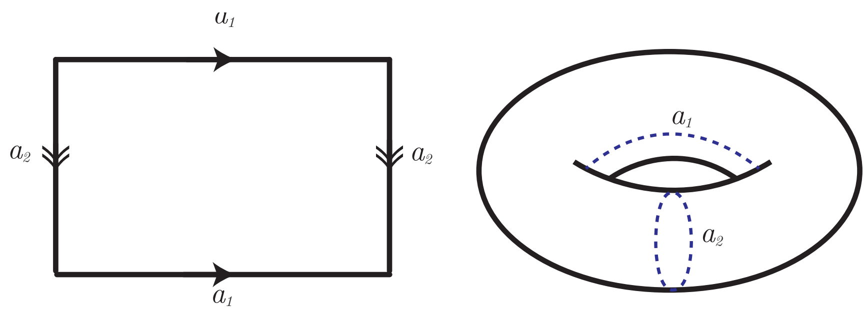

Figure 1.16: The boundary of the the square maps to the “figure-8” space, while its interior maps to all other points..

where $x_0 = h(0)$ and $V = X \setminus A$. We claim that A is a deformation retract of U and V is homeomorphic to an open disk. Furthermore, the intersection $U \cap V$ is homeomorphic to $B^2 \setminus 0$ for which $\pi_1(U \cap V, h(b)) = \mathbb{Z}$ for some $b \in B^2$.

Let O be the center of B^2 and $x_0 = h(O)$. A is a deformation retract of U and so define $C = h(B^2)$ for $\pi : B^2 \to C$ is the map obtained by restricting the range of h.

$$\begin{tikzcd}
B^2 \setminus \{0\} \times I \arrow[r, "d"] \arrow[d] \arrow[dr] & B^2 \setminus \{0\} \arrow[d, "\pi'"] \\
X \setminus \{x_0\} \times I \arrow[r] & X \setminus \{x_0\}
\end{tikzcd}$$

where d is the deformation retraction. We would like to create the deformation retraction from $X \setminus \{x_0\} \times I$ to $X \setminus \{x_0\}$. Let $\pi \times \mathrm{id} : B^2 \times I \to C \times I$ be a quotient map, and its restriction $\pi' : B^2 \setminus \{0\} \times I \to (C \setminus \{x_0\}) \times I$ which is also a quotient map since its domain is open in $B^2 \times I$. Extend this deformation to A. Now $V = X \setminus A$ is homeomorphic to $\mathrm{Int}(B^2)$ and so it is simply connected. Likewise, $U \cap V$ is homeomorphic to $\mathrm{Int}(B^2) \setminus \{0\}$. Hence $\pi_1(U \cap V, h(b)) = \mathbb{Z}$. We will use Seifert-van Kampen to deduce the fundamental group of X.

Consider the map $\psi_{1*} : \pi_1(U, h(b)) \to \pi_1(X, h(b))$ induced by inclusion, which is surjective and whose kernel will be generated by the image of $\pi(U \cap V, b)$. We can move the basepoint to the boundary but considering loops at $h(b)$ and adjoining the line between a basepoint and

$h(b)$.

$$\begin{tikzcd}
\pi_1(U,h(b)) \arrow[r] \arrow[d] & \pi_1(X,b) \arrow[d] \\
\pi_1(U,b) \arrow[r] & \pi_1(X,a)
\end{tikzcd}$$

so $h \circ \sigma = \gamma$ is a homomorphism of $\pi_1(U,a)$ into $\pi_1(X,a)$ is surjective and its kernel is in S^1 containing $\tilde{\gamma}([g])$ and is homotopic to $h \circ \hat{\sigma} f_0$ homotopic to $h \circ f$ where f is a generator of $\pi_1(S^1, p)$. □

Example 1.7.2. Consider the sphere with three distinct points $p_1, p_2, p_2 \in S^2$ identified. A is a figure 8.

Example 1.7.3. Let us compute the fundamental group of a double torus, $\pi(\mathbb{T}^2 \# \mathbb{T}^2, p)$. Set $A = a \cup b \cup c \cup d$ so that the fundamental group is

$$\pi(\mathbb{T}^2 \# \mathbb{T}^2, p) = \big\langle a, b, c, d : aba^{-1}b^{-1}cdc^{-1}d^{-1} = e \big\rangle. \tag{1.45}$$

Example 1.7.4. The n-fold dunce hat is constructed as a quotient by breaking the sphere into n-pieces with the same orientation and identifying these pieces. The fundamental group of this space in $\mathbb{Z}^n$.

Example 1.7.5. Consider $X = S^2 \wedge S^1$, for which we can either adjoin a two-cell, or use Seifert-van Kampen. For Seifert-van Kampen, define U to be $X \setminus \{p\}$ where $p \in S_2$ and $V = X \setminus q$ where $q \in S_1$. Then $U \cap V = X \setminus \{p, q\}$, **Finish:**

Example 1.7.6. Consider the torus with two internal points in the "hole" identified. Let A be a wedge of three circles, so that for some point $p \in A$ we have that $\pi_1(A, p) \cong F_3 = \langle a, b, c | \emptyset \rangle$. Let us adjoin a two-cell and examine how to boundary acts under this transformation.

$$\pi_1(X, p) = \big\langle a, b, c | abca^{-1}c^{-1}b^{-1} = e \big\rangle. \tag{1.46}$$

By setting $bc = d$ we could instead write this as

$$\pi_1(X, p) = \big\langle a, b, d | ada^{-1}d^{-1} = e \big\rangle \tag{1.47}$$

which is easier to recognize as $F_1 \times \mathbb{Z}^2$.

1.8 Fundamental Group of A Wedge of Circles

Definition 1.8.1. Let X be a Hausdorff space that is the union of $S_1, \ldots, S_n$ where each S_1 is homeomorphic to S^1. Assume that there is a point $p \in X$ such that $S_i \cap S_j = \{p\}$ for all $i \neq j$. Then X is said to be a *wedge of circles*.

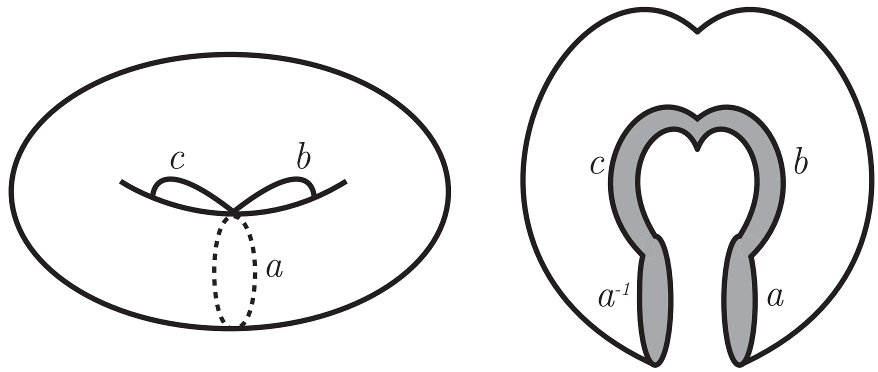

Figure 1.17: Adjoining a two-cell to the torus with two interior points identified.

Theorem 1.8.2. *Let X be the wedge of the circles $S_1, \ldots, S_n$ sharing common point p. Then $\pi_1(X, p)$ is a free group with n-generators.*

Proof. We proceed by induction on the number of circles. Assume that the fundamental group of $n-1$ circles is F_{n-1} and let $f_1, \ldots, f_n$ be the generators of the circles S_i. We will use Seifert-van Kampen, so let us choose a point q on the circle S_n and define $U = X \setminus q$. Define V to be X with one point removed for each circle $S_1, \ldots, S_{n-1}$. In the intersection, we get X with one point removed from each circle. Hence $\pi_1(U, p) = F_{n-1}$, $\pi_1(V, p) = \mathbb{Z}$ and $\pi(U \cap V, p) = \emptyset$. By Seifert-van Kampen $\pi_1(X) = F_n$. $\square$

Definition 1.8.3. Let X be a space which is the union of the subspaces $X_\alpha, \alpha \in J$. The topology of X is said to be *coherent with the subspaces* X_α provided that a subset $C \subseteq X$ is closed in X if and only if $C \cap X_\alpha$ is closed in X_α for all $\alpha \in J$.

Definition 1.8.4. Let X be a space that is the union of subspaces $S_\alpha, \alpha \in J$ where each S_α is homeomorphic to S^1. If there is further a $p \in X$ such that $S_\alpha \cap S_\beta = \{p\}$ whenever $\alpha \neq \beta$ and the topology of X is coherent with S_α. Then we say that X is a *wedge of circles.*

Example 1.8.5. A *Hawaiian ring* or *infinite earing space* is a collection of circles of radius $\frac{1}{n}, n \in \mathbb{N}$ with intersection at the origin. This space is not simply connected, and F_∞ is a subspace of its fundamental group. Note that this is not a wedge of circles since the space topology is not coherent with the circles.

Take the unit interval $[0, 1]$ and divide it into an infinite partition with endpoints $\frac{1}{n}, n \in \mathbb{N}$. Map each sub-interval to the circle whose radius is given by the right-endpoint of each interval. Define the map $h : X \to C_N$ such that $h(x) = x$ for $x \in C_N$ and $h(x) = p$ otherwise; that is, h takes all points not in the N^{th} circle to the origin. This is a continuous map, so let us consider the induced homomorphism $h_* : \pi_1(X, p) \to \pi_1(C_N, p)$, which is not trivial.

Theorem 1.8.6. *Let X be the wedge of circles $S_\alpha, \alpha \in J$ and p be the common point of these circles. Then $\pi_1(X, p)$ is a free group generated by $\{f_\alpha\}_{\alpha \in J}$ where f_α is a generator of $\pi_1(S_\alpha, p)$.*

Theorem 1.8.7. *Let X, Y be topological spaces with $(x_0, y_0) \in X \times Y$. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0). \tag{1.48}$$

Proof. Define projections $p : X \times Y \to X$ and $q : X \times Y \to Y$ and consider the induced homomorphisms $p_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$ and $q_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$. Define

$$\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0), \qquad \phi([f]) = (p_*[f], q_*[f]) = ([p \circ f], [q \circ f]). \tag{1.49}$$

We claim that this map is surjective. Indeed, let $([g], [h]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ and define $f(s) = (g(s), h(s))$ for which $\phi[f] = ([g], [h])$. Secondly, let us show that this map is injective. Assume that $\phi[f] = ([p \circ f], [q \circ g])$ is trivial. Then there is a homotopy H between $p \circ f$ and e_{x_0} and a homotopy G between $q \circ f$ and e_{y_0}. Then $F = H \times G$ is a homotopy in $X \times Y$ between f and $e_{(x_0, y_0)}$ so the kernel is trivial and ϕ is an isomorphism. $\square$

Corollary 1.8.8. *The fundamental group of the torus can now be computed quite easily as* $\pi_1(\mathbb{T}^2, p) \cong \pi_1(S^1, p) \times \pi_1(S^1, p) \cong \mathbb{Z} \oplus \mathbb{Z}$.

1.9 Recovering A Covering

We recall that p is a covering map if $p : E \to B$ is surjective, continuous and every point in B has a neighbourhood which is evenly covered by p. We have seen how to lift paths and homotopies

$$\begin{tikzcd} & E, e_0 \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B, b_0 \end{tikzcd} \qquad \begin{tikzcd} & E, e_0 \arrow[d, "p"] \\ I \times I \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & B, b_0 \end{tikzcd}$$

We have seen that if E is path connected the lifting correspondence $\phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$ is bijective.

From now on, we will assume that E, B are both path connected and locally path connected.

Definition 1.9.1. We say that $p : E \to B$ and $p' : E' \to B$ are equivalent covering spaces if there exists $h : E \to E'$ a homeomorphism such that $p = p' \circ h$.

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"'] & E' \arrow[dl, "p'"] \\ B & \end{tikzcd}$$

Consider for example two coverings of S^1 by $z^2 : S^1 \to S^1$ and $z^3 : S^1 \to S^1$. These two covering maps are not equivalent.

Lemma 1.9.2 (General Lifting Lemma)**.** *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map with $f(y_0) = b_0$. If Y is path connected and locally path connected then the map f can be lifted to $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$. If such a lifting exists, it is uniquely determined.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. Suppose that the lifting $\tilde{f}$ exists so that $(f_*) = (p \circ \tilde{f})_* = p_* \circ \tilde{f}_*$ in which case

$$f_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(E, e_0)). \tag{1.50}$$

If $\tilde{f}$ exists then we claim it is unique, for which we shall use the uniqueness of path-liftings. Given a point $y_1 \in Y$ choose a path α that connects y_0 to y_1. The composition $f \circ \alpha$ is a path in B, while the composition $\tilde{f} \circ \alpha$ is a path in E which corresponds to a lifting of $f \circ \alpha$. However, the lift of paths in B are unique and so $\tilde{f}(y) = \tilde{f} \circ \alpha(1)$ is uniquely determined since $\tilde{f} \circ \alpha$ is uniquely determined. Since Y is path connected, this procedure can be applied for every point $y \in Y$ and so $\tilde{f}$ is unique.

To show existence, we begin by defining $\tilde{f}$ to send y_0 to e_0. For a general point $y \in Y$ let α be a path connected y_0 to y. Once again, $f \circ \alpha$ is a path in B starting at b_0 and so there exists a path γ in E that lifts $f \circ \alpha$ and starts at the point e_0. Define $\tilde{f}(y) = \gamma(1)$. We need to show that $\tilde{f}$ is well-defined; that is, it does not depend on the choice of path α connecting y_0 to y. To do this, let β be some other path connecting y_0 to y and consider the path $f \circ \alpha \star \overline{f \circ \beta} = f \circ (\alpha \star \bar{\beta})$ which is a loop in B. By hypothesis, we know that $f(\alpha \star \bar{\beta}) \subseteq p_*(\pi_1(E_1, e_0))$ and so this lifts as a loop. Thus if σ is a lift of $f \circ \beta$, σ must have the same endpoint as $\tilde{f} \circ \alpha$ to preserve the loop.

To show continuity, let N be an open neighbourhood of $\tilde{f}(y_1)$, for which we want to show that there is a neighbourhood W of y_1 such that $\tilde{f}(W) \subseteq N$. To begin, let U be a neighbourhood of $f(y_1)$ that is evenly covered by p. Then there exists V_0 a slice of $p^{-1}(U)$ that contains $\tilde{f}(y_1)$. By choosing U sufficiently small so that $V_0 \subseteq N$ we know there exists a neighbourhood W of y_1 such that $f(W) \subseteq N$. We claim that $\tilde{f}(W) \subseteq V_0$. Indeed, if $v \in W$ choose a path β from y_1 to y. By the well-definedness of $\tilde{f}$, the point $\tilde{f}(y)$ can be obtained by lifting $f \circ (\alpha \star \beta)$ in E and letting $\tilde{f}(y)$ be the endpoint of this map. Set $\delta = p_0^{-1} \circ f \circ \beta$

that ends at $\tilde{f}(y_1)$; then $\gamma \star \delta$ is a lift of $f \circ (\alpha \star \beta)$ that begins at de_0 and ends at $\delta(1)$ of V_0, so $\tilde{f}(W) \subseteq V_0$ as desired. $\square$

Theorem 1.9.3. *Let $p : E \to B$ and $p' : E' \to B$ be covering maps such that $p(e_0) = p'(e_0') = b_0$. Then there is an equivalence $h : E \to E'$ such that $h(e_0) = e_0'$ if and only if*

$$H_0 = p_*(\pi_1(E,e_0)) = p'_*(\pi_1(E', e_0')) = H_0'. \tag{1.51}$$

If h exists, it is unique.

Proof. Suppose that there exists such an equivalence. Since h is a homeomorphism, we know that $h_*(\pi_1(E, e_0)) = \pi_1(E', e_0)$ and since $p' \circ h = p$ then $H_0 = H_0'$.

On the other hand, suppose that $H_0 = H_0'$ and show that h exists. Since the image of the subgroups are in fact equal, we can apply the general lifting lemma in either direction; that is, there is a map $h : E \to E'$ and a map $k : E' \to E$ such that $k \circ h : E \to E$ is a lift of p and $h \circ k : E' \to E'$ is a lift of p', which follows since $p \circ k \circ h = p' \circ h = p$ with $p(e_0) = e_0$. By uniqueness, since the identity is also a lift we must have that $k \circ h$ is the identity on E and $h \circ k$ is the identity on E'. There are mutually inverse continuous maps and so are bijective, hence homeomorphisms. $\square$

Lemma 1.9.4. *Let $p : E \to B$ be a covering map and let $e_0, e_1 \in p^{-1}(b_0)$. Setting $H_i = p_*(\pi_1(E, e_i))$ we have that*

1. *If γ is a path in E from e_0 to e_1 and α is a loop $p \circ \gamma$ in B then the equation $[\alpha] \star H_1 \star [\alpha]^{-1} = H_0$ holds so that H_0 and H_1 are conjugate.*
2. *On the other hand, given an e_0 and a subgroup $H \leq \pi_1(B, b_0)$ which is conjugate to H_0, there exists a point $e_1 \in p^{-1}(b_0)$ such that $H_1 = H$.*

Proof.

1. Given $[h] \in H_1$ we know there is some loop $\bar{h} \in \pi_1(E, e_1)$ such that $p_*([\bar{h}]) = [h]$. Define $\bar{k} = (\gamma \star \bar{h}) \star \gamma \in \pi_1(E, e_0)$ so that

$$p_*[\bar{k}] = [(\alpha \star h) \star \bar{\alpha}] = [\alpha] \star [h] \star [\bar{\alpha}] \in p_*(\pi_1(E, e_0)) = H_0 \tag{1.52}$$

and so $[\alpha] \star H_1 \star [\bar{\alpha}] \subseteq H_0$. An identical argument with the indices interchanged gives the opposite inclusion and we conclude our result.

2. Since H and H_0 are conjugate, we know there exists $[\alpha]$ such that $H_0 = [\alpha] \star H \star [\alpha]^{-1}$, so let γ be a lifting of α to a path in E beginning at e_0, defining $e_1 = \gamma(1)$. Employing

part (a) we then have that $H_0 = [\alpha] \star H_1 \star [\alpha]^{-1}$ and so comparing both conjugates, we conclude that $H = H_1$.

□

Theorem 1.9.5. *Let $p : E \to B', p : E' \to B$ be covering maps with $p(e_0) = p'(e_0') = b_0$. Then the covering maps p and p' are equivalent if and only if $H_0 = p_*(\pi_1(E, e_0))$ and $H_0' = p_*'(\pi_1(E', e_0))$ are conjugate.*

Proof. Assume that the two covering maps are equivalent so that there exists a homeomorphism $h : E \to E'$. Set $e_1' = H(e_0)$ and $H_1' = p_*(\pi_1(E', e_1'))$. Then theorem 1.9.3 tells us that $H_0 = H_1'$ and lemma 1.9.4 implies that H_1' is conjugate to H_0' and so H_0 is conjugate to H_0'.

Conversely, now assume that that H_0 and H_0' are conjugate. Then Lemma 1.9.4 tells us there is a point e_1' of E' such that $H_1' = H_0$, and Theorem 1.9.3 then implies that the covering maps are equivalent. □

Example 1.9.6. Let us classify all the covering spaces of S^1. Since $\pi_1(S^1, b) = \mathbb{Z}$ we know that all subgroups are of the form $n\mathbb{Z}$ and that conjugacy classes are actually equalities since the group is abelian. Hence the only such covering maps are of the form $z \mapsto z^n$.

1.10 Universal Covering Spaces

Definition 1.10.1. If $p : E \to B$ is a covering map with $p(e_0) = b_0$, then if E is simply connected we say that E is a *universal covering space* of B.

Note that if $p : E \to B$ and $p : E' \to B$ are universal covering maps then $\pi_1(E, e_0) \cong \pi_1(E', e_0') = \{0\}$ and so their image under their corresponding induced homomorphisms will be conjugate. Thus all universal coverings are equivalent up to homeomorphism.

