MAT401 - 2008 Spring - Term Test Solutions

Problem 1: First of all, since multiplication (mod 10) is commutative, our ring is commutative. This is a prerequisite for the existence of a unity. Now to find out whether the ring has a unity, we simply construct the multiplication table with all possible pairs of elements. There are $5+4+3+2+1=15$ such independent pairs.

$\times$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

It is clear from this table that $6x = x \pmod{10}$ $\forall$ x $\in \{0,2,4,6,8\}$ so 6 is the unity of the ring. Also we see that no other element satisfies the requirements of being a unity, which is consistent with the property that a unity, if it exists in a ring, is unique.

Problem 2: refer to Chapter 13 of the textbook, in particular the definitions on p. 248 and 250, as well as Theorem 13.2

Problem 3: refer to Theorem 14.4 in the textbook and Theorem 2 as presented in the 23 Jan lecture

Problem 4: Suppose we have a ring homomorphism $\phi : \mathbb{Z}_6 \to \mathbb{Z}_{10}$. Let $\phi(1) = t \in \mathbb{Z}_{10}$. By the fact that ϕ preserves the operation of addition, for any $n \in \mathbb{Z}_6$ we have

$$\phi(n) = \phi(n1) = n\,\phi(1) = nt\,(mod\,10)$$

which completely defines the action of the homomorphism on all elements of $\mathbb{Z}_6$. We can narrow down the possibilities for t because

$$\phi(1) = t = \phi(1^2) = \phi(1)\phi(1) = t^2$$

so t must satisfy $t^2 = t\,(mod\,10)$ meaning $t \in \{0,1,5,6\}$. Finally we must impose the requirement that for any a, $b \in \mathbb{Z}_6$, $\phi(a+b) = \phi(a) + \phi(b)$ and

$\phi(ab) = \phi(a)\phi(b)$. Take, for example, $a + b = 6q_1 + r_1$.

$$\begin{aligned}
\phi(a+b) &= \phi(r_1) \\
&= t[(a+b) - 6q_1] \\
&= ta + tb - 6q_1 t \\
&= \phi(a) + \phi(b) - 6q_1 t
\end{aligned}$$

Likewise with $ab = 6q_2 + r_2$

$$\begin{aligned}
\phi(ab) &= \phi(r_2) \\
&= t[(ab) - 6q_2] \\
&= tab - 6q_2 t \\
&= t^2 ab - 6q_2 t \\
&= \phi(a)\phi(b) - 6q_2 t
\end{aligned}$$

so if we require $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$, where the equalities are $(mod\, 10)$, then we must have $6q_1 t = 6q_2 t = 0\ (mod\, 10)\ \forall\ q_1$, q_2. This implies $10 \mid 6t$ which further restricts our choices to $t \in \{0, 5\}$ ■

Problem 5: Let us define

$$g(x) \equiv f(x) - [\frac{f(b) - f(a)}{b - a}(x - a) + f(a)]$$

and observe that

$$g(a) = f(a) - f(a) = 0$$

$$g(b) = f(b) - [f(b) - f(a) + f(a)] = 0$$

Now because $g(a) = 0$ we can write

$$g(x) = (x - a)q_1(x)$$

In addition we have

$$g(b) = (b - a)q_1(b) = 0$$

and since $a \neq b$, this means that $q_1(b) = 0$ because $F[x]$ is a domain. Therefore we can write

$$q_1(x) = (x - b)q_2(x)$$

$$g(x) = (x-a)(x-b)q_2(x)$$

and hence $f(x)$ can be written as

$$\begin{aligned} f(x) &= g(x) + \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \\ &= (x-a)(x-b)q_2(x) + \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \end{aligned}$$

Finally, we know when $f \in F[x]$ is divided by $(x-a)(x-b)$ the expression

$$f(x) = (x-a)(x-b)q(x) + r(x)$$

is unique, where $r(x)$ is the remainder and $deg[r(x)] < deg[(x-a)(x-b)] = 2$. Therefore it is clear by inspection that the remainder is

$$r(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

So if we divide $f(x) = x^{2008}$ by $x^2 - 1 = (x+1)(x-1)$ we have $a = -1$ and $b = 1$ so

$$r(x) = \frac{f(1)-f(-1)}{1-(-1)}(x-(-1)) + f(-1)$$

But clearly $f(1) = f(-1) = 1$ so $r(x) = 1$ ■

Problem 6: Eisenstein's criterion says that for $f \in \mathbb{Z}[x]$ where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ...a_1 x + a_0$$

if $\exists$ a prime p such that p $\mid a_n$, p $\mid a_j$ for $j \in \{0, 1, 2...n-1\}$ and $p^2 \nmid a_0$, then f is irreducible over $\mathbb{Z}$. Since polynomials irreducible over $\mathbb{Z}$ are also irreducible over $\mathbb{Q}$, f is thus also irreducible over $\mathbb{Q}$. Now we simply observe that in our case f satisfies Eisenstein's criterion for $p = 5$, so it is irreducible over $\mathbb{Q}$ ■