Repeated Moral Hazard with Worker Mobility via Directed On-the-Job Search

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Abstract

This paper develops a search theoretic model of employment contracts with repeated moral hazard and analyzes how workers' incentives inside a firm interact with their mobility in the labor market. In equilibrium, the firm's incentives to induce effort as well as to retain the worker generate an optimal long-term wage contract that has an increasing wage-tenure profile. The optimal incentive compatible effort level and the resulting worker productivity both increase with tenure. In addition, the theory makes predictions about how the contractual structure interacts with macroeconomic behaviors. In particular