Lemma 1.10.2. *Let $p : X \to Z, q : X \to Y, r : Y \to Z$ be continuous maps with $p = r \circ q$ then*

1. *If p and r are covering maps then q is a covering map.*
2. *If p and q are covering maps then r is a covering map.*

Theorem 1.10.3. *Let $p : E \to B$ be a covering map and let E be simply connected. Given any covering map $q : Y \to B$ there exists a covering map $r : Y \to E$ such that $q = p \circ r$.*

Proof. Via the general lifting lemma and Lemma 1.10.2 this follows immediately. $\square$

Lemma 1.10.4. *Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$. If E is simply connected then b_0 has a neighbourhood U such that $\iota : U \hookrightarrow B$ induces the trivial homomorphism $\iota_* : \pi_1(U, b_0) \to \pi_1(B, b_0)$.*

Proof. Let U be a neighbourhood of b_0 that is evenly covered by p. If we look at the inverse image of U, one of these sets must contain the point e_0: call this set V_0. Let α be a loop based at b_0 in U so that $\tilde{\alpha}$ is a lift starting at e_0. Since $p|_V$ is a homeomorphism, $\tilde{\alpha}$ is contained entirely within V_0 and is a loop itself. Since E is simply connected, there exists a homotopy $\tilde{F}$ that connects $\tilde{\alpha}$ to the constant map at e_0; that is, $\tilde{\alpha}$ is contractible. Hence $p \circ \tilde{F}$ is a path homotopy in B between f and the constant loop at b_0. $\square$

Example 1.10.5. Consider the infinite Hawaiian earing, which we claim does not have a covering space. Let C_n be the circle of radius $\frac{1}{n}$ with center $(\frac{1}{n}, 0)$ so that $X = \bigcup_n C_n$ is our space. If b_0 is the origin, let U be a neighbourhood of b_0 and consider N sufficiently large so that C_n is contained entirely within U. There is a retraction $r : X \to C_n$ by keeping C_n fixed and sending everything else to b_0. Then the following commutative diagram implies that i^* cannot be trivial:

$$\begin{tikzcd}
\pi_1(C_n, b_0) \arrow[rr, "j_*"'] \arrow[dr, "k_*"] & & \pi_1(X, b_0) \\
& \pi_1(U, b_0) \arrow[ur, "i_*"] &
\end{tikzcd}$$

Indeed, i, j, k are all induced by inclusion and the diagram is commutative. Since j_* is injective it is non-trivial and so i_* is non-trivial.

1.11 Covering Transformations

Definition 1.11.1. Given a covering map $p : E \to B$ the *group of covering transformations* or *group of deck transformations* is the set of all equivalences of covering spaces and is denoted $\mathcal{C}(E, p, B)$.

Recall that two covering maps $p : E \to B$ and $p' : E' \to B$ are said to be equivalent if there exists $h : E \to E'$ a homeomorphism such that $p = p' \circ h$. Hence $\mathcal{C}(E, p, B)$ consists of the mappings $h : E \to E'$ that gives these equivalences. The group $\mathcal{C}(E, p, B)$ is entirely determined by the group $\pi_1(B, b_0)$ and $H_0 = p_*(\pi(E, e_0))$.

Definition 1.11.2. If $H \leq G$ are group, the *normalizer* of H in G is

$$N_G(H) = \left\{ g : gHg^{-1} = H \right\}. \tag{1.53}$$

Let $F = p^{-1}(b_0)$ and recall from Theorem 1.3.18 that there is a map $\Phi : \pi_1(B, b_0)/H_0 \to F$. Define $\Psi : \mathcal{C}(E, p, B) \to F$ by $\Psi(h) = h(e_0)$. Ψ is injective since h is uniquely determined by $h(e_0)$.

Lemma 1.11.3. *The image of Ψ is the image of $N_G(H_0)/H_0 \subseteq \pi_1(B, b_0)/H_0$ under Φ.*

Proof. In order to make sense of this, we will have to recall how Φ acts on loops. Let α be a loop in B based at b_0 and let γ be its lift to E beginning at E_0. If $e_1 = \gamma(1)$ then the lifting corresponds sends $\alpha \to e_1$. We want to show that there is a covering map equivalence $h : E \to E$ with $h(e_0) = e_1$ if and only if $[\alpha] \in N_G(H_0)$. To see this, recall that Theorem 1.9.3 tells us that such a map will exist if $H_0 = p_*(\pi_1(E, e_0)) = p_*(\pi_1(E, e_1)) = H_1$. But also we know that such an equivalence exists if and only if $[\alpha] \star H_0[\alpha]^{-1} = H_0$ and $[\alpha] \star H_0[\alpha]^{-1} = H_0$. This will be true if and only if $[\alpha] \in N_G(H_0)$, which is precisely our case. □

Theorem 1.11.4. *The map $\Phi^{-1} \circ \Psi : C(E, p, B) \to N_G(H_0)/H_0$ is a group isomorphism.*

Proof. We have already shown that this map is bijective, hence all that remains to be shown is that the map is a group homomorphism. Let $h, k \in \mathcal{C}(E, p, B)$ so that $h, k : E \to E$ are both covering transformations with $h(e_0) = e_1$ and $k(e_0) = e_2$. By definition of Ψ we then have that $\Psi(h) = e_1$ and $\Psi(k) = e_2$, and so we can choose a path γ taking e_0 to e_1 and a path δ taking e_0 to e_2. Notice then that

$$\Phi([p \circ \gamma]H_0) = e_1, \qquad \Phi([p \circ \delta]H_0) = e_2 \tag{1.54}$$

so if $e_3 = h(k(e_0))$ then $\Psi(h \circ k) = e_3$. Then $h \circ \delta$ is a path from $h(e_0) = e_1$ to $h(e_2) = h(k(e_0)) = e_3$ and so $\gamma \star (h \circ \delta)$ is a path from e_0 to e_3. It is a lifting of $\alpha \star \beta$ and so $\Phi([\alpha \star \beta]H_0) = e_3$. □

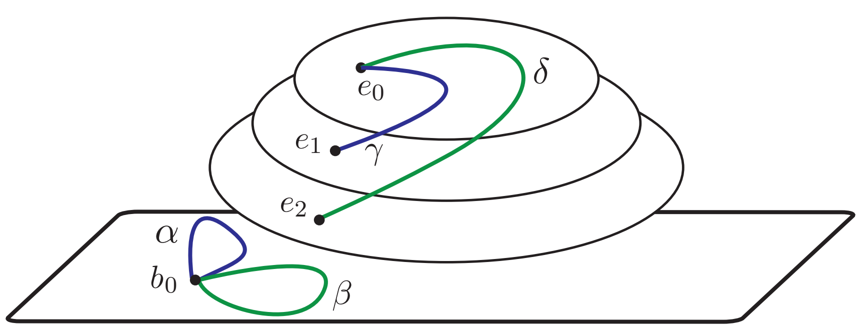

Figure 1.18: The paths used in the proof of Theorem 1.11.4.

This equivalence tells us that we may study universal covering maps simply by studying the subgroups and corresponding quotient groups of the fundamental group of B.

Definition 1.11.5. If $H_0 \lhd \pi_1(B, b_0)$ then $p : E \to B$ is said to be a *regular cover.*

Note that alternatively, we may say that a covering space $p : E \to B$ is *normal* if for any two points $e_0, e_1 \in p^{-1}(b_0)$ there is a deck transformation $h : e_0 \to e_1$. The following propositions shows that normal covers and regular covers are actually equivalent notions.

Proposition 1.11.6. *Let $p : E \to B$ with $p(e_0) = b_0$ be a path-connected covering space of the path connected, locally path connected space X, and let $H = p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$. Then*

1. *The covering space is normal if and only if $H \lhd \pi_1(B, b_0)$.*
2. *If the covering space is normal then $C(E, p, B) \cong \pi_1(B, b_0)/H$ and if the covering space is universal then $C(E, p, B) \cong \pi_1(B, b_0)$.*

Proof. Hatcher pg 71, proposition 1.39. □

Theorem 1.11.7. *Let B be path connected, locally path connected, and semi-locally simply connected and take a point $b_0 \in B$. Given H of $\pi_1(B, b_0)$ there exists a covering map $p : E \to B$ and a point $e_0 \in p^{-1}(b_0)$ such that $p_*(\pi_1(E, e_0)) = H$.*

Proof. Let S be the set of all paths starting at b_0 and define an equivalence relation on S by setting $\alpha \sim \beta$ if α and β end at the same endpoint in B and $[\alpha \star \bar{\beta}] \in H$. We will use $\alpha^\#$ to denote equivalence classes in $E = S/\sim$. Define $p : E \to B$ as $p(\alpha^\#) = \alpha(1)$, and since B is path connected, p is surjective. Note that if $[\alpha] = [\beta]$ then α is pat homotopic to β and so they must have the same endpoints. Thus $[\alpha \star \bar{\beta}]$ is the identity element and obviously contained within H. On the other hand, if $\alpha^* = \beta^*$ and that if δ is any path beginning at $\alpha(1)$ then $(\alpha \star \delta)^\# = (\beta \star \delta)^\#$.

We now need to endow E with a topology. For any $\alpha \in S$ let U be any path connected neighbourhood of $\alpha(1)$. Define

$$B(U, \alpha) = \left\{ (\alpha \star \delta)^\# : \delta \text{ a path beginning at } \alpha(1) \right\}. \tag{1.55}$$

We need to show that $B(U, \alpha)$ is a basis for a topology. Notice that if $\beta^\# \in B(U, \alpha)$ then $\alpha^\# \in B(U, \beta)$. This is because $\beta^\# \in B(U, \alpha)$ implies that $\beta^\# = (\alpha \star \delta)^\#$ for some path in δ

and so

$$(\beta \star \bar{\delta})^{\#} = \big((\alpha \star \delta) \star \bar{\delta}\big)^{\#} = \alpha^{\#}. \tag{1.56}$$

Now we claim that this implies that $B(U, \beta) = B(U, \alpha)$. A general element of $B(U, \beta)$ is of the form $(\beta \star \gamma)^{\#}$ for some path $\gamma \in U$, so that

$$(\beta \star \gamma)^{\#} = ((\alpha \star \delta) \star \gamma)^{\#} = (\alpha \star (\delta \star \gamma))^{\#} \in B(U, \alpha) \tag{1.57}$$

implying that $B(U, \beta) \subseteq B(U, \alpha)$. Precisely the same argument holds for the other direction and equality immediately follows.

Let $\beta^{\#} \in B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$ and choose a path connected neighbourhood of $\beta(1)$ contained in $U_1 \cap U_2$ so that $B(V, \beta) \subseteq B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$.

Next we want to show that p is continuous. Let $\alpha^{\#}$ be an element in E and consider a neighbourhood W of $p(\alpha^{\#}) = \alpha(1)$. Let U be a path connected neighbourhood of $\alpha(1)$. Then a neighbourhood $B(U, \alpha)$ of $\alpha^{\#}$ is mapped to U under p by our previous observation, hence p is continuous.

Next our goal is to show that every point in B has a neighbourhood that is evenly covered by p. Fix a point $b_1 \in B$ and let U be a path connected neighbourhood of b_1. Assume further that $\pi_1(U, b_1) \to \pi_1(B, b_1)$ is trivial. Since any path from b_0 to b_1 will work, we note that $p^{-1}(U)$ contains the union of $B(U, \alpha)$ over all possible paths from b_0 to b_1. On the other hand, if $\beta^{\#}$ is in $p^{-1}(U)$ then then $\beta(1) \in U$. Since U is path connected, let δ be a path between $\beta(1)$ and b_0 so that we get a path α between b_0 and b_1; that is, $\beta^{\#} = (\alpha \star \bar{\delta})^{\#}$ which implies that $p^{-1}(U)$ is contained in the union of $B(U, \alpha)$. Hence equality must hold. All that remains to be shown is that the $B(U, \alpha)$ are disjoint and map homeomorphically onto U be p. Disjoint follows immediately since if $\beta^{\#} \in B(U, \alpha_1) \cap B(U, \alpha_2)$ then $B(U, \alpha_1) = B(U, \beta) = B(U, \alpha_2)$. Bijectivity will follow from injectivity since we have already shown surjectivity. Suppose then that $p((\alpha \star \delta_1)^{\#}) = p((\alpha \star \delta_2)^{\#})$ for two paths δ_1, δ_2 in U. By using semi-local simple connectivity we know that δ_1 is path homotopic to δ_2 which implies that $\alpha \star \delta_1$ is path homotopic to $\alpha \star \delta_2$. which implies that $(\alpha \star \delta_1)^{\#} = (\alpha \star \delta_2)^{\#}$.

Now we can lift paths in B to those in E. To see this, let e_0 denote the equivalence class of the constant path at b_0. Let $c \in [0, 1]$ and define the map $\alpha_c(t) = \alpha(tc)$ for $0 \leq t \leq 1$. Define the map $\tilde{\alpha} : I \to E$ by $\tilde{\alpha}(c) = (\alpha_c)^{\#}$. One needs to show that this map is continuous, and that is left as an exercise.

To see that $p : E \to B$ is a covering map, we only need to show that E is path connected. However, this follows quite quickly since if $\alpha^{\#}$ is any point in E then $\tilde{\alpha}$ is a lift of α which is a path from e_0 to $\alpha^{\#}$.

Finally, we claim that $H = p_*(\pi_1(E, e_0))$. Let α be a loop at b_0, for which we want to show that $[\alpha] \in p_*(\pi_1(E, e_0)) \Leftrightarrow [a] \in H$. We know that $[\alpha] \in p_*(\pi_1(E, e_0)) \Leftrightarrow \tilde{\alpha}$ is a loops. But $\tilde{\alpha}$ is a loop if and only if $\alpha^{\#}$ is e_0 if and only if α and a constant map both have the same endpoints and $[\alpha \star \bar{c_{b_0}}] \in H$ if and only if $[\alpha] \in H$. □

Note that it can be quite difficult to check the relation $[\alpha \star \bar{\beta}] \in H$, making this construction difficult to actually carry through. Hence it is useful theoretically, but not practically.

If E is the universal covering space of B and Γ is its group of deck transformations which is isomorphic to $\pi_1(B, b_0)$. Let G be a subgroup of $\pi_1(B, b_0)$ and take Γ_G to the be corresponding group of deck transformations. Define an equivalence relation on E as follows: We will say that two points are equivalent if one can be mapped to the other by an element of Γ_G. By considering the quotient space E/Γ_G there is a map $\pi : E/\Gamma_G \to B$ which is induced by $p : E \to B$.

Example 1.11.8. Let $B = S^1$ and $E = \mathbb{R}$. We know that $\pi_1(S^1, b_0) = \mathbb{Z}$ in which case the corresponding deck transformations are given by $x \mapsto x + n$. Consider the set of transformations given by $x \mapsto x + 3n$ which corresponds to a subgroup of $\mathbb{Z}$. In this case we then have that $E/\Gamma_g = [0, 3]/\sim = S^1$.

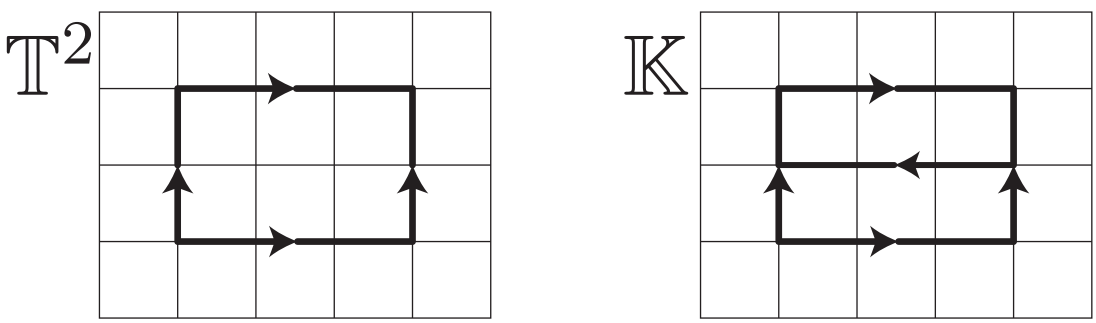

Figure 1.19: The rectangles describing the quotient group of deck transformations for the torus (left) and the Klein bottle (right).

Example 1.11.9. Consider the torus $T = S^1 \times S^1$. We know that the universal covering space of T is the plane $\mathbb{R}^2$ and the fundamental group is $\pi_1(S^1 \times S^1, b_0) \cong \mathbb{Z} \oplus \mathbb{Z}$. For an element $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ the corresponding deck transformation is given by $(x, y) \mapsto (x + m, y + n)$. Now consider the subgroup Γ_G given by $(x, y) \mapsto (x + 3k, y + 2\ell)$ which corresponds to $3\mathbb{Z} \times 2\mathbb{Z}$. Any point is equivalent to a point in a rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 2$ and in particular, we can identify edges of the induced rectangle in $\mathbb{R}^2$ to get the 6-fold cover of the torus (See Figure 1.19, left caption).

Example 1.11.10. Let B be the Klein bottle and $E = \mathbb{R} \times \mathbb{R}$. Recall that the Klein bottle can be can be given by adjoining a two-cell with the relation $aba^{-1}b$, which we claim is equivalent to $\mathbb{Z} \rtimes \mathbb{Z}$ with $(m_1, n_1) \cdot (m_2, n_2) = (m_1 + (-1)^{n_1} m_2, n_1 + n_2)$. Note that in our original presentation, we have $ab^{-1} = ba$ and so $ab^{-\ell} = b^{\ell}a$. Hence

$$\begin{aligned} b^{m_1} a^{n_1} \cdot b^{m_2} a^{n_2} &= b^{m_1} b^{(-1)^{n_1} m_2} a^{n_1} a^{n_2} \\ &= b^{m_1 + (-1)^{n_1} m_2} a^{n_1 + n_2} \end{aligned}$$

so that the map $(n, m) \mapsto b^m a^n$ is the required isomorphism.

Now consider the subgroup of $G \leq \mathbb{Z} \rtimes \mathbb{Z}$ generated by $(2,0)$ and $(1,3)$; that is, $G = \langle (2,0), (1,3) \rangle$. The G can be realized as a union

$$G = \bigcup_{n \text{ even}} \{(2m, 3n)\} \cup \bigcup_{n \text{ odd}} \{(2m+1, 3n)\}.$$

We want a covering map that corresponds to G. In particular, if we take (m, n) as a general element in $\mathbb{Z} \rtimes \mathbb{Z}$ the corresponding deck transformation acts as

$$(x, y) \mapsto ((-1)^n x + m_1, y + n). \tag{1.58}$$

Hence generator of G correspond to $(x, y) \mapsto (x + 2, y)$ and $(x, y) \mapsto (-x + 1, y + 3)$. A 6-fold cover of the Klein bottle (See Figure 1.19, right caption).

Chapter 2

Homology

2.1 Cell-Complexes

We have already seen how to construct the torus from a square with opposite sides appropriately identified. Let us extend this to oriented surfaces of genus g.

Start with a discrete set X^0 and inductively form the n-skeleton X^n from the $n-1$-skeleton by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$ where X^n is $X^{n-1} \bigsqcup D^n_\alpha / \sim$ where the equivalence class is given by $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. We may choose to terminate this process after finitely many steps, or we may consider $X = \bigcup_n X^n$. In this case, we endow X with the weak topology as follows: the set $A \subseteq X$ is open in X if and only if $A \cap X^n$ is open for all n in X^n.

Definition 2.1.1. Any space constructed in the sense described above is called a *CW-complex*.

Example 2.1.2.

1. The set of one-dimensional cell-complexes are known as graphs.
2. The two sphere S^2 can be viewed as a two-complex. Begin with a single point and attach a two-cell by mapping the boundary of the two-cell to that single point. Note that this construction is note unique. We also could have attached a one cell to the point (creating a circle) and then attached two 2-cells to the one-cell to get the sphere.
3. Recall that the real projective space is the set of all lines through the origin. In $\mathbb{R}^{n+1}$ we can view $\mathbb{RP}^n$ as an embedding of S^n in $\mathbb{R}^{n+1}$ under the antipodal identification. Alternatively, consider an n-disk with diametrically opposing boundary points identified. These are actually equivalent, since $\partial D^n = S^{n-1}$. Hence $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n$, since

we have just adjoined an n-cell. This gives the general decomposition of the projective space as

$$\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n, \qquad \mathbb{RP}^\infty = \bigcup_{n=0}^{\infty} \mathbb{RP}^n.$$

4. The complex projective space is defined in the same way as the real projective space; namely, the quotient space of the identification of lines through the origin. That is, $\mathbb{CP}^n = \mathbb{C}^{n+1}/\sim$ where $v \sim \lambda v$ for all $\lambda \neq 0$. Again we can view this as a quotient of a sphere $S^{2n+1}/\sim$ under a similar identification, $v \sim \lambda v$ with $|\lambda| = 1$.

   There is a natural homeomorphism $S : \mathbb{C}n + 1 \to \mathbb{R}^{2n+2}$ which is called the realification operator, which acts as

$$(z_1, \ldots, z_{n+1}) \mapsto (x_1, y_1, \ldots, x_{n+1}, y_{n+1}). \tag{2.1}$$

   Since $|z| = [\sum z_i \bar{z}_i]^{\frac{1}{2}}$ we know that the complex sphere is simply the subspace $|z| = 1$ of $\mathbb{C}^{n+1}$.

Proposition 2.1.3. *The complex projective space $\mathbb{CP}^n$ is a CW-complex of* $\dim 2n$ *and has open cells of only even dimension, and its j-skeleton is $\mathbb{CP}^j$.*

Proof. We would like to show that $\mathbb{CP}^n - \mathbb{CP}^{n-1}$ is an open cell of dimension $2n$. Note that $\mathbb{CP}^0$ is just a point, since all points can be identified with the point $z = 1$ by multiplying by its complex conjugate. On the other hand, consider the subset of a all points in S^{2n+1} wherein z_{n+1} is real, that is

$$\{(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, 0)\} \tag{2.2}$$

which is a sphere of dimension $2n$. If we take E_+^{2n} to be the subset of S^{2n+1} such that z_n is real and non-negative, this yields an "upper hemisphere" of S^{2n}. We would like to show that a quotient map (rather, its restriction to E_+^{2n}) maps the interior of E_+^{2n} bijectively onto $\mathbb{CP}^n - \mathbb{CP}^{n-1}$.

We begin by showing surjectivity. Suppose that we are given a point $y \in \mathbb{CP}^n - \mathbb{CP}^{n-1}$. We know that $p(z) = y$ for some z in $S^{2n+1} - S^{2n-1}$ where $|z| = 1$ and $z_{n+1} \neq 0$. Write $z_{n+1} = re^{i\theta}$ for some $r > 0$. We would like to show that this lies in the upper hemisphere. Since we have the freedom to multiply by units of modulus one, we take $\lambda = e^{-i\theta}$ so that $\lambda z = (\lambda z_1, \ldots, \lambda z_n, r, 0)$.

To show injectivity, let $p(z) = p(w)$ for $z, w \in \operatorname{Int} E_+^{2n}$, so that $z = \lambda w$ for some $|\lambda| = 1$. Since $z_{n+1} = x_{n+1} > 0$ and $w_{n+1} = \tilde{x}_{n+1} > 0$ we know that $\lambda \equiv 1$ and so $z = w$ so injectivity holds. ∎

2.1.1 Simplices

Definition 2.1.4. An *n-simplex* is the smallest convex set in $\mathbb{R}^n$ containing $n+1$-points $v_0, \dots, v_{n+1}$ that do not lie in a hyperplane of dimension less than n.

The points v_i are called vertices, and their orders define an orientation. Two orientations are equivalent if the ordering of their indices differ by an even permutation.

Definition 2.1.5. The *standard simplex* is the set

$$\Delta^n = \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1, t_i \geq 0, \forall i.\right\} \tag{2.3}$$

There exists a canonical linear homeomorphism from Δ^n to any other n-simplex $[v_0, \dots, v_n]$ that preserves the order of the vertices. This homeomorphism is given as

$$(t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i. \tag{2.4}$$

Definition 2.1.6. A *face* of a simplex is the subsimplex with vertices that is any non-empty subset of $\{v_i\}$.

Definition 2.1.7. A Δ-*complex* is a quotient space of a collection of disjoint simplices obtained by identifying some of their faces via the canonical linear homeomorphisms.

Definition 2.1.8. A *simplicial complex* is a Δ-complex whose subsimplices are uniquely determined by their vertices.

Definition 2.1.9. Let $C_n = \Delta_n(x)$ be a free abelian group whose basis is the open simplices of X. Elements of $\Delta_n(x)$ are called n-chains.

Note that a typical element of $\Delta_n(x)$ looks like finite sums of the form $\sum_\alpha n_\alpha e_\alpha^n$ for $n_\alpha \in \mathbb{Z}$ and e_α^n open simplices.

Definition 2.1.10. The boundary of an n-simplex $[v_0, \dots, v_n]$ as

$$\partial[v_0, \dots, v_n] = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots v_n] \tag{2.5}$$

where $\hat{v}_i$ denotes the omission of v_i.

Example 2.1.11. 1. For a 1-simplex we have $\partial[v_0, v_1] = v_1 - v_0$.

2. For a 2-simplex

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]. \tag{2.6}$$

Definition 2.1.12. We define the *boundary homeomorphism* $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by

$$\partial \sum_\alpha n_\alpha e_\alpha^n = \sum_\alpha n_\alpha \partial e_\alpha^n. \tag{2.7}$$

Lemma 2.1.13. *The map $\partial_{n-1}\partial_n : \Delta_n(X) \to \Delta_{n-2}(X)$ is the trivial homomorphism.*

Example 2.1.14. For example, consider the simplex $[v_0, v_1, v_2, v_3]$ in which case

$$\partial_3[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]. \tag{2.8}$$

Applying ∂_2 we then get

$$\begin{aligned}\partial_2\partial_3 &= -[v_2,v_3]+[v_1,v_3]-[v_0,v_3]-[v_1,v_2]+[v_0,v_2]-[v_0,v_1]\\ &\quad +[v_2,v_3]-[v_1,v_3]+[v_1,v_2]+[v_0,v_3]-[v_0,v_2]+[v_0,v_1]\\ &= 0.\end{aligned}$$

The proof of the lemma follows precisely the same idea:

Proof.

$$\begin{aligned}\partial_{n-1}\partial_n[v_0,\ldots,v_n] &= \sum_{j<i}(-1)^i(-1)^j[v_0,\ldots,\hat{v}_i,\cdots\hat{v}_j,\cdots v_n]+\\ &\qquad + \sum_{j>i}(-1)^i(-1)^{j-1}[v_0,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_n]\\ &= 0\end{aligned}$$

□

Definition 2.1.15. A *chain complex* is a sequence of the form

$$\to \Delta_{n+1} \xrightarrow{\partial_{n+1}} \Delta_n \xrightarrow{\partial_n} \Delta_{n-1} \to \cdots \to \Delta_1 \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} 0 \tag{2.9}$$

wherein $\partial_n\partial_{n+1} = 0$, so that $\operatorname{im}(\partial_{n+1}) \subseteq \ker \partial_n$. Denote by $Z_n = \ker \partial_n$ which are the cycles, and $B_n = \operatorname{im}\partial_{n+1}$ to be the boundaries. Then the n^{th} *homology group* is defined as $H_n = Z_n/B_n$.

It is hopeful that the reason why Z_n are the cycles and B_n are the boundaries are clear. By definition, a cycle is a (non-trivial) element of C_n which evaluates to zero, and since all n-cells occur as the boundary of $n+1$ cells, it then makes sense that cycles are precisely those elements which vanish in the boundary map. On the other hand, the image of ∂_{n+1} is precisely the collection of n-boundaries. We consider the quotient by B_n since we are only interested in cycles up to an identification of the boundaries of neighbouring faces. More specifically, any n-cycle which occurs as the boundary of a $n+1$-cycle can be homotopically identified as a single n-cell. Thus we are only interested in cycles up to boundaries. A good example of this is given by Figure 2.1, which is a CW complex of two vertices x and y, four edges a, b, c and d, and a single 2-cell called A. Note that the edges a and b are homotopically equivalent because they are the boundary of A; that is, $\partial A = b - a$. Hence when we consider one-cycles in X we are only interested in those that contain either the edge a or b since they are identified via the homotopy.

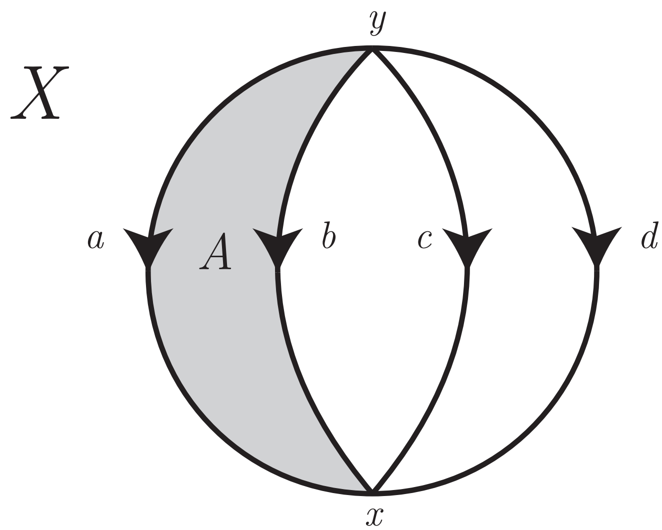

Figure 2.1: .

Example 2.1.16. Consider Figure 2.2 which consists of three vertices (0-cells) and three edges (1-cells). We endeavour to examine the homology groups of this complex and note that $C_n(X) = 0$ for $n \geq 3$ since there are n-cells for these numbers. Hence we have the following chain complex for this Δ-complex

$$0 \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Note that $C_0(X)$ is generated by the number of vertices, so that $C_0(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. On the other hand, $C_1(X)$ is generated by the number of edges, of which there are again three so $C_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Hence

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_0} 0.$$

Now $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2$ and while it is not explicitly given in the above diagram, it is clear that $\partial_2 = 0$ and so $\operatorname{im} \partial_2 = \{0\}$ so $H_1(X) = \ker \partial_1$. Let us now look at elements of the kernel. Pick a typical element in C_1 and notice that

$$\begin{aligned} 0 = \partial(me_0 + ne_1 + ke_2) &= m(v_1 - v_0) + n(v_2 - v_1) + k(v_0 - v_2) \\ &= (k - m)v_0 + (m - n)v_1 + (n - k)v_2 \end{aligned}$$

This means in particular that $k = m = n$ and so up to a choice of integer, we have the element $e_0 + e_1 + e_2$ which generates Z_1 implying that $H_1(X) = Z_1(X) = \mathbb{Z}$. It turns out that $H_0(X) = \mathbb{Z}$ since X is path connected, but the proof of this result must be delayed until we look at singular homology.

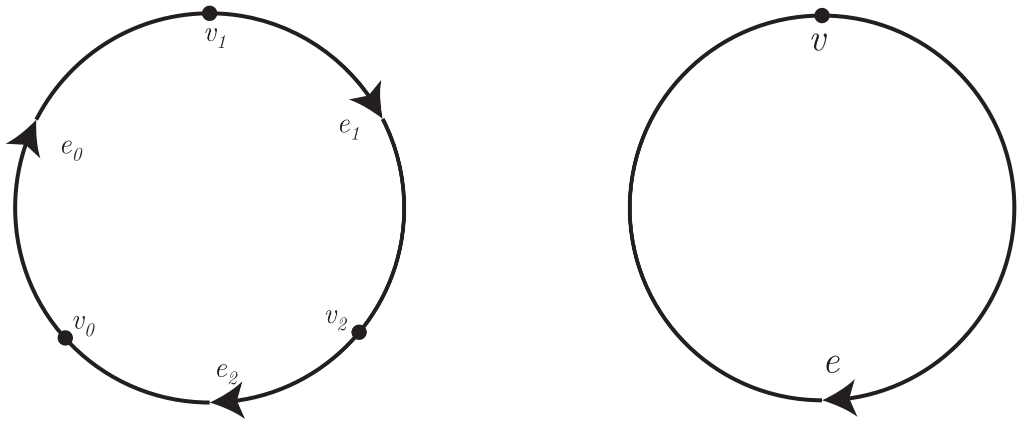

Figure 2.2: The cell complexes for Example 2.1.16 (left) and for the sphere (right).

Example 2.1.17. In this example, we compute the homology of the circle. Note that this is a much more simplified version of Example 2.1.16. Indeed, we can visualize the sphere as a single vertex and a single edge as in Figure 2.2.

This diagram gives us the following chain complex

$$0 \to \underset{\mathbb{Z}}{C_1(X)} \xrightarrow{\partial_1} \underset{\mathbb{Z}}{C_0(X)} \to 0.$$

By precisely the same arguments as those given in Example 2.1.16 we see that $\operatorname{im}\partial_2 = \{0\}$ and $\partial_1(me) = 0$ so that ∂_1 is the zero map. Thus $H_1(X) = \ker\partial_1/\operatorname{im}\partial_2 = C_0(X) = \mathbb{Z}$. On the other hand, $H_0(X) = Z_0(X)/B_0(X) = \mathbb{Z}$ by the same path connectedness argument as made before.

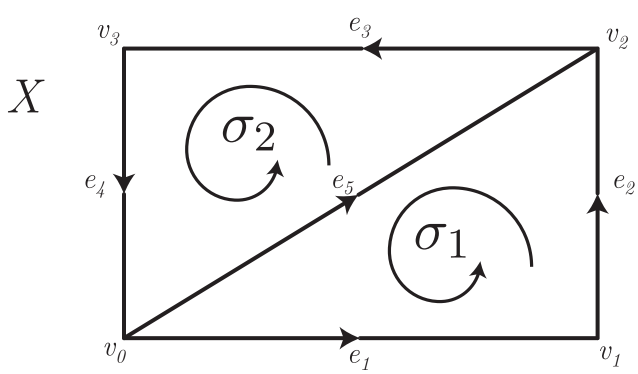

Figure 2.3: The cell complexes for Example 2.1.18

Example 2.1.18. Consider the simplicial complex X given by Figure 2.3 which is homeomorphic to a disk. This complex consists of four vertices, five edges, and two faces, hence we get the following chain complex

$$0 \to \underset{\bigoplus_{i=2}^{4}\mathbb{Z}}{C_2(X)} \xrightarrow{\partial_2} \underset{\bigoplus_{i=1}^{5}\mathbb{Z}}{C_1(X)} \xrightarrow{\partial_1} \underset{\bigoplus_{i=1}^{4}\mathbb{Z}}{C_0(X)} \to 0.$$

We begin by calculating the second homology group, $H_2(X) = \ker \partial_2 / \operatorname{im} \partial_3 = \ker \partial_2$. Consider the cycles $\sigma_1 = [v_0, v_1, v_2]$ and $\sigma_2 = [v_0, v_2, v_3]$. A general cycle of C_2 will thus be of the form $m\sigma_1 + n\sigma_2$ and is contained in the kernel of ∂_2 if

$$\begin{aligned} 0 = \partial(m\sigma_1 + n\sigma_2) &= m\partial\sigma_1 + n\partial\sigma_2 \\ &= m(e_1 + e_2 - e_5) + n(e_5 + e_3 + e_4) \\ &= me_1 + me_2 + ne_3 + ne_4 + (n-m)e_5 \end{aligned}$$

which implies that $m = n = 0$. Hence the only element in the kernel of ∂_2 is the trivial element implying that $Z_2(X) = \{0\}$ so $H_2(X) = \{0\}$. For the one dimensional cycles, we have a general element $\alpha = n_1e_1 + n_2e_2 + n_3e_3 + n_4e_4 + n_5e_5$ in which case one can compute

$$\partial\alpha = v_0(-n_1 + n_4 - n_5) + v_1(n_1 - n_2) + v_2(n_2 - n_3 + n_5) + v_3(n_3 - n_4) = 0 \tag{2.10}$$

which gives us a system of four equations with five unknowns, yielding a two-parameter family of solutions given by

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{pmatrix} = n_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + n_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{2.11}$$

Hence we have two generators, which can be read off from the column vectors in (2.11) as $e_1 + e_2 - e_5$ and $e_3 + e_4 + e_5$ which correspond precisely to the boundaries of the cycles σ_1 and σ_2. It is then clear that $Z_1(X) = \mathbb{Z} \oplus \mathbb{Z}$ but that $\operatorname{im} \partial_2 = \ker \partial_1$ and so $H_1(X) = (\mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z} \oplus \mathbb{Z}) = \{0\}$. Finally $H_0(X) = \mathbb{Z}$ since X is path connected.

Definition 2.1.19. If K is a simplicial complex then the *underlying space* of K or the *polytopes* of K, denoted $|K|$, is the subset of $\mathbb{R}^\infty$ that is the union of all simplicies of K. Each simplex is given its natural topology; that is, A is closed in $|K|$ if and only if $A \cap \sigma$ is closed for each σ in K.

Definition 2.1.20. A subcomplex of K is a subcollection of K that contains all faces of its elements.

Definition 2.1.21. A chain c is *carried by a subcomplex* L of K if it has value 0 on each simplex that is not in L.

Definition 2.1.22. Two subcomplexes c_p and c'_p are said to be *homologous* if their difference occurs as the boundary of a $p+1$ complex; that is, there exists d_{p+1} a $p+1$ complex such that $c_p - c'_p = \partial_{p+1} d_{p+1}$.

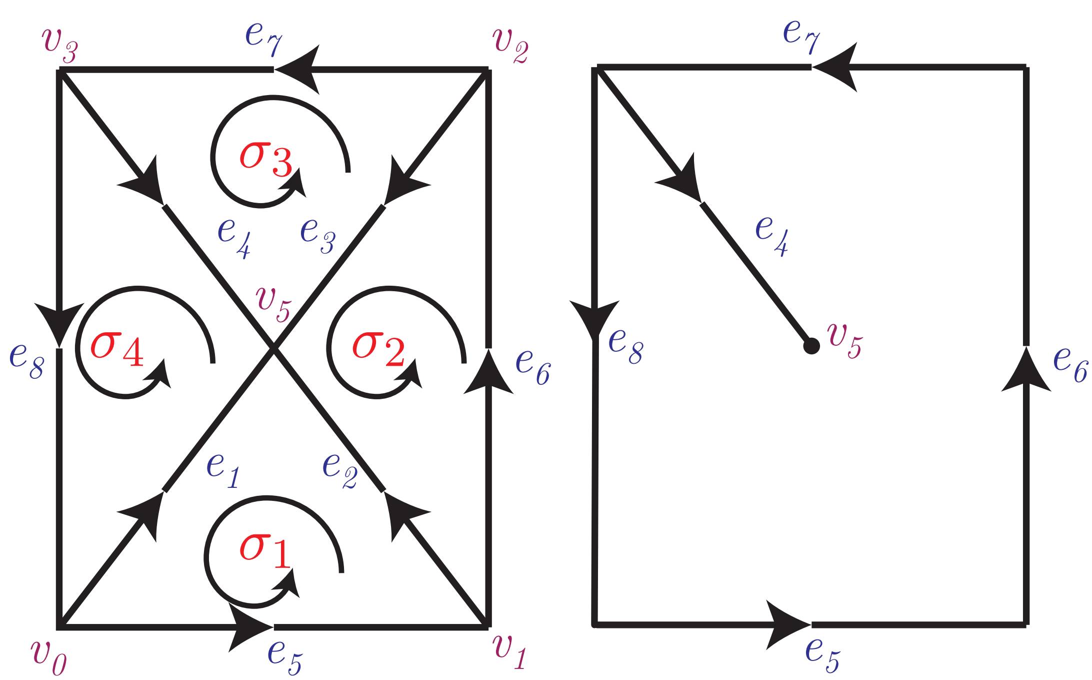

Figure 2.4: An intricate cell complex (left) and a subcomplex to which every 1-chain is homeomorphic (right), used in Example 2.1.23

Example 2.1.23. Consider the simplex given in Figure 2.4 in which case $|X|$ is homeomorphic to the square. Our goal is to show that any 1-chain is in fact homeomorphic to the figure given in the right caption of Figure 2.4. Let c be an arbitrary 1-chain which can be written as $c = \sum_i n_i e_i$. Now define $c_1 = c + n_1\partial(\sigma_1) = c + \partial(n_1\sigma_1)$ and notice that the coefficient of e_1 in c_1 is zero (since σ_1 is negatively oriented with respect to e_1). Hence c and c_1 are homologous, and in particular c_1 does not contain the simplex e_1. Similarly, if a is the coefficient of e_2 in c_1 we may define $c_2 = c_1 + \partial(a\sigma_2)$ which is homologous to c_1 (and hence c) and if b is the coefficient of e_3 in c_2 then $c_3 = c_2 + \partial(b\sigma_3)$ is homologous to c_2 (and hence c). Thus every 1-chain is actually homologous to complex given in the right diagram of Figure 2.4. To this point however, we only assumed that c was a chain. If we instead suppose that c is a cycle then c_3 is homologous to c, but cannot contain a non-trivial value for e_4. Thus c is homologous to a chain that is carried by the subcomplex given purely in terms of the boundary of X, which we denote $|X|$.

The purpose of this example is to demonstrate that despite the intricate structure that may be seen on the interior of a complex, cycles and hence homology groups depend only on the underlying polytope $|X|$. This can be proven in general, but is currently beyond the focus of this lecture.

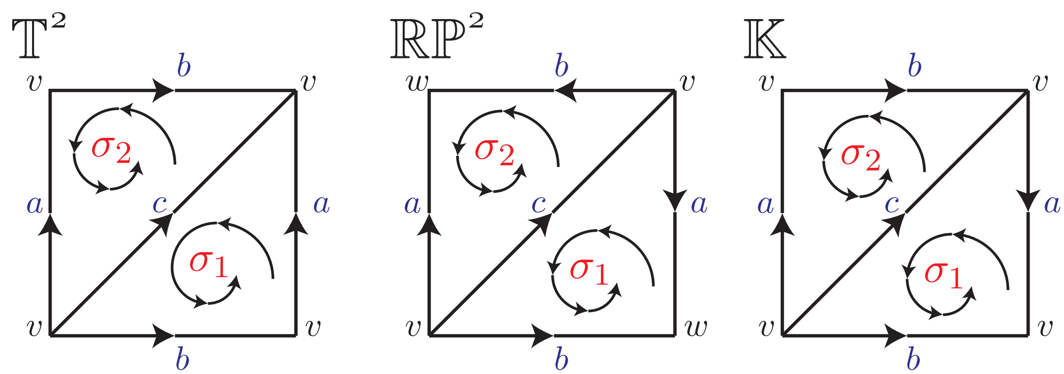

Figure 2.5: A series of Δ-complexes corresponding the torus (left), the real projective plane (middle) and the Klein bottle (right).

Example 2.1.24 (Homology of the Torus)**.** Consider the Δ-complex given in the left caption of Figure 2.5, which gives us the following chain complex

$$0 \to \underset{\mathbb{Z}^2}{C_2(\mathbb{T}^2)} \xrightarrow{\partial_2} \underset{\mathbb{Z}^3}{C_1(\mathbb{T}^2)} \xrightarrow{\partial_1} \underset{\mathbb{Z}}{C_0(\mathbb{T}^2)} \to 0.$$

Now $H_2(\mathbb{T}^2) = \ker \partial_2 / \operatorname{im} \partial_3 = \ker \partial_2$ and so we begin by analyzing elements of the kernel of ∂_2. Given an arbitrary element in the kernel, we have

$$\begin{aligned} \partial(m\sigma_1 + n\sigma_2) &= m(b + a - c) + n(c - b - a) \\ &= a(m-n) + b(m-n) + c(n-m) = 0 \end{aligned}$$

which implies that $m = n$. Thus $Z_2(\mathbb{T}^2) = \langle \sigma_1 + \sigma_2 \rangle \cong \mathbb{Z}$ and so $H_2(\mathbb{T}^2) = \mathbb{Z}$. For our next instance, notice that since all vertices of $|\mathbb{T}^2|$ are identified we have $\partial a = \partial b = \partial c = 0$ and so

$$\partial(ma + nb + kc) = 0$$

independent of any choice in m, n, k. We conclude that $Z_1(\mathbb{T}^2)$ is generated by three elements, for which the most obvious choice would be simply $\{a, b, c\}$. However, let us choose the elements $\{a, c, c-b-a\}$, the choice of which will be evident in a moment. In order to compute the first homology group, we need to consider the image of ∂_2, but we notice $B_1(\mathbb{T}^2) = \operatorname{im}\partial_2 = \langle a+b-c\rangle \cong \mathbb{Z}$ and so in fact this third generator vanishes, so $H_1(\mathbb{T}^2) = \mathbb{Z}^2$. Note that it was not necessary to choose our basis in this way as the homology group would have been the quotient on additive groups and hence would have resulted in the same result all the same. Finally, the torus is obviously path connected and so $H_0(\mathbb{T}^2) = \mathbb{Z}$.

Example 2.1.25 (Homology of the Klein Bottle)**.** Again, consider the following simplicial complex

$$0 \to \underset{\mathbb{Z}^2}{C_2(\mathbb{K}^2)} \xrightarrow{\partial_2} \underset{\mathbb{Z}^3}{C_1(\mathbb{K}^2)} \xrightarrow{\partial_1} \underset{\mathbb{Z}}{C_0(\mathbb{K}^2)} \to 0.$$

Note that

$$\partial(m\sigma_1 + n\sigma_2) = (m-n)c + (-m-n)b + (n-m)a = 0$$

for which the first and second equations imply the $m = n = 0$ so that $Z_2(\mathbb{K}) = \{0\}$ and $H_2(\mathbb{K}) = 0$. Note again that $\partial_1 = 0$ and choose generators a, b and $c - b - a$. Now $\partial\sigma_1 = c - b - a$ so this generator disappears. On the other hand, $\partial(\sigma_1 + \sigma_2) = -2b$ and so $B_1(\mathbb{K}) = \mathbb{Z} \oplus 2\mathbb{Z}$ and we get

$$H_1(\mathbb{K}) = \frac{\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}\oplus 2\mathbb{Z}} \cong \mathbb{Z}\oplus\mathbb{Z}/2.$$

Example 2.1.26 (Homology of $\mathbb{RP}^2$)**.**

$$\partial(m\sigma_1 + n\sigma_2) = m(c-b-a) + n(-b-a-c) = 0 \tag{2.12}$$

and so $m = n = 0$. Note that we have cycles given by c and $a + b$. On the other hand, the boundaries are $\partial\sigma_1 = c - b - a = c - (b + a)$ and so c and $(b + a)$ are homologous. Furthermore, $\partial(\sigma_1 + \sigma_2) = -2(a + b)$ and so $H_1(\mathbb{RP}^2) = Z_1(\mathbb{RP}^2)/B_1(\mathbb{RP}^2) = \mathbb{Z}_2$.

2.2 Singular Homology

Let Δ_p be the standard p-simplex as defined by Hatcher.

Definition 2.2.1. A *singular p-simplex* is a continuous map $T : \Delta_p \to X$ where X is a topological space.

Note that this is less restrictive than simplicial homology, wherein we equivalently required that $T : \Delta_p \to X$ was more than just continuous, but was rather an embedding of the complex. Many of our concepts from simplicial homology will translate to singular homology. Below, we define the equivalent of the $C_n(X)$:

Definition 2.2.2. A *singular chain group* $S_p(X)$ is the free abelain group generated by the singular p-simplices of X, wherein $s_p \in S_p(X)$ is given by $s_p = \sum n_\alpha T_\alpha$ for $n_\alpha \in \mathbb{Z}$ and T_α a singular p-simplex.

For the sake of simplicity, let $\mathbb{R}^\infty$ denote the set of infinite sequences of $\mathbb{R}$ and in particular set

$$\epsilon_i = (\underbrace{0, \cdots, 1}_{i-\text{times}}, 0 \cdots).$$

The standard simplex Δ_p is simply the simplex consisting of $\{\epsilon_i\}_{i=1}^p$. Given points $a_0, \ldots, a_p$ in $\mathbb{R}^\infty$ there exists an affine map $C : \Delta_p \to \mathbb{R}^\infty$ that maps $\epsilon_i \mapsto a_i$, where

$$C(x_1, \ldots, x_p, 0, \ldots, 0) = a_0 + \sum_{i=1}^{p} x_i(a_i - a_i)$$

in which case $C(\epsilon_0, \ldots, \epsilon_p)$ is the inclusion map, so $C|_{\Delta_p} : \Delta_p \to \Delta_p$.

Definition 2.2.3. If $T : \Delta_p \to X$ then notice that $T \circ C : \Delta_p \to X$ is a well-defined may, in which case we define the *boundary map*

$$\partial T = \sum_{i=0}^{p} (-1)^i T \circ C(\epsilon_0, \ldots, \hat{\epsilon}_i, \cdots, \epsilon_p).$$

Definition 2.2.4. If $f : X \to Y$ is a continuous map we define the *chain map* as $f_\# : S_p(X) \to S_p(Y)$ by $f_\#(T) = f \circ T$.

Note that this definition makes sense, since if $T : \Delta_p \to X$ then $f \circ T : \Delta_p \to X \to Y$ and so $f \circ T \in S_p(Y)$.

Theorem 2.2.5. *1. If $f : X \to Y$ is a continuous map then $f_\#$ commutes with the the boundary operator ∂.*

2. $\partial^2 = 0$.

Proof. 1. This is simply a matter of computation. Indeed, notice that we have

$$f_\#(\partial(T)) = \sum_{i=0}^{p} (-1)^i f \circ (T \circ C(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p))$$
$$\partial f_\#(T) = \sum_{i=0}^{p} (-1)^i (f \circ T) \circ C(\epsilon_0, \cdots, \hat{\epsilon}_i, \cdots, \epsilon_p)$$

and as function composition is associative, we are done.

2. Note that the proof of this result is identical to that given for simplicial complexes with the new boundary map. Hence we omit it here.

□

Definition 2.2.6. The family of groups and homomorphisms $\partial_p : S_p(X) \to S_{p-1}(X)$ is called the singular chain complex.

Proposition 2.2.7. *If X is nonempty and path-connected, then $H_0(X) \cong \mathbb{Z}$.*

Proof. Recall that $H_0(X) \cong \ker \partial_0 / \operatorname{im} \partial_1$. Since ∂_0 is the zero map, $\ker \partial_0 = C_0(X)$ and so $H_0(X) = C_0(X)/\operatorname{im} \partial_1$. Define the map $\epsilon : C_0(X) \to \mathbb{Z}$ by $\epsilon\left(\sum_i n_i \sigma_i\right) = \sum_i n_i$. This map is clearly surjective, and hence it suffices to show that $\ker \epsilon = \operatorname{im} \partial_1$ in which case the first isomorphism theorem for groups will imply that $\mathbb{Z} \cong C_0(X)/\operatorname{im} \partial_1$ and we will be done.

Now $\operatorname{im} \partial_1 \subseteq \ker \epsilon$ since for any singular 1-simplex we have $\epsilon \partial_1(T) = \epsilon(v_1 - v_0) = 1 - 1 = 0$. On the other hand, suppose that $\epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = 0$ so that $\sum_\alpha n_\alpha = 0$. Let $\tau_i : I \to X$ be a path from a base point x_0 to $\sigma_i(v_0)$ and σ_0 be the singular 0-simplex whose image is x_0. Then $\tau_i : [v_0, v_1] \to X$ and $\partial \tau_i = \sigma_i - \sigma_0$ so $\partial(\sum_\alpha n_\alpha \tau_\alpha) = \sum_\alpha n_\alpha \sigma_\alpha - \sum_\alpha n_\alpha \sigma_0 = \sum_\alpha n_\alpha \sigma_\alpha$ since $\sum_\alpha n_\alpha = 0$. □

Definition 2.2.8. Given a singular chain complex

$$\to S_n(X) \to S_{n-1}(X) \to S_{n-2}(X) \to \cdots \to S_0(X) \to 0$$

we define the n^{th} *singular homology group* to be $H_p(X) = \frac{Z_p(X)}{B_p(X)}$ where $Z_p(X) = \ker \partial_p$ and $B_p(X) = \operatorname{im} \partial_{p+1}$.

Given a singular chain complex, we define the *augmentation map* $\epsilon : S_0(X) \to \mathbb{Z}$ to be precisely the same ϵ defined in the proof of Proposition 2.2.7. Notice that for any singular 0-simplex $\epsilon(T) = 1$ and that in this case $\epsilon(\partial T) = 0$.

Definition 2.2.9. The homology groups defined by the chain complex $\{S(X), \epsilon\}$ is called the set of *reduced singular homology groups*. We often denote these groups by $\tilde{H}_p(X)$.

Note that while we denote all the homology groups as $\tilde{H}_p(X)$ as compared to $H_p(X)$, the only difference occurs in the 0^{th} homology group, and in this case our sequence looks like

$$\to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Hence $\tilde{H}_0(X) = \ker \epsilon / \operatorname{im} \partial_1$.

Theorem 2.2.10. *Let X be a topological space. Then $H_0(X)$ is a free abelian group. If $\{X_\alpha\}$ is a collection of path components of X and T_α is a singular 0-simplex whose image is in X_α for all α then the homology classes of the chains T_α form a basis for $H_0(X)$. Furthermore, $\tilde{H}_0(X)$ is free abelian, trivial when X is path connected, and if X is not path connected then we may choose α_0 such that $\tilde{H}_0(X)$ is generated by the homology classes of $T_\alpha - T_{\alpha_0}$ where $\alpha \neq \alpha_0$.*

Proof. Let $x_\alpha = T_\alpha(\Delta_0)$ and let $T : \Delta_0 \to X$ be any singular 0-simplex. Then $T(\Delta_0)$ must lie in X_α for some α. Suppose that $f : [0,1] \to X$ is a path from $T(\Delta_0)$ to x_α then f is a singular 1-simplex. In this case $T - T_\alpha = \partial f$ so that T is homologous to T_α. Hence any singular 0 chain is homologous to a linear combination of T_α, say $T = \sum n_\alpha T_\alpha$.

We claim that $\sum n_\alpha T_\alpha$ is not the boundary of any simplex. For the sake of contradiction, suppose that $\sum n_\alpha T_\alpha$ does bound a simplex, so that we may write $\sum n_\alpha T_\alpha = \partial d$ for some singular simplex d. Write $d = \sum_\alpha d_\alpha$ where the image of each d_α lies in X_α, so that $\partial d_\alpha = n_\alpha T_\alpha$. Now by applying the ϵ operator to either side, we find that

$$0 = \epsilon(\partial d_\alpha) = \epsilon(n_\alpha T_\alpha) = n_\alpha \epsilon(T_\alpha) = n_\alpha$$

hence $n_\alpha = 0$. This must hold for all such α, which is a contradiction.

In the case of reduced singular homology, we can proceed precisely as above. We note that in this case however, we cannot claim that every singular chain of 0-simplices is a cycle. Applying ϵ to our linear combination of T_α we then get

$$\epsilon\left(\sum_\alpha n_\alpha T_\alpha\right) = 0, \qquad \sum n_\alpha = 0.$$

If X is path connected, then $\epsilon(nT) = n = 0$ so that $\tilde{H}_0(X) = \{0\}$. On the other hand, if X is not path connected then fix some α_0 and write $n_{\alpha_0} = -\sum_{\alpha \neq \alpha_0} n_\alpha$. This implies that $\{T_\alpha - T_{\alpha_0}\}$ generates $\tilde{H}_0(X)$ as required. □

Definition 2.2.11. If the reduced homology of X vanishes in all dimensions, then we say that X is *acyclic*.

If $f : X \to Y$ and $f_\# : S_p(X) \to S_p(Y)$ then since $f_\#$ commutes with ∂ we know that it induces a group homomorphism $f_* : H_p(X) \to H_p(Y)$. To see this, let α be a cycle so that $\partial\alpha = 0$. Then $\delta f_\#(\alpha) = f_\#(\delta\alpha) = f_\#(0) = 0$ so cycles are preserved. Precisely the same argument shows that boundaries are preserved.

Definition 2.2.12. A set X in $\mathbb{R}^\infty$ is said to be *star-convex* relative to the point ω if for each $x \in X \setminus \{\omega\}$ we have that the line segment from x to ω lies entirely within X.

Definition 2.2.13. Let K be a simplicial complex in $\mathbb{R}^\infty$ and ω a point in $\mathbb{R}^\infty$ such that each ray emanating from ω intersects $|K|$ in one point. Then we define $\omega \star K$, the cone on K with vertex ω, as the collection of all simplices of the form $\omega, a_0, \ldots, a_p$ where $a_0, \ldots, a_p$ is a simplex.

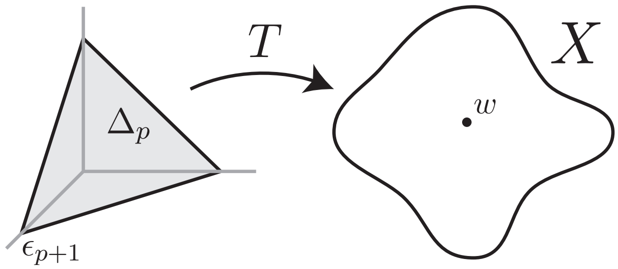

Figure 2.6: Defining the bracket operator to extend simplicial maps.

Suppose that X is star-convex relative to a point ω. Define a bracket operation on singular chains of X as follows: Let $T : \Delta_p \to X$ be a singular p-simplex and $\omega \in \mathbb{R}^\infty$. Define our bracket operation as $[T, \omega] : \Delta_{p+1} \to X$ where the line segment from ω to ϵ_{p+1} is mapped linearly onto the line segment from $T(x)$ to ω. We may extend this to arbitrary p-chains.

Note that $[T, \omega]\Big|_{\Delta_p} = T$. Furthermore, if T is a linear simplex $C(a_0, \ldots, a_p)$ then $[T, \omega] = C(a_0, \ldots, a_p, \omega)$.

Lemma 2.2.14. *Let X be star-convex with respect to ω and let C be a singular p-chain of X, then*

$$\partial[C, \omega] = \begin{cases} [\partial C, \omega] + (-1)^{p+1} C & \text{if } p > 0 \\ \epsilon(C) T\omega - C & \text{if } p = 0 \end{cases}. \tag{2.13}$$

Proof. Let T be a singular 0-simplex, so that $[T, \omega]$ is a singular 1-simplex that takes Δ_1 linearly onto the line segment that connects ω and $T(\Delta_0)$. Now $\partial[T, \omega] = T_\omega - T$. Suppose that C is a 0-cycle so that $C = \sum n_\alpha T_\alpha$ in which case

$$\begin{aligned} \partial[C, \omega] &= \sum n_\alpha (T_\omega - T_\alpha) \\ &= \sum n_\alpha T_\omega - \sum n_\alpha T_\alpha \\ &= T_\omega \sum n_\alpha - C \\ &= \epsilon(C) T_\omega - C \end{aligned}$$

Suppose now that $p > 0$. It is sufficient to show that the statement is true for a singular p-simplex T

$$\partial[T, \omega] = \sum_{i=0}^{p+1} (-1)^i [T, \omega] \circ C_i, \qquad C_i = C(\epsilon_0, \cdots \hat{\epsilon}_i, \cdots, \epsilon_{p+1}). \tag{2.14}$$

Indeed, by simply calculating we have

$$\sum_{i=0}^{p}(-1)^i[T,\omega]\circ C_i + (-1)^{p+1}T[T,\omega]\circ C_i \qquad = \left[T\circ\left(\frac{\rho_i}{\Delta_p - 1}\right),\omega\right]$$

□

Corollary 2.2.15. *Let X be a subspace of $\mathbb{R}^\infty$ that is star convex relative to ω. Then X is acyclic.*

Proof. We want to show that $\tilde{H}_p(X)$ is zero for all p. If X is path connected then $\tilde{H}_0(X) = \{0\}$ and so this case is easily taken care of. On the other hand, for $p > 0$ let z be a singular p-cycle. Then $[z,\omega]$ is a $p+1$ cycle and its image under ∂ is given by $\partial[z,\omega] = [\partial z,\omega] + (-1)^{p+1}z = (-1)^{p+1}z$ since z is a cycle and so $\partial z = 0$. Hence $\operatorname{im}\partial_{p+1} = \ker\partial_p$ and we conclude that $\tilde{H}_p(X) = 0$. □

Note for example that the n-simplex is acyclic since it is convex and hence star convex about any point in its interior.

Theorem 2.2.16. *Let σ be an n-simplex. Then the simplicial complex K_σ that consists of σ and its faces is acyclic. If Σ^{n-1} denotes the subcomplex whose underlying space is the boundary of σ then $\tilde{H}_{n-1}(\Sigma^{n-1}) = \mathbb{Z}$ and is generated by $\partial\sigma$. In particular $\tilde{H}_i(\Sigma^{n-1}) = \{0\}$ for all $i \neq n-1$.*

Proof. Note that K_σ is a cone and hence is star-convex so acyclic. Now the chain groups of K_σ and Σ^{n-1} are equal save in dimension n, yielding the following chain complex

$$\begin{tikzcd}
C_n(K_\sigma) \arrow[r, "\partial_n"] & C_{n-1}(K_\sigma) \arrow[r, "\partial_{n-1}"] \arrow[d, equal] & C_{n-2}(K_\sigma) \arrow[r, "\partial_{n-2}"] \arrow[d, equal] & \cdots \arrow[r] & C_0(K_\sigma) \arrow[r, "\epsilon"] \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\
0 \arrow[r] & C_{n=1}(\Sigma^{n-1}) \arrow[r, "\partial'_{n-1}"] & C_{n-2}(\Sigma^{n-1}) \arrow[r, "\partial'_{n-2}"] & \cdots \arrow[r] & C_0(\Sigma^{n-1}) \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

Hence for $i \neq n-1$ it follows that $\tilde{H}_i(\Sigma^{n-1}) = \tilde{H}_i(K_\sigma) = \{0\}$ by acyclicity of K_σ. For the $n-1$ case we have

$$\begin{aligned} H_{n-1}(\Sigma^{n-1}) &= Z_{n-1}(\Sigma^{n-1}) && \text{since there are no } n-1 \text{ boundaries} \\ &= \ker\partial_{n-1} = \operatorname{im}\partial_n \end{aligned}$$

where in the last line we have used the fact that $\tilde{H}_{n-1}(K_\sigma) = 0$. But $\operatorname{im}\partial_n = C_n(K_\sigma) \cong \mathbb{Z}$ and so we conclude that $\tilde{H}_{n-1}(\Sigma^{n-1}) \cong \mathbb{Z}$ as required. □

2.3 Relative Homology

Let K be a simplicial complex and let K_0 be a subcomplex. Recall that $C_p(K)$ is a group of p-chains and $C_p(K_0)$ is a group of p chains on K_0, hence $C_p(K_0)$ is a subgroup of $C_p(K)$. We shall denote the group $C_p(K)/C_p(K_0) = C_p(K, K_0)$ which is called the *group of relative chains*. Note that it is generated by simplices that are in K but not in K_0. Namely, the set $\{\sigma_i\} = \sigma_i + C_p(K_0)$ where σ_i is a simplex in K but not in K_0. We still have the boundary maps $\partial : C_p(K) \to C_{p-1}(K)$ and $\partial : C_p(K_0) \to C_{p-1}(K_0)$ which induces a homomorphism $\partial : C_p(K, K_0) \to C_{p-1}(K, K_0)$. We still have that $\partial^2 = 0$ and so we can consider the following chain

$$\cdots \to C_p(K, K_0) \xrightarrow{\partial} C_{p-1}(K, K_0) \xrightarrow{\partial} C_{p-2}(K, K_0) \to \cdots \to C_0(K, K_0) \to 0.$$

Defining $Z_p(K, K_0) = \ker(\partial : C_p(K, K_0) \to C_{p-1}(K, K_0))$ the group of relative cycles and $B_p(K, K_0) = \operatorname{im}(\partial : C_{p+1}(K, K_0) \to C_p(K, K_0))$ the group of relative boundaries. We can now take the quotient $H_p(K, K_0) = \frac{Z_p(K,K_0)}{B_p(K,K_0)}$ the n^{th} relative homology group.

A p-chain is a relative p-cycle if its boundary is in K_0; that is, relative cycles consist of all n-chains $\alpha \in C_n(X)$ such that $\partial_\alpha \in C_{n-1}(K_0)$. A p-chain α is a relative p-boundary if there exists a $(p+1)$-chain β_{p+1} such that $\alpha - \partial\beta$ is carried by K_0; that is, $\alpha = \partial\beta + \gamma$ for some $\gamma \in C_n(K_0)$.

Example 2.3.1. If K_σ where σ is an n-simplex, so that it consists of σ and all of its faces. Let K_0 be a subcomplex that consists of all the proper faces of K. Note that if $i \neq n$ the $H_i(K, K_0) = \{0\}$, so the only interesting case is the top level case. Consider the chain

$$0 \to C_n(K, K_0) \to C_{n-1}(K, K_0)$$

in which case $H_n(K, K_0) \cong Z_n(K, K_0) = \mathbb{Z}$.

Example 2.3.2. If K is a complex and K_0 is a subcomplex consisting of a single vertex of K, then $H_0(K, v)$ is a free abelian group generated by every vertex of K which is not the vertex K_0. Hence $H_0(K, K_0) \cong \tilde{H}_0(K)$. It can be shown that $H_p(K, v) \cong H_p(K)$ for all $p > 0$.

Example 2.3.3. Let K_0 be the simplex considered in Example 2.1.23 pictured in Figure 2.4 (left) such that $|K_0| = \partial|K|$. In this case $C_0(K, K_0) = \mathbb{Z}$ and the 0^{th} relative homology group can be immediately seen to be $H_0(K, K_0) = \{0\}$. To deduce the structure of $H_1(K, K_0)$, recall that if C_1 is a one-chain of K then it is homologous to a chain that is carried by the subcomplex in Figure 2.4 (right). Suppose that c_1 is also a cycle so that it is homologous to a chain that is carried by K_0. Then $\partial(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) = -(e_1 + e_2 + e_3 + e_4) \in K_0$. Thus any relative 2-cycle is of the form $m\gamma$ where $\gamma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$.

Example 2.3.4. Let $|K_0| = \partial|K|$. How $H_0(K, K_0) = \{0\}$, $H_2(K, K_0) = \mathbb{Z}$ which is generated by the sum of the two-cycles between the annulus, and $H_1(K, K_0) = \mathbb{Z}$.

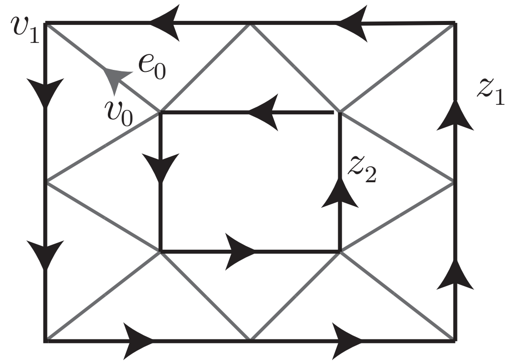

Figure 2.7: A complex whose boundary is an annular region: the union of the interior and outer edges.

2.4 Relative Singular Chains

If X is a space and A is a subspace of X there exists a natural inclusion $S_p(A) \to S_p(X)$ and we define $S_p(X,A) = S_p(X)/S_p(A)$ called the *relative group of singular p-chains*. Define the boundary map $\partial : S_p(X,A) \to S_{p-1}(X,A)$ and the relative singular homology group $H_p(X,A) = Z_p(X,A)/B_p(X,A)$.

$T + S_p(A)$ where T is a singular p-simplex that is not in A.

Theorem 2.4.1. *Any n-simplex in a Δ-complex decomposition of X may be viewed as a singular n-simplex $\Delta_n(X,A) \to S_n(X,A)$. The induced homomorphisms $H_n^{\Delta}(X,A) \to H_n(X,A)$ is an isomorphism. We can take $A = \emptyset$.*

If $f : (X,A) \to (Y,B)$ is a continuous map then $f_\# : S_p(X,A) \to S_p(Y,B)$; that is $f_\#$ takes chains on A to chains on B. Note that $f_\#$ commutes with ∂, so $f_* : H_p(X,A) \to H_p(Y,B)$.

Theorem 2.4.2. *If $\iota : (X,A) \to (X,A)$ is the identity, then ι_* is the identity. If $h : (X,A) \to (Y,B)$ and $k : (Y,B) \to (Z,C)$ are two continuous maps then $(k \circ h)_* = k_* \circ h_*$.*

Theorem 2.4.3. *If $f, g : (X,A) \to (Y,B)$ are homotopic then $f_* = g_*$. Furthermore, if $f : X \to Y$ is a homotopy equivalence then f_* is an isomorphism.*

Example 2.4.4. If X is contractible then X is acyclic.

Example 2.4.5. The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$. Then $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $\mathbb{R}^m \setminus \{0\}$, both of which have deformation retractions to S^n and S^m respectively. However, $H_j(S^n) = 0$ for $n \neq j$ and $H_j(S^m) = 0$ for $m \neq j$. If these were homeomorphic, they would have to agree, and this is not the case.

Example 2.4.6. Consider the Mobius band M. Note that there is a deformation retraction of this to a circle, so $H_0(M) = \mathbb{Z} = H_1(M)$.

Example 2.4.7. Consider the solid torus $X = B^2 \times S^1$. There is a deformation retraction of the solid torus to S^1. Hence $H_1(B^2 \times S^1) = H_0(B^2 \times S^1) = \mathbb{Z}$.

Example 2.4.8. Consider the space $X = \mathbb{T}^2 \setminus \{p\}$. Recall that the fundamental group of X is the same as that of the figure 8, hence $H_0(X) = \mathbb{Z}$ and $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$.

Example 2.4.9. Consider the space $X = \mathbb{T}^2 \setminus \{p, q\}$. We can deformation retract this to a wedge of three circles. Hence $H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(X) = \mathbb{Z}$.

Example 2.4.10. Consider the space $X = \mathbb{R}^3 \setminus \{(0, 0, z) : z \in \mathbb{R}\}$. Which we can deformation retract to S^1.

Example 2.4.11. Consider $X = \mathbb{R}^3 \setminus C$ where $C = \{x^2 + y^2 = 1, z = 0\}$. By thickening the circle, we can remove a torus instead (Heegaard decomposition). Let V be a solid torus, W be a solid torus, and consider $V \cup_f W = V \sqcup W/\sim$ where $f : \partial V \to \partial W$ which reverses the orientation. Hence $\mathbb{R}^3 \setminus C$ is equivalent to a punctured torus, which is equivalent to $S^2 \wedge S^1$, so $H_0(X) = H_1(X) = H_2(X) = \mathbb{Z}$.

Example 2.4.12. Consider $X = \mathbb{R}^3 \setminus C \cup z$, which is equivalent to a torus. Hence $H_2(X) = \mathbb{Z}, H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(X) = \mathbb{Z}$.

Example 2.4.13. Let X be two unlinked circles in $\mathbb{R}^3$. Since removing a circle is equivalent to a punctured torus, we do this for the first of the unlinked circles. However, we must remove another circle, resulting in the punctured torus with a circle removed. Hence this can be retracted to a ball with a circle removed, wedged with a circle, wedged with a sphere. However, the ball with the circle removed can be retracted to the sphere containing a diameter. Hence this is a wedge of two spheres and two circles, so $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}, H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(X) = \mathbb{Z}$.

Theorem 2.4.14 (Excision Theorem)**.** *Let K be a complex and K_0 be a subcomplex. Let U be an open set contained in $|K_0|$ such that $|K| - U$ is the polytope of a subcomplex L of K. Let L_0 be a subcomplex whose polytope is $|K_0| - U$. Then the inclusion induces an isomorphism between $H_p(L, L_0)$ and $H_p(K, K_0)$.*

Proof.

$$C_p(L) \xrightarrow{\text{inclusion}} C_p(K) \xrightarrow{\text{projection}} C_p(K)/C_p(K_0).$$

If ϕ is the composition of these maps, then $\ker \phi = C_p(L_0)$. Hence $C_p(L)/C_p(L_0) \cong C_p(K)/C_p(K_0)$ implies that $H_p(L, L_0) \cong H_p(K, K_0)$. ☐

2.5 Degree of a Map and Applications

Definition 2.5.1. Let $n \geq 1$, $f : S^n \to S^n$ be a continuous map, and recall that $H_n(S^n) = \mathbb{Z}$. Let $f_* : H_n(S^n) \to H_n(S^n)$, and α be one of the generators of $H_n(S^n)$, so that $f_*(\alpha) = d\alpha$, in which case the *degree* of f is d.

Proposition 2.5.2. *The degree has the following properties*

1. *$f \simeq g$ implies that* $\deg f = \deg g$.
2. *If f extends to a continuous map $h : B^{n+1} \to S^n$ then* $\deg f = 0$.
3. $\deg \mathrm{id} = 1$
4. $\deg(f \circ g) = \deg f \cdot \deg g$

Theorem 2.5.3. *There is no retraction $r : B^n \to S^{n-1}$.*

Proof. By the proposition above, and map which extends to a continuous map $h : B^n \to S^{n-1}$ must have degree zero. By the same hand, any such retraction would be homotopic to the identity and thus have degree one. This is a contradiction, so no retraction can exist. ☐

Theorem 2.5.4 (General Brouwer Fixed Point Theorem)**.** *Every continuous map $\phi : B^n \to B^n$ has a fixed point.*

Proof. Suppose there is no fixed point, so that we can define $h : B^n \to S^{n-1}$ via $h(x) = \frac{x-\phi(x)}{\|x-\phi(x)\|}$. This is well defined since $x - \phi(x) \neq 0$ by the fact that there are not fixed points. Note then that $h|_{S^{n-1}}$ must have degree zero since it extends to a continuous map. On the other hand, f is homotopic to the identity via $H(u,t) = \frac{u-t\phi(u)}{\|u-t\phi(u)\|}$ and since the denominator is non-zero $\deg f = 1$, a contradiction. □

Recall that antipodal map $a : S^n \to S^n$ taking $a(x) = -x$. What is the degree of a? Note that this depends on n. Define the map $\rho(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$ so $\deg \rho = -1$. Let h be a homeomorphism that exchanges x_i and x_{n+1} so that $S_i = h^{-1} \circ \rho \circ h$ in which case $\deg S_i = \deg h^{-1} \cdot \deg \rho \cdot \deg h = \deg \rho = -1$. The antipodal map can be written as $a = S_1 \circ \cdots \circ S_{n+1}$ so $\deg a = (-1)^{n+1}$.

Theorem 2.5.5. *If $h : S^n \to S^n$ has degree different from $(-1)^{n+1}$ then h has a fixed point.*

Proof. Suppose for the sake of contradiction that h contains no fixed point. We then have that h is homotopic to $a(x)$ via a homotopy $H : S^n \times I \to S^n$ defined as

$$H(x,t) = \frac{(1-t)h(x) + t(-x)}{\|(1-t)h(x) + t(-x)\|}.$$

Now notice that the denominator does not vanish, since if it did then $(1-t)h(x) = tx$ for some x and t. Norming both sides reveals that $1-t = t = \frac{1}{2}$ which would imply that $h(x) = x$ which cannot happen. Hence this homotopy is well defined, but this too is a contradiction since otherwise the degree of h and of the antipodal map would be the same. □

Corollary 2.5.6. *If $h : S^n \to S^n$ has degree not 1 then h takes some point x to its antipode $-x$.*

Proof. Consider $a \circ h$. Since $\deg h \neq 1$ then $\deg a \circ h \neq (-1)^{n+1}$ so $a \circ h$ has a fixed point. Thus there exists $x \in S^n$ such that $a \circ h(x) = x$ or $h(x) = -x$. □

Corollary 2.5.7. *S^n has a non-zero tangent vector field if and only if n is odd.*

Proof. Suppose that n is odd, so that $S^{2n-1} = \{(x_1, \ldots, x_{xk}) : x_1^2 + \cdots x_{2k}^2 = 1\}$. Let $v(x) = (-x_2, x_1, -x_4, x_3, \cdots, -x_{2k}, x_{2k-1})$ which is clearly non-zero. On the other hand, suppose that $v(x)$ is a non-zero tangent vector field on S^n, and define $h = \frac{v}{\|v\|} : S^n \to S^n$. Note that this map must be perpendicular to x for all x and so we cannot have $h(x) = \pm x$. Thus this map has no fixed points which is only possible if if $\deg h = (-1)^{n+1} = 1$ so n must be odd. □

2.6 Lefschetz Fixed Point Theorem

Let G be a free abelian group with basis $e_1, \dots, e_n$. Let $\phi : G \to G$ be a homomorphism. As $\mathbb{Z}$-modules, ϕ is a matrix so we can define the trace of ϕ relative to this given basis. If K is a finite simplicial complex and $\phi : C_p(K) \to C_p(K)$ then the trace map on ϕ is well defined, and is denoted $\mathrm{Tr}(\phi, C_p(K))$. Note that while $H_p(K)$ need not be free abelian, if we take $T_p(K)$ to be its torsion subgroup then $H_p(K)/T_p(K)$ is torsion free and hence free abelian. Furthermore, ϕ induces a homomorphism on chains by $\phi_* : H_p(K) \to H_p(k)$ and we denote the trace of this map as $\mathrm{Tr}(\phi_*, H_p(K)/T_p(K))$.

Theorem 2.6.1 (Hopf Trace Theorem)**.** *Let K be a finite simplicial complex and $\phi : C_p(K) \to C_p(K)$ be a chain map. Then*

$$\sum_p (-1)^p \mathrm{Tr}(\phi, C_p(K)) = \sum_p (-1)^p \mathrm{Tr}(\phi_*, H_p(K)/T_p(K)).$$

The proof of this theorem is quite long and technical, so we refer to Munkres (page 122) for the necessary information.

Definition 2.6.2. Let K be a finite simplicial complex. Define the *Euler characteristic* $\chi(K) = \sum_p (-1)^p \operatorname{rank} C_p(K)$. The p *Betti number* is $\beta_p = \operatorname{rank} H_p(K)/T_p(K)$.

The Euler number as defined above generalizes the Euler characteristic used in graph theory, given by $\chi(K) = V - E + F$ where V is the number of vertices, E the edges, and F the faces. Notice that in our case, the rank of $C_n(X)$ is precisely the number of n-cells of our simplex and so for surfaces the Euler characteristics agree entirely.

Theorem 2.6.3. *The Euler characteristic is a topological invariant and is given by*

$$\chi(K) = \sum_p (-1)^p \beta_p$$

where β_p are the Betti numbers.

Proof. Note that the first claim will follow once the second has been shown since the homology group is clearly topologically invariant. To see this, let $\phi = \text{id}$ be the identity chain map, in which case $\text{Tr}(\phi, C_p(K)) = \text{rank}\, C_p$. Similarly, because ϕ_* is the identity map we must have $\text{Tr}(\phi_*, H_p(K)/T_p(K)) = \text{rank}\, H_p(K)/T_p(K)$. The result then follows from the Hopf trace theorem. $\square$

Example 2.6.4. Note that S^2 has one 2-cell, no 1-cells, and one 0-cell so that $\chi(\S^2) = 1 - 0 + 1 = 2$. For the torus $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$.

2.6.1 Triangulation

A topic somewhat ignored to this point is the number of triangles required to completely specify a simplicial complex. A key characterization of triangulation is that $3F = 2E$, which can be seen as follows: Note that every face (which is a triangle) consists of precisely three edges. By summing up over all edges of a simplex, we have counted each edge twice, hence $3F = 2E$ as required.

Now using the Euler characteristic, we know that $\chi(X) = V - E + F$ so in particular, $3\chi(X) = 3V - 3E + 3F = 3V - 3E + 2E = 3V - E$ so that $E = 3(V - \chi(X))$. Finally, it can be shown using combinatorial graph theory that the number of corresponding vertices must be bounded below as $v \geq \frac{1}{2}\left(\sqrt{49 - 24\chi(X)} + 7\right)$ which we can then use to trace backwards and discover the minimal number of faces.

Example 2.6.5. Let us find the smallest number of simplices required to triangulate a torus. That is, to find a simplicial complex whose underlying space is the torus. The Euler characteristic is $V - E + F = 0$ and $3F = 2E$ thus $3V - 3E + 3F = 3V - 3E + 2E = 0$ implies that $3V = E$. Let E_i be the number of edges that come out of v_i. Hence $\sum E_i = 2E = 6V$. Thus $\frac{1}{V}\sum E_i = 6$. Any triangulation must contain this vertex plus six others emanating from it and hence must contain 7 vertices. This corresponds to at least 21 edges which corresponds to at least 14 faces.

2.6.2 The Fixed Point Theorem

Definition 2.6.6. Let K be a finite simplicial complex and $h : |K| \to |K|$ be continuous. Define the *Lefschetz number* to be the quantity

$$\Lambda(h) = \sum (-1)^p \mathrm{Tr}\left(h_*, \frac{H_p(K)}{T_p(K)}\right). \tag{2.15}$$

Theorem 2.6.7 (Lefschetz Fixed Point Theorem)**.** *If K is a finite simplicial complex and $h : |K| \to |K|$ is a continuous map with non-zero Lefschetz number then h has a fixed point.*

This proof is lengthy and technical and its details are not particularly enlightening. Consequently, we refer the reader to Munkres (page 125) for the proof.

Theorem 2.6.8. *If K is a finite connected simplicial complex that is acyclic, and $h : |K| \to |K|$ a continuous map, then h has a fixed point.*

Proof. By the Lefschetz Fixed Point Theorem it is sufficient to show that h has a non-zero Lefschetz number. However, if v is any vertex of K the since continuous maps preserve points and augmentation, we must have that $h\{v_0\} = \{v_0\}$ so that in particular, $h_* : H_0(K) \to H_0(K)$ is the identity map on $H_0(K) = \mathbb{Z}$. Since $\mathbb{Z}$ is certainly torsion free, we may then compute the Lefschetz constant as

$$\Lambda(h) = \mathrm{Tr}\left(h_*, \frac{H_0(K)}{T_0(K)}\right) = \mathrm{Tr}(h_*, H_0(K)) = 1$$

which is clearly non-zero, so h has a fixed point. □

Theorem 2.6.9. *For any $n > 1$ the antipodal map $a : S^n \to S^n$ has degree $(-1)^{n+1}$.*

Proof. By this point it is well known that the n^{th} homology group of S^n is simply $\mathbb{Z}$. For any degree d map $h : S^n \to S^n$, the matrix representation of h_* is then one-dimensional with a single entry whose value is d. Hence the Lefschetz value of h is $\Lambda(h) = 1 + (-1)^n d$. In particular, let $a : S^n \to S^n$ be the antipodal map and let the degree of a be d. Since a has no fixed points it is necessary that $\Lambda(a) = 0 = 1 + (-1)^n d$ for which we can solve to find that $d = (-1)^{n+1}$ as required. $\square$

Example 2.6.10. Every continuous map $h : \mathbb{RP}^2 \to \mathbb{RP}^2$ has a fixed point. To see this, recall that the homology groups for $\mathbb{RP}^2$ $H_0(\mathbb{RP}^2) = \mathbb{Z}, H_1(\mathbb{RP}^2) = \mathbb{Z}_2$ and $H_2(\mathbb{RP}^2) = \{0\}$ whose only non-trivial torsion free group is given by $\frac{H_0(\mathbb{RP}^2)}{T_0(\mathbb{RP}^2)} = \mathbb{Z}$. Thus the Lefschetz number is determined entirely by the 0^{th} homology group and is calculated as

$$\Lambda(h) = \mathrm{Tr}(h_*, H_0(\mathbb{RP}^2)) = 1 \neq 0,$$

hence h has a fixed point.

2.7 Chain Maps and Chain Complexes

Consider $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \left\{C'_p, \partial'_p\right\}$ chain complexes, which we recall are just families of free abelian groups with ∂ homomorphisms

$$\cdots \to C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \to \cdots$$

A chain map $\phi : \mathcal{C} \to \mathcal{C}'$ is a family of homomorphisms $\phi_p : C_p \to C'_p$ such that $\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p$ for all p.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[d, "\phi_{p+1}"'] & C_p \arrow[r, "\partial_p"] \arrow[d, "\phi_p"'] & C_{p-1} \arrow[r, "\partial_{p-1}"] \arrow[d, "\phi_{p-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{p+1} \arrow[r, "\partial'_{p+1}"] & C'_p \arrow[r, "\partial'_p"] & C'_{p-1} \arrow[r, "\partial'_{p-1}"] & \cdots
\end{tikzcd}$$

Furthermore, these chain maps induces a homomorphism $(\phi_p)_*$ from the corresponding homology groups of C_p and C'_p.

Lemma 2.7.1 (Zig-Zag Lemma)**.** *Suppose one is given chain complexes $\mathcal{C} = \{C_p, \partial_c\}$, $\mathcal{D} = \{D_p, \partial_d\}$, $\mathcal{E} = \{E_p, \partial_e\}$ and chain maps ϕ, ψ such that*

$$0 \to \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$$

is exact. Then there exists a long exact sequence of homological groups satisfying

$$\cdots \xrightarrow{\partial_p} \mathcal{H}_p(\mathcal{C}) \xrightarrow{\phi_*} H_p(\mathcal{D}) \xrightarrow{\psi_*} H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \to \cdots$$

Proof. The proof to this theorem is an exercise in diagram chasing, corresponding to the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial_c"] & D_{p+1} \arrow[r, "\psi"] \arrow[d, "\partial_d"] & E_{p+1} \arrow[r] \arrow[d, "\partial_e"] & 0 \\
0 \arrow[r] & C_p \arrow[r, "\phi"] \arrow[d, "\partial_c"] & D_p \arrow[r, "\psi"] \arrow[d, "\partial_d"] & E_p \arrow[r] \arrow[d, "\partial_e"] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "\phi"] & D_{p-1} \arrow[r, "\psi"] & E_{p-1} \arrow[r] & 0
\end{tikzcd}$$

□

We can use this theorem to prove a very useful result on relative homology groups. If K is a simplicial complex and K_0 is a subcomplex then we have a map $\iota : C_p(K_0) \to C_p(K)$ by inclusion and a map $\pi : C_p(K) \to C_p(K, K_0)$ by projection (since $C_p(K, K_0)$ is the quotient group $C_p(K)/C_p(K_0)$.) This can be summarized by the following theorem:

Theorem 2.7.2. *Let K be a simplicial complex, K_0 its subcomplex. Then there is a long exact sequence*

$$\cdots \to H_p(K_0) \xrightarrow{i_*} H_p(K) \xrightarrow{\pi_*} H_p(K, K_0) \xrightarrow{\partial_*} H_{p-1}(K_0) \to \cdots$$

Example 2.7.3. Consider the simplex given in Example 2.1.23 presented in Figure 2.4 where $|K_0| = \partial|K|$. Theorem 2.7.2 implies that we have the following long exact sequence

$$\underset{0}{H_2(K_0)} \xrightarrow{i_*} \underset{0}{H_2(K)} \xrightarrow{\pi_*} H_2(K, K_0) \xrightarrow{\partial_*} \underset{\mathbb{Z}}{H_1(K_0)} \to \underset{0}{H_1(K)} \xrightarrow{\pi_*} H_1(K, K_0) \xrightarrow{\partial_*} \underset{0}{\tilde{H}_0(K_0)}.$$

For which we endeavor to find the relative homology groups $H_2(K, K_0)$ and $H_1(K, K_0)$. Immediately we see that $H_1(K, K_0) = 0$. On the other hand, for $H_2(K, K_0)$ we may consider the smaller exact sequence given by

$$0 \xrightarrow{\pi_*} H_2(K, K_0) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{\iota^*} 0.$$

Being exact, we know that $H_2(K, K_0)/\ker \pi_* = \operatorname{im} \partial_*$ and that $\ker \pi_* = \{0\}$. Now $\operatorname{im} \partial_* = \ker \iota_* = \mathbb{Z}$ and so $H_2(K, K_0) = \mathbb{Z}$.

Example 2.7.4. Let us reconsider the simplex given in Example 2.3.4 given in Figure 2.7, with $|K_0| = \partial|K|$. Theorem 2.7.2 gives us the long exact sequence

$$0 \to H_2(K, K_0) \xrightarrow{\partial_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(K_0)} \xrightarrow{i_*} \underset{\mathbb{Z}}{H_1(K)} \xrightarrow{\pi_*} H_1(K, K_0) \xrightarrow{\partial_*} \underset{\mathbb{Z}}{\tilde{H}_0(K_0)} \to 0$$

Now $H_1(K_0) = \mathbb{Z} \oplus \mathbb{Z}$ has two generators, say z_1 and z_2. $\tilde{H}_0(K_0)$ is generated by $\{v_1 - v_0\}$. $H_1(K)$ is generated by either z_1 or z_2 since they are homologous. Note that ι_* is surjective, its kernel is $\mathbb{Z}$ and so $\operatorname{im}\delta_* = \mathbb{Z}$. Hence $H_2(K, K_0) = \mathbb{Z}$. On the other hand, note that $\ker \pi_*$ is everything and so $\ker \partial_* = 0$ and ∂_* is surjective hence $H_1(K, K_0) = \mathbb{Z}$.

Example 2.7.5. For this example, we consider both a cylinder relative to its top edge, and the Mobius band relative to its edge. Notice in both cases K_0 is homotopic to a circle.

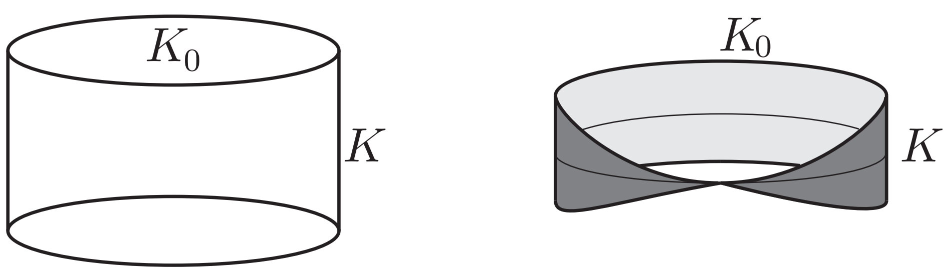

Figure 2.8: Computing the relative homology of the cylinder and the Moebius band.

In both cases we can find a deformation retract from K to S^1, so that $H_2(K) = 0$. Since K_0 is also homotopic to a circle, $H_1(K) = H_1(K_0) = H_0(K_0) = \mathbb{Z}$ and we get the following exact sequence

$$0 \to \underset{0}{H_2(K)} \to \overset{\{0\}}{H_2(K, K_0)} \xrightarrow{\partial_*} \underset{\mathbb{Z}}{H_1(K_0)} \xrightarrow{i_*} \overset{2\mathbb{Z}}{\underset{\mathbb{Z}}{H_1(K)}} \xrightarrow{\pi_*} \overset{\mathbb{Z}_2}{H_1(K, K_0)} \xrightarrow{\partial_*} \underset{0}{\tilde{H}_0(K_0)} \to 0$$

Now since $\ker \pi_*$ is trivial we have $H_2(K, K_0)/\ker \pi_* = H_2(K, K_0) = \operatorname{im}\partial_* = \ker i_*$ and $H_1(K)/\operatorname{im} i_* = \operatorname{im}\pi_* = \ker \partial_* = H_1(K, K_0)$, or more succinctly

$$H_1(K, K_0) = H_1(K)/\operatorname{im} i_*, \qquad H_2(K, K_0) = \ker i_*$$

so it is sufficient to examine the map i_* to determine everything about the relative homology groups. For the cylinder, i_* is an isomorphism as the retraction is a homotopy equivalence, implying that $\ker i_* = 0$ and $\operatorname{im} i_* = H_1(K)$ so we have $H_1(K, K_0) = \{0\}, H_2(K, K_0) = \{0\}$. On the other hand, for the Moebius strip i_* correspond to a multiplication $\cdot 2$ while is injective and has an image isomorphic to $2\mathbb{Z}$, implying that $H_1(K, K_0) = \mathbb{Z}/(2\mathbb{Z})$ while $H_2(K, K_0) = \{0\}$.

2.8 Mayer-Vietoris Sequences

Theorem 2.8.1. *Let K be a simplicial complex; K_0 and K_1 be subcomplexes such that $K_0 \cup K_1 = K$. Let $A = K_0 \cap K_1$. Then there exists a long exact sequences*

$$\cdots H_p(A) \to H_p(K_0) \oplus H_p(K_1) \to H_p(K) \to H_{p-1}(A) \to \cdots$$

Proof. Construct a short exact sequence

$$0 \to \mathcal{C}(A) \xrightarrow{\phi} \mathcal{C}(K_0) \oplus \mathcal{C}(K_1) \xrightarrow{\psi} \mathcal{C}(K) \to 0.$$

Now apply the Zig-Zag lemma.

$$\begin{tikzcd}
& K_0 \arrow[dr, "k"] & \\
A \arrow[ur, "i"] \arrow[rr, "m"] \arrow[dr, "j"'] & & K \\
& K_1 \arrow[ur, "\ell"'] &
\end{tikzcd}$$

where $\phi(c) = (i_\#(c), -j_\#(c))$ and $\psi(d, e) = k_\#(d) + \ell_\#(e)$. $\square$

Example 2.8.2. Let K be the union of complexes K_0 and K_1, $|K_0|$ and $|K_1|$ are path connected and $K_0 \cap K_1$ is nonempty and acyclic. Then $H_p(K) \cong H_p(K_1) \oplus H_p(K_2)$.

Definition 2.8.3. Let K be a simplicial complex and $\omega_0 \star K$ and $\omega_1 \star K$ be two cones on K whose polytopes intersect only in $|K|$. Then we define a suspension of K as $S(K) = \omega_0 \star K \cup \omega_1 \star K$.

Theorem 2.8.4. *If K is a simplicial complex then for all p there exists an isomorphism $\tilde{H}_p(S(K)) \to \tilde{H}_{p-1}(K)$.*

Proof. Set $K_0 = \omega_0 \star K$ and $K_1 = \omega_1 \star K$ so that $K_0 \cup K_1 = S(K_1)$ and $K_1 \cap K_2 = K$. Hence we can use Meyer Veitoris to get the following sequence

$$\tilde{H}_p(K_0) \underset{0}{\oplus} \tilde{H}_p(K_1) \to \tilde{H}_p(S(k)) \xrightarrow{\cong} \tilde{H}_{p-1}(K) \to \tilde{H}_{p-1}(K_0) \underset{0}{\oplus} \tilde{H}_{p-1}(K_1).$$

□

Example 2.8.5. We can use Meyer-Veitoris to compute the homology of $S^1 \times S^2$. Let $D^2_\pm$ be the two-cells that represent the upper and lower hemispheres of S^2. Set $X_\pm = S^1 \times D^2_\pm$ and note that each $X_\pm$ equivalent to a torus. The set $A = X_+ \cap X_-$ is just a torus, and so Meyer-Veitoris gives us the following sequence

$$\begin{tikzcd}
H_3(X_+) \overset{0}{\oplus} H_3(X_-) \arrow[r] & \overset{?}{H_3(X)} \arrow[r] & \overset{\mathbb{Z}}{H_2(A)} \arrow[r] & H_2(X_0) \overset{0}{\oplus} H_2(X_1) \arrow[dlll] & \\
\underset{?}{H_2(X)} \arrow[r] & \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A)} \arrow[r] & H_1(X_0) \underset{\mathbb{Z}^{\oplus 4}}{\oplus} H_1(X_1) \arrow[r] & \underset{?}{H_1(X)} \arrow[r] & \underset{0}{\tilde{H}_0(A)}
\end{tikzcd}$$

Note that immediately we can see that $H_3(X) \cong \mathbb{Z}$. Now let $H_1(A)$ be generated by α and β. Then $\alpha \mapsto (0,0)$ and $\beta \mapsto (\tilde{\beta}, \tilde{\beta})$ and so $H_2(X) = \mathbb{Z}$. Finally, $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} / \operatorname{im} \varphi_* \cong \mathbb{Z}$.

Example 2.8.6. Consider the double torus which can be written as the quotient of a 2-cell (octagon) with border identifications $[a,b][c,d]$ (see Figure 2.9). Let $X_0 = \mathbb{T}\#\mathbb{T} \setminus \{p\}$ and $X_1 = \mathbb{T}\#\mathbb{T} \setminus a \cup b \cup c \cup d$ and note that X_0 can be deformation retracted to the wedge of four circles while X_1 is contactable and hence acyclic. Now $X_0 \cap X_1$ is deformation retractable to a circle, implying that $H_1(X_0 \cap X_1) \cong \mathbb{Z}$, yielding the following Mayer-Vietoris sequence

$$H_2(X_0) \underset{0}{\oplus} H_2(X_1) \to \underset{?}{H_2(X)} \to \underset{\mathbb{Z}}{H_1(A)} \to H_1(X_0) \underset{\mathbb{Z}^{\oplus 4}}{\oplus} H_1(X_1) \to \underset{?}{H_1(X)} \to \underset{0}{\tilde{H}_0(A)}$$

Now we would like to consider the map $H_1(X_0 \cap X_1) \to H_1(X_0) \oplus H_1(X_1)$. Since X_1 is acyclic we have $H_1(X_1) = 0$ and so the image under the inclusion of γ is clearly zero. Similarly, since $H_1(X_1)$ is abelian, the image of γ is $(a+b-a-b)+(c+d-c-d) = 0$ so $\gamma \mapsto (0,0)$. We conclude that $H_2(X_1 \cup X_1) = \mathbb{Z}$. Similarly, $H_1(X) = \mathbb{Z}^4$.

Example 2.8.7. Consider the following abstraction construction: Define X_0 and X_1 to be double tori so that $X_0, X_1 = \mathbb{T}\#\mathbb{T}$. If $\mathrm{id} : X_0 \to X_1$ is the identity map, define the space $X = X_0 \bigsqcup X_1/\sim$, so that X is two double-tori whose boundaries are identified. We can very naturally apply Mayer-Veitoris to this problem, as our decompositions have already been specified. Notice that $A = X_1 \cap X_2$ is also a double torus, resulting in the following exact sequence

$$\begin{tikzcd}
H_3(X_+) \overset{0}{\oplus} H_3(X_-) \arrow[r] & \overset{?}{H_3(X)} \arrow[r] & \overset{\mathbb{Z}}{H_2(A)} \arrow[r] & H_2(X_0) \overset{0}{\oplus} H_2(X_1) \arrow[dlll] & \\
\underset{?}{H_2(X)} \arrow[r] & \underset{\mathbb{Z}^{\oplus 4}}{H_1(A)} \arrow[r] & H_1(X_0) \underset{\mathbb{Z}^{\oplus 4}}{\oplus} H_1(X_1) \arrow[r] & \underset{?}{H_1(X)} \arrow[r] & \underset{0}{\tilde{H}_0(A)}
\end{tikzcd}$$

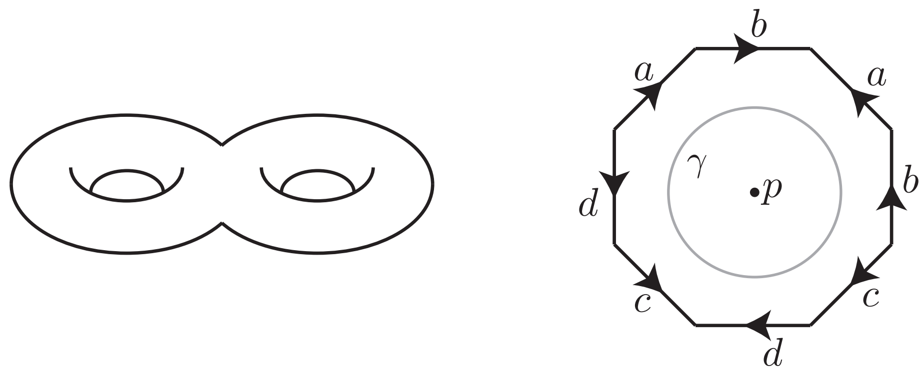

Figure 2.9: The double torus as a two-cell.

The first sequence surrounding $H_3(X)$ implies that the map $H_3(X) \to H_2(A)$ is an isomorphism so $H_3(X) = \mathbb{Z}$. Let $\phi : H_1(A) \to H_1(X_0) \oplus H_1(X_1)$. $\ker \phi_* = \mathbb{Z} \oplus \mathbb{Z}$ and $\gamma \mapsto (\tilde{\gamma}, 0, \hat{\gamma}, 0)$ and $\sigma \mapsto (0, \tilde{\sigma}, 0, \hat{\sigma})$ so $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$.

2.9 Jordan Curve Theorem

Definition 2.9.1. Let A be a subspace of X. We say that A *separates* X if the $X \setminus A$ is not connected.

Note that $\tilde{H}_0(X \setminus A) = \{0\}$ if and only if $X \setminus A$ is path connected, hence all connected components are path connected components.

Theorem 2.9.2. *Let B be a k-cell in S^n for $k \leq n$. Then $S^n \setminus B$ is acyclic. In particular, $\tilde{H}_0(S^n \setminus B) = \{0\}$ implies that B does not separate S^n.*

Proof. We shall proceed by induction on k. Suppose that $k = 0$ in which case B is a point. If $n = 0$ then S^n is two points so $S^n \setminus B$ is a single point and hence acyclic. If $n > 0$ then $S^n \setminus B$ is homeomorphic to $\mathbb{R}^n$ which is acyclic.

Now suppose that the statement is true for a $k-1$ cell in S^n. Let B be a k-cell in S^N and let $h : I^k \to B$ be a homeomorphism. Write $I^k = I^{k-1} \times \left[0, \frac{1}{2}\right] \cup I^{k-1} \times \left[\frac{1}{2}, 1\right]$. The restriction h to $I^{k-1} \times \left\{\frac{1}{2}\right\}$ is now a $k-1$ cell, so let C be this image. Apply Meyer-Veitoris to $S^n \setminus C$. Let $h(I^{k-1} \times [0, \frac{1}{2}]) = B_1, h(I^{k-1} \times [\frac{1}{2}, 1]) = B_2$ and set $X_1 = S^n \setminus B_1$ and $X_2 = S^n \setminus B_2$.

Then $A = X_1 \cap X_2 = S^n \setminus B$. Thus we get the following sequence

$$\to \tilde{H}_i(A) \to \tilde{H}_i(X_1) \oplus \tilde{H}_i(X_2) \to 0$$

Let α be a non-trivial element in $\tilde{H}_i(A)$ in which case $(i_*(\alpha), -j_*(\alpha))$ must be non-trivial in one of these components. We may continue in this fashion, dividing $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ in which case $\bigcap_i [a_i, b_i] = e$ for some point e. Define $E = h(I^{k-1} \times \{e\})$. Then $\tilde{H}_i(S^n \setminus E) = \{0\}$ implies that the image of α in this group vanishes.

Let $D_i = h(I^{k-1} \times [a_m, b_m])$ and notice then that there exists a compact subsets F of $S^n \setminus E$ such that the image of α vanishes in $\tilde{H}_i(F)$. Then $S_n \setminus E$ is the union of open sets $\bigcup_i S^n \setminus D_i$. Choose a finite subcover $S^n - D_{i_1}, \dots, S^n - D_{i_k}$ so that for some k, α vanishes in $\tilde{H}_i(S^n - D_{i_k})$ and this is a contradiction. □

Theorem 2.9.3. *Let $n > k \geq 0$ and let $h : S^k \to S^n$ be an embedding. Then*

$$\tilde{H}_i(S^n \setminus h(S^k)) \cong \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \textit{otherwise} \end{cases}. \tag{2.16}$$

Proof. Let n be fixed and proceed by induction on k. If $k = 0$ then $h(S^0) = \{p, q\}$ and so we have $S^n \setminus \{p, q\}$ which is equivalent to a punctured S^{n-1} sphere which is equivalent to $\mathbb{R}^{n-1} \setminus 0$ and so $\tilde{H}_i(S^n \setminus \{p, q\})$ is infinite cyclic for $i = n-1$ and vanishes everywhere else.

Now suppose that the result is true in dimension $k-1$ with $h : S^k \to S^n$ an embedding. We shall proceed by using the Mayer-Vietoris sequences on X_1 and X_2 where $X = S^n \setminus h(S^{k-1})$

$$X_1 = S^n - h(E_k^+), \qquad X_2 = S^n - h(E_k^-), \qquad A = X_1 \cap X_2 = X - h(S^k).$$

Since X_1 and X_2 are open in X, we can apply Mayer-Vietoris to get the following sequence

$$\tilde{H}_{i+1} \underset{0}{\oplus} \tilde{H}_{i+1}(X_2) \to \tilde{H}_{i+1}(X) \to \tilde{H}_i(A) \to \tilde{H}_i(X_1) \underset{0}{\oplus} \tilde{H}_i(X_2)$$

Thus $\tilde{H}_{i+1}(X)$ is isomorphic to $\tilde{H}_i(A)$ by exactness. We note that in this case $i + 1 = n - (k-1) - 1$ yields $\mathbb{Z}$ which we can solve for i as $i = n - k - 1$. □

Theorem 2.9.4 (Generalized Jordan Curve Theorem)**.** *Let $n > 0$ and let C be a subset of S^n homeomorphic to S^{n-1}. Then $S^n \setminus C$ has precisely two components, of which C is the common (topological) boundary.*

Proof. Applying the previous theorem to the $k = n-1$ case we see that

$$\tilde{H}_i(S^n \setminus h(S^{n-1})) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus $S^n \setminus C$ has precisely two path components. Let W_1, W_2 be these path components, for which our goal is to show that $\overline{W_1} - W_1 = C = \overline{W_2} - W_2$. We will only show the first of these identities as the second one follows in a symmetrical manner. Since W_1 is open, no point of W_1 is a limit point of W_2 and hence $\overline{W_1} - W_1 \subseteq C$, and so it is sufficient to show the other direction. Hence let $x \in C$ and let U be a neighbourhood of x. Since C is homeomorphic to S^{n-1} we can write C as the union of two $n-1$ cells, say C_1 and C_2 and we may choose C_1 to lie entirely within U. Let $p \in W_1$ and $q \in W_2$. Since C_2 does not separate S^n we can find a path α in $S^n \setminus C_2$ joining p and q. Now α must contains a point of $\overline{W_1} \setminus W_1$ say y, and $y \in \overline{W_1} \setminus W_1 \cap \alpha$ implies that $y \in C$ so that $y \in U$. Hence $U \cap \overline{W_1} \setminus W_1 \neq \emptyset$, as required. □

As a general rule, the connected components W_1 and W_2 given above are not n-cells. We can give a criteria of when they are indeed n-cells as follows:

Theorem 2.9.5 (Morton-Brown or Mazur-Mores)**.** *Suppose that $h : S^{n-1} \to S^n$ can be collared; that is, there exists an embedding $H : S^{n-1} \times I \to S^n$ such that $H\left(x, \frac{1}{2}\right) = h(x)$ for all x. If W_1 and W_2 are as above, then W_1 and W_2 are n-cells.*

Corollary 2.9.6. *Let $n > 1$ and C be a subset of $\mathbb{R}^n$ homeomorphic to S^{n-1}. Then $\mathbb{R}^n \setminus C$ has precisely two components of which C is the common boundary.*

We define θ-space to be two vertices connected by three edges.

Theorem 2.9.7. *Let C_1 and C_2 be closed connected subsets of S^2 whose intersection consists of two points. If neither C_1 nor C_2 separates S^2 then $C_1 \cup C_2$ separates S^2 into precisely two components.*

Proof. By Mayer-Vietoris let $X_i = S^2 \setminus C_i$ so that $A = S^2 \setminus C_1 \cup C_2$ so that we get the following sequence

$$0 \to \mathbb{Z} \to \tilde{H}_0(A) \to 0$$

which implies that $\tilde{H}_0(A) \cong \mathbb{Z}$. Thus A has precisely two connected components. □

Theorem 2.9.8. *Let X be the θ-space visualized as a subspace of S^2. If A, B, C are the arcs of θ-space then X separates S^2 into three components whose boundaries are $A \cup B, B \cup C$ and $A \cup C$.*

Proof. Through exhaustive use of the previous theorem, we can divide S^2 into two components, then divide one of these components again into three components. □

Recall the utilities graph which consists of three houses and three utilities. The objective is to connect all three utilities to all three houses without crossing any of the lines. This is in general impossible, as we shall show next.

Theorem 2.9.9. *Let X be the utilities graph, then X cannot be embedded in the plane.*

Proof. We will show that the utilities graph cannot be embedded into S^2. Let $A = gh_1w, B = gh_2w$ and $C = gh_3w$. Note that $Y = A \cup B \cup C$ is a θ-space and Y separates S^2 into three components U, V, W whose boundaries are $A \cup B, B \cup C$ and $A \cup C$. Note that e cannot be in U because h_3 is not in U. □

Theorem 2.9.10 (Invariance of Domain)**.** *if U be open in $\mathbb{R}^n$ (resp. S^n) and $f : U \to \mathbb{R}^n$ is continuous and injective then $f(U)$ is open in $\mathbb{R}^n$ and f is an embedding.*

Proof. Without loss of generality, we shall proceed on S^n. We want to show that $f(U)$ is open so for any point $y \in f(U)$ we shall find an open neighbourhood of y contained entirely in $f(U)$. Let $x \in U$ be some point such that $f(x) = y$ and choose an open ball B_ϵ of radius ϵ centred at x such that $\overline{B_\epsilon} \subseteq U$. Define $S_\epsilon = \overline{B_\epsilon} - B_\epsilon$ in which case $f(S_\epsilon)$ is homeomorphic to S^{n-1} and therefore $f(S_\epsilon)$ separates S^n into two components, say W_1 and W_2. Since $f(B_\epsilon)$ is connected and $f(B_\epsilon) \cap f(S_\epsilon) = \emptyset$ it must lie in one of the components W_1 or W_2. Without loss of generality, suppose that it lies is W_1. Now if $f(B_\epsilon)$ were a strict subset then $S_n - f(B_\epsilon) - f(S_\epsilon) = S^n - f(\overline{B_\epsilon})$ would contain W_2 and points of W_1 contradicting the fact that $f(\overline{B_\epsilon})$ does not separate S^n. Hence $f(B_\epsilon) = W_1$ and so $f(U)$ contains a neighbourhood W_1 of y as required. □

Note that any open set of U is mapped to an open set of S^n and so is open in $f(U)$. We conclude that f is a homeomorphism onto its image and hence is an embedding.

2.10 Cellular Homology

Let X be a CW complex and denote by X^n the n-skeleton of X. Note that $H_k(X^n, X^{n-1})$ vanishes for $k \neq n$ and in the case of $k = n$ we get a free abelian group whose generators are the components of the n-skeleton. Furthermore, $H_k(X^n) = 0$ for $k > n$. Finally, if $i : X^n \hookrightarrow X$ is the inclusion map then this induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ for $k < n$. Intuitively, this is because we are only concerned with the k-skeleton, and X_n contains the k-skeleton.

We can for example write down the following chain

$$H_{k+1}(X, X^n) \to H_k(X^n) \xrightarrow{\iota_*} H_k(X) \to H_k(X, X^n)$$

for which we would like ι_* to be an isomorphism, requiring that $H_{k+1}(X, X^n) = H_k(X, X^n) = 0$ and $k + 1 \leq n$. This is non-trivial to show, but can be done by showing that $H_k(X, X^n) \cong \tilde{H}_k(X/X_n)$.

Consider the following cellular chain complex

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

where $d_{n+1} = j_n \circ \partial_{n+1}$. That is, the arrows above hide the natural sequence of maps

$$H_{n+1}(X^{n+1}, x^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}).$$

Thus we are now in a position to define the CW-homology groups $H_n^{CW} = Z_n/B_n$ where Z_n are cellular cyclies and B_n are cellular boundaries, defined as usual. We claim that $H_n^{CW} \cong H_n(X)$ so that CW-homology agrees with typical homology (Hatcher page 140, Munkres 222).

The following is the cellular boundary formula:

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1} \tag{2.17}$$

where $d_{\alpha\beta}$ is the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$ which is the composition of the attaching map of e_α^n with the quotient map which collapses $X^{n-1} \setminus \left\{e_\beta^{n-1}\right\}$ to a point.

Proposition 2.10.1. *Suppose that $f : S^n \to S, n > 0$ has the property that for some point $y \in S^n, f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_n$. Let $U_1, \ldots, U_m$ be disjoint neighbourhoods of $x_1, \ldots, x_n$ all mapped by f into V a neighbourhood of y.*

$$\begin{tikzcd}
 & \overset{\mathbb{Z}}{H_n(U_i, U_i \setminus \{x\})} \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & \overset{\mathbb{Z}}{H_n(V, V \setminus \{y\})} \arrow[d, "\cong"] \\
\underset{\mathbb{Z}}{H_n(S^n, S^n \setminus \{x_i\})} & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

Recall the excision property which tells us that given subspaces $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq \text{int}(A)$ then the inclusion map $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$. Hence this yields the following short chain

$$\underbrace{H_n(S^n \setminus \{x_i\})}_{=0} \to H_n(S^n) \to H_n(S^n, S^n \setminus \{x_i\}) \to \underbrace{H_{n-1}(S^n \setminus \{x_i\})}_{=0}$$

which implies that $H_n(S^n) \cong H_n(S^n, S^n \setminus \{x_i\})$. Referring to the above diagram, the define the *local degree* as the degree of $f|_{X_i}$.

Proposition 2.10.2. *The degree of f is the sum of the local degrees; that is,* $\deg f = \sum_i \deg f|_{X_i}$.

Proof. The middle term $H_n(S^n, S^n \setminus f^{-1}(y))$ is the direct sum of $H_n(U_i, U_i \setminus X_i)$ by excision. The map k_i is the inclusion of the i^{th} summand, and p_i is the projection onto the i^{th} summand. Since we may identify all of the exterior groups in the diagram above with $\mathbb{Z}$, looking at the lower triangle we have $p_i j(1) = 1$ so that $j(1) = (1, 1, \ldots, 1) = \sum_i k_i(1)$. The upper square says that f_* takes $k_i(1)$ to $\deg f|_{x_i}$ and so $\sum_i k_i(1) = j(1) \mapsto \sum_i \deg f|_{x_i}$. Using commutativity of the lower square, we conclude that $\deg f = \sum_i \deg f|_{x_i}$. $\square$

Example 2.10.3. Let M_g be the closed orientable surface of genus g consisting of one 0-cell, $2g$ 1-cells, and one 2-cell attached via commutators $[a_1, b_1] \cdots [a_g, b_g]$. This yields the following associated cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

where we note that $d_1 = 0$ since we are in a path connected space. Hence all that remains is to compute $d_2(e^2)$. Note for example that we can compute the coefficient of a_1 by squeezing everything but a_1 to a point. Then we get a $+1$ contribution and a -1 contribution since the identifications have opposite orientations. Thus d_2 is also the zero map. Hence $H_2(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}^{2g}$, $H_0(M) = \mathbb{Z}$.

Example 2.10.4. Consider the Klein bottle as a two-cell with boundary identification $abab^{-1}$. This yields the following associated cellular chain complex

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

Hence $d_2(e^2) = 2a + 0b$. Thus $H_2(\mathbb{K}) = 0, H_1(\mathbb{K}) = \mathbb{Z} \oplus \mathbb{Z}_2$ and $H_0(\mathbb{K}) = \mathbb{Z}$.

Example 2.10.5. Let X be obtained from $S^1 \vee S^1$ by attached two 2-cells by the words a^5b^{-3} and $b^3(ab)^{-2}$. Thus we get the following associated cellular chain complex

$$0 \to \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0.$$

Now $d_2(e_1^2) = 5a - 2b$ and $d_2(e_2^2) = -2a + b$ so we get the matrix $\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}$ which has non-zero determinant.

Example 2.10.6. We can realize $\mathbb{CP}^n$ as a CW-complex of dimension $2n$ which has on open cell in each even dimension, yielding the following associated cellular chain

$$\mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots \to 0 \to \mathbb{Z} \to 0$$

Hence all the maps are trivial, and so $H_{2j}(\mathbb{CP}^n) = \mathbb{Z}$ and $H_{2j-1}(\mathbb{CP}^n) = 0$. Now let $\mathbb{CP}^\infty = \bigcup_n \mathbb{CP}^n$. Its homology can be given as

$$H_i(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}. \tag{2.18}$$

Example 2.10.7. Recall that we can build $\mathbb{RP}^n$ by adjoining to $\mathbb{RP}^{n-1}$ two n-cells along the boundary. Hence $\mathbb{RP}^n$ has a single n-cell for each dimension, yielding the associated cellular chain

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0.$$

Now recall that $\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$ via $S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ the composition of two quotient maps. Label the upper and lower hemispheres of S^{k-1} as E_+^{k-1} and E_-^{k-1} respectively. Write $h = h|_{\text{int}(E_+^{k-1})}$ and $\tilde{h}|_{\text{int}(E_-^{k-1})}$ so that $\tilde{h} = a \circ h$ where a is the antipodal map. Then $\deg q \circ \varphi = 1 + (-1)^k$ so that d_k is either 0 or 2. Thus we get

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i \text{ odd }, n \text{ even}, \\ 0 & \text{even} \end{cases} \qquad H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \text{ odd }, n \text{ odd}. \\ \mathbb{Z}_2 & \text{even} \end{cases} \tag{2.19}$$

Example 2.10.8. Consider $\mathbb{RP}^3 \wedge S^2$. Using Mayer-Vietoris this is the direct sum of the homology.

2.11 Homology with Arbitrary Coefficients

Let G be an abelian group and let K be a simplicial or Δ-complex. Let our chain elements be given by $c_p = \sum g_i \sigma_i^p$ where $g_i \in G$. Notice that the difference between this and what has been done in the past is that we had always taken $G = \mathbb{Z}$.

Example 2.11.1. Consider the homology of the torus with coefficients in $\mathbb{Z}/2$, in which we replace $\mathbb{Z}$ with $\mathbb{Z}/2$ to get the chain complex

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0.$$

Notice then in this case all the maps are trivial. Similarly, the homology with coefficients in $\mathbb{Q}$ yield all zero maps.

Example 2.11.2. Consider the Klein bottle, for which we shall compute the homology with coefficients in $\mathbb{Z}/2$ and $\mathbb{Q}$.

For $\mathbb{Z}/2$ we get the following chain complex

$$0 \to \mathbb{Z}_2 \xrightarrow{d_2} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{d_1} \mathbb{Z}_2 \to 0$$

and $d_1 = 0$. If γ is the two-cell, then in $\mathbb{Z}$ we would have had $d_2(\gamma) = 2b$; however, in $\mathbb{Z}/2$ $2b = 0$ and so d_2 is the trivial map. Thus $\text{Hom}_2(\mathbb{K}, \mathbb{Z}_2) = \mathbb{Z}_2, H_1(\mathbb{K}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, H_0(\mathbb{K}) = \mathbb{Z}_2$.

With coefficients in $\mathbb{Q}$ we have a similar chain complex as above, given by

$$0 \to \mathbb{Q} \xrightarrow{d_2} \mathbb{Q} \oplus \mathbb{Q} \xrightarrow{d_1} \mathbb{Q} \to 0$$

and once again $d_1 = 0$. Notes that $Z_1(\mathbb{K}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ and that $d_2(\gamma) = 2b$. Hence $b = \frac{1}{2} d_2(\gamma) = d_2\left(\frac{\gamma}{2}\right)$. Thus $H_1(\mathbb{K}, \mathbb{Q}) = \mathbb{Q}, H_2(\mathbb{K}, \mathbb{Q}) = \{0\}$ and $H_0(\mathbb{K}, \mathbb{Q}) = \mathbb{Q}$.

Chapter 3

Cohomology

3.1 Preliminaries

We define $\operatorname{Hom}(A, G)$ to be the set of all homomorphisms of A into G where A and G are abelian groups. Now $\operatorname{Hom}(A, G)$ is also an abelian group under pointwise addition since if $h, f \in \operatorname{Hom}(A, G)$ then $(h + f)(a) = h(a) + f(a) \in G$.

Definition 3.1.1. If $f : A \to B$ is a homomorphism of abelian groups, we may define the *dual homomorphism* $\tilde{f} : \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G)$ as follows: if $\phi \in \operatorname{Hom}(B, G)$ then $\tilde{f}(\phi)(x) = \phi(f(x))$. ($f$ is a contravariant function on the set of abelian groups).

3.2 Simplicial Cohomology Groups

Definition 3.2.1. Let K be a simplicial complex and G an abelian group. The group of *p-dimensional cochains with coefficients in* G is denoted $C^p(K; G)$ and consists of the group of homomorphisms $\operatorname{Hom}(C_p(K); G)$.

Our goal will be to work with cochain complexes which requires us to define a dual map to the boundary map. More precisely, we define

$$C_{p+1}(K; G) \xrightarrow{\partial} C_p(K), \qquad C^{p+1}(K; G) \xleftarrow{\delta} C^p(K; G).$$

Similar to before, we then define the set of *cocycles* as $Z^p(K;G) = \ker \delta$ and the *coboundaries* as $B^{p+1}(K;G) = \operatorname{im} \delta$. It is easy to show that $\delta^2 = 0$ and we then define the p^{th} simplicial cohomology group $H^p(K;G) = Z^p(K;G)/B^p(K;G)$.

If C^p is a cochain then C_p is chain and we denote the value of the cochain applied to the chain as $\langle c^p, c_p \rangle = c^p(c_p)$. To see that this makes sense, recall that $C_p(K)$ is the free abelian group whose generators consists of p-simplices, say $\{\sigma_\alpha\}_{\alpha \in J}$. Fix $c_p \in C_p(K)$ and write $c_p = \sum_\alpha n_\alpha \sigma_\alpha$. Let $c^p \in C^p(K)$, then $\langle c^p, c_p \rangle = \displaystyle\sum_\alpha n_\alpha \langle c^p, \sigma_\alpha \rangle$.

Definition 3.2.2. An *elementary cochain with coefficients in* $\mathbb{Z}$ is an element such that $\langle \sigma^*_\alpha, \sigma_\beta \rangle = \delta_{\alpha\beta}$; that is, the elementary cochains are precisely the dual basis to σ_α.

More generally, if G is any abelian group and $g \in G$ then we define $g\sigma^*_\alpha$ to be the chain such that $\langle g\sigma^*_\alpha, \sigma_\beta \rangle = g\delta_{\alpha\beta}$. Hence we can write elements in C^p as $\sum_\alpha g_\alpha \sigma^*_\alpha$.

We claim that $\delta c^p = \sum g_\alpha(\delta\sigma^*_\alpha)$, for which we omit the proof but note that it is sufficient to show that for each p simplex

$$\langle \delta c^p, \tau \rangle = \left\langle \sum_\alpha g_\alpha(\delta\sigma^*_\alpha), \tau \right\rangle \tag{3.1}$$

Example 3.2.3. Consider the diagram, and let us try to determine δe^*_5. Note that by definition we must have

$$\langle \delta e^*_5, \sigma_1 \rangle = \langle e^*_5, \partial\sigma_1 \rangle = \langle e^*_5, e_5 - e_1 - e_4 \rangle = 1 \tag{3.2}$$

and applying this to σ_1 we get

$$\langle \delta e^*_5, \sigma_2 \rangle = \langle e^*_5, \partial\sigma_2 \rangle = \langle e^*_5, -e_5 - e_2 - e_3 \rangle = -1. \tag{3.3}$$

Thus $\delta e^*_5 = \sigma^*_1 - \sigma^*_2$.

Example 3.2.4. Let us compute the cohomology group of the diagram. We know that the cochain complex is given by

$$0 \leftarrow C^1(K) \leftarrow C^0(K) \leftarrow 0$$

and set $c^0 = \sum n_i v^*_i$. To determine when this is a cocycle notice that $\delta c^0 = \sum_i n_i \delta v^*_i$ and so it is sufficient to determine the action of δv^*_i for each i. Much of the computations are repetitive, so we make due with an example of v^*_0. Notice that

$$\begin{aligned}
\langle \delta v^*_0, e_1 \rangle &= \langle v^*_1, \partial e_1 \rangle = \langle v^*_1, v_1 - v_0 \rangle = -1 \\
\langle \delta v^*_0, e_2 \rangle &= \langle v^*_0, v_2 - v_1 \rangle = 0 \\
\langle \delta v^*_0, e_3 \rangle &= \langle v^*_0, v_3 - v_2 \rangle = 0 \\
\langle \delta v^*_0, e_4 \rangle &= \langle v^*_0, v_0 - v_3 \rangle = 1
\end{aligned}$$

and so we conclude that $\delta v_0^* = e_4^* - e_1^*$. Thus

$$\begin{aligned}0 = \delta c^0 &= n_1(e_4^* - e_1^*) + n_2(e_1^* - e_2^*) + n_3(e_2^* - e_3^*) + n_4(e_3^* - e_4^*) \\ &= e_1^*(n_2 - n_1) + e_2^*(n_3 - n_2) + e_3^*(n_4 - n_3) + e_4^*(n_1 - n_4)\end{aligned}$$

which implies that $n_1 = n_2 = n_3 = n_4$. Thus $c^0 = n(\sum_i v_i^*)$ is the cocycle. Thus $H^0(K) = \mathbb{Z}$ and is generated by $\sum v_i^*$. It is clear that if c^1 is a cochain then c^1 is a cocycle since $0 \xleftarrow{\delta} C^1(K)$. We next claim that c^1 is cohomologous to some multiple of e_1^* and that e_i^* is cohomologous to e_1^*. Indeed, $\delta(v_3^* + v_0^*) = e_3^* - e_4^* + e_4^* - e_1^* = e_3^* - e_1^*$. Furthermore, no non-zero multiple of e_1^* is a coboundary. For the sake of contradiction, assume that $ne_1 = \delta c^0$ for some c^0. There are two ways to see this. The first is that we know that all coboundaries must have the same coefficients for e_i^* and since the coefficient of e_j^* is zero for $j \neq 1$ then $n = 0$. On the other hand, we could also note that $n = \langle ne_1^*, e_1 + e_2 + e_3 + e_4 \rangle = 0$ thus $H^1(K) = H^0(K) = \mathbb{Z}$.

Example 3.2.5. We have the following cochain complex

$$0 \leftarrow \mathbb{Z} \oplus \mathbb{Z} \leftarrow \delta_2 \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \leftarrow \delta_1 \leftarrow 0$$

and computing the δv^* we see

$$\langle \delta v^*, a \rangle = 0, \quad \langle \delta v^*, b \rangle = 0, \langle \delta v^*, c \rangle = 0 \tag{3.4}$$

thus $H^0(\mathbb{T}^2) = \mathbb{Z}$. Set $c^1 = n_1 a^* + n_2 b^* + n_3 c^*$ in which case

$$\langle \delta a^*, \sigma_1 \rangle = \langle a^*, c - b - a \rangle = -1, \quad \langle \delta a^*, \sigma_2 \rangle = \langle a^*, a - c + b \rangle = 1 \tag{3.5}$$

so that $\delta a^* = \sigma_2^* - \sigma_1^*$. In a sense, this can be gleamed from the figure by realizing that a occurs in the boundary of σ_1 with negative orientation and σ_2 with positive orientation. More generality, if c^1 is a cochain cocycle then

$$\begin{aligned}0 = \delta c^1 &= n_1(-\sigma_1^* + \sigma_2^*) + n_2(-\sigma_1^* + \sigma_2^*) + n_3(\sigma_1^* - \sigma_2^*) \\ &= \sigma_1^*(-n_1 - n_2 + n_3) + \sigma_2^*(n_1 + n_2 - n_3)\end{aligned}$$

so we have a two-parameter family with $n_3 = n_1 + n_2$ with

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = n_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + n_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Hence we have that the basis for the kernel is given by the vectors above, translated as $e_1^* + e_3^*$ and $e_2^* + e_3^*$ and we conclude that $H^1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$. Now σ_1^*, σ_2^* are cocycles and σ_1^* is cohomologous to σ_2^*. In particular, set $\sigma_1^* - \sigma_2^* = \delta c^*$. We claim that no multiple of σ_1^* is a coboundary. For the sake of contradiction, assume that $n\sigma_1^* = \delta c^1$ and notice that

$$n = \langle n\sigma_1^*, \sigma_1 + \sigma_2 \rangle = \left\langle \delta c^1, \sigma_1 + \sigma_2 \right\rangle = \left\langle c^1, \partial(\sigma_1 + \sigma_2) \right\rangle = 0$$

as required. Thus $H^2(\mathbb{T}^2) = \mathbb{Z}$.

Up to this point, the homology and the cohomology groups have agreed. The next example indicates that this is not always the case.

Example 3.2.6. Consider the Klein bottle which yields the following chain

$$0 \leftarrow \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\delta_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\delta_1} \mathbb{Z} \leftarrow 0.$$

Let $c^1 = n_1 a^* + n_2 b^* + n_3 c^*$ so that

$$\delta c^1 = n_1(-\sigma_1^* + \sigma_2^*) + n_2(-\sigma_1^* - \sigma_2^*) + n_3(\sigma_1^* - \sigma_2^*) = 0$$

so that $n_1 = n_3$ and $n_2 = 0$ telling us that we have a one-parameter family

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = n_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

so that $H^1(\mathbb{K}) = \mathbb{Z}$. σ_1^* is cohomologous to σ_2^* so $\delta\sigma_1^* = \delta(-b^* + c^*)$ so $H^2(\mathbb{K}) = \mathbb{Z}_2$. Summarizing, $H^0(\mathbb{K}) = \mathbb{Z}, H^1(\mathbb{K}) = \mathbb{Z}, H^2(\mathbb{K}) = \mathbb{Z}/2$. We compare this to the homology groups $H_0(\mathbb{K}) = \mathbb{Z}, H_1(\mathbb{K}) = \mathbb{Z} \oplus \mathbb{Z}_2$ and $H_2(\mathbb{K}) = 0$.

3.3 Zero-Dimensional Cohomology

Theorem 3.3.1. *If K is a complex then $H^0(K;G)$ is the group of all 0-cochains c^0 such that $\langle c^0, v\rangle = \langle c^0, w\rangle$ whenever v, w are in the same connected component. In particular, if $|K|$ is connected the $H^0(K) \cong \mathbb{Z}$ which is generated by the cochain c^0 which is 1 on each vertex.*

Proof. We have the following short chain

$$\leftarrow C^0(K;G) \leftarrow 0$$

wherein $H^0(K;G) = \frac{Z^0(K;G)}{B^0(K;G)} = Z^0(K;G)$. Suppose that c^0 is a cocycle and let c_1 be a one-chain that connects v and w. Then

$$\begin{aligned} 0 &= \langle \delta c^0, c_1\rangle = \langle c^0, \partial c_1\rangle \\ &= \langle c^0, w - v\rangle = \langle c^0, w\rangle - \langle c^0, v\rangle \end{aligned}$$

which implies that $\langle c^0, w\rangle = \langle c^0, v\rangle$. Now suppose that c^0 is a cochain such that $\langle c^0, v\rangle = \langle c^0, w\rangle$ whenever v and w are in the same connected component. Our goal is to show that $\delta c^0 = 0$. Let e_{1i} be a one simplex, so that

$$\langle \delta c^0, e_{1i}\rangle = \langle c^0, \partial e_{1i}\rangle = \langle c^0, w - v\rangle = 0$$

which implies that $\delta c^0 = 0$ so that c^0 is a cocycle. ∎

Definition 3.3.2. If ϵ is the augmentation map $C_0 \xrightarrow{\epsilon} \mathbb{Z}$ then we can create the dual augmentation map $C^0(K, G) \xleftarrow{\tilde{\epsilon}} G$ called the *co-augmentation map*. We then define the *reduced cohomology* $\tilde{H}^0(K; G) = \frac{\ker \delta_1}{\operatorname{im} \tilde{\epsilon}}$.

Theorem 3.3.3. *If $|K|$ is connected then $\tilde{H}^0(K; G) = 0$. In general, for any complex K we have $H^0(K; G) \cong \tilde{H}^0(K; G) \oplus G$.*

Definition 3.3.4. If K is a complex and K_0 is a subcomplex, the group of relative cochains is $C^p(K, K_0; G) = \operatorname{Hom}(C_\phi(K, K_0); G)$. Given the relative boundary homomorphism $C_p(K, K_0) \xrightarrow{\partial} C_{p-1}(K, K_0)$ we can define the relative coboundary homomorphism $C^p(K, K_0; G) \xleftarrow{\delta} C^p(K, K_0; G)$, which allows us to define the p^{th} relative cohomology groups $H^p(K, K_0; G) = \frac{Z^p(K, K_0; G)}{B^p(K, K_0; G)}$.

It is natural to consider $C^p(K, K_0; G)$ as a subgroup of $C^p(K, G)$ by restricting cochains of K to K_0. This given the following chain complexes

$$0 \to C_p(K_0) \xrightarrow{i} C_p(K) \xrightarrow{j} C_p(K, K_0) \to 0$$

and

$$0 \leftarrow C^p(K_0) \xleftarrow{\tilde{i}} C^p(K) \xleftarrow{\tilde{j}} C^p(K, K_0) \leftarrow 0.$$

There is a continuous map $f : (K, K_0) \to (L, L_0)$ such that $f_* : H_p(K, K_0) \to H_p(L, L_0)$ and its dual map $f^* : H^p(K, K_0) \to H^p(L, L_0)$.

These dual maps satisfy the properties of a contravariant functor. In particular, if i is the identity map that i^* is the identity map, and $(f \circ g)^* = g^* \circ f^*$.

Theorem 3.3.5. *If K is a complex and K_0 is a complex then there exists an exact sequence*

$$\cdots \leftarrow H^p(K_0; G) \leftarrow H^p(K; G) \leftarrow H^p(K, K_0) \leftarrow \partial^* H^{p-1}(K_0; G) \leftarrow \cdots$$

Example 3.3.6. Consider the figure given in Example 2.1.18 illustrated in Figure 2.3, with $K_0 = |K|$. We have the following long exact sequence of cohomology

$$H^2(K_0) \leftarrow H^2(K) \leftarrow H^2(K; K_0) \leftarrow H^2(K_0) \leftarrow H^1(K) \leftarrow H^1(K; K_0).$$

Now $C^2(K, K_0)$ is generated by σ_1^* and σ_2^* each of which is a cocycle. Furthermore, they are cohomologous as $\delta e_5^* = \sigma_1^* - \sigma_2^*$ and so $H^2(K, K_0) \cong \mathbb{Z}$. To compute $H^1(K, K_0)$ we note that $C_1(K, K_0)$ is generated by e_5^* whose coboundary is $\sigma_1^* - \sigma_2^*$ which is certainly non-trivial, hence $H^1(K, K_0) = \{0\}$. Finally $H^0(K, K_0) = \{0\}$. Translating this into our long exact sequence gives

$$0 \leftarrow H^2(K) \leftarrow \mathbb{Z} \xleftarrow{\delta} \mathbb{Z} \leftarrow H^1(K) \leftarrow 0.$$

In order to fill in the details regarding $H^1(K)$ and $H^2(K)$ we need to know the details of δ. Let e_1^* be the generator for $H^1(K_0)$ in which case $\delta e_1^* = -\sigma_1^*$ which is a generator for $H^2(K, K_0)$ and hence δ is an isomorphism. Thus $H^1(K) = H^2(K) = \{0\}$.

3.4 Cup Products

Definition 3.4.1. Let X be a topological space and define $S^p(X; R) = \mathrm{Hom}(S_p(X), R)$. We define a map $\sqcup : S^p(X, R) \times S^q(X, R) \to S^{p+q}(X, R)$ called the *cup product* as follows: if $T : \Delta_{p+q} \to X$ is a singular $(p+q)$-simplex then

$$\langle c^p \cup c^q, T \rangle = \langle c^p, T \circ C(\epsilon_0, \ldots, \epsilon_p) \rangle \langle c^q, T \circ C(\epsilon_p, \ldots, \epsilon_{p+q}) \rangle . \tag{3.6}$$

Theorem 3.4.2. *The cup product is bilinear and associative. As a consequence, the cochain z^0 whose value is one on each singular zero-simplex acts as the unity element. Furthermore,*

$$\delta(c^p \sqcup c^q) = (\delta c^p) \sqcup c^q + (-1)^p c^p \sqcup (\delta c^q).$$

Thus the cochain cup product induces an operation

$$H^p(X, R) \times H^q(X, R) \xrightarrow{\sqcup} H^{p+q}(X, R)$$

and defining $H^*(X; R) = \bigoplus H^i(X; R)$ combined with the cup-product is a ring.

Example 3.4.3 (Cohomology Ring of a Torus)**.** Let α be the middle column and β be the middle row. Now α, β generate $H^1(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$ generated by $\lambda = \{\sigma^*\}$ where σ is any

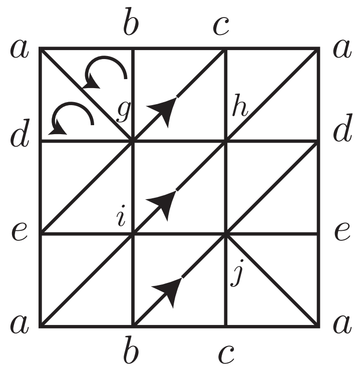

Figure 3.1: .

of two simplices. Then the multiplication table for the cohomology ring is as follows:

	α	β	Λ
α	0	$-\Lambda$	0
β	Λ	0	0
Λ	0	0	0

Now $w^1 \sqcup z^1$ acts on each two-simplex as

$$\begin{aligned}\left\langle w^1 \sqcup z^1, [g,h,i]\right\rangle &= \left\langle w^1, [g,h]\right\rangle \left\langle z^1, [h,i]\right\rangle = 1 \cdot 1 = 1 \\ \left\langle w^1 \sqcup z^1, [h,i,j]\right\rangle &= \left\langle w^1, [h,i]\right\rangle \left\langle z^1, [i,j]\right\rangle = (-1) \cdot 0 = 0\end{aligned}$$

and so $w^1 \bigsqcup z^1 = [g,h,i]^* = -\Lambda$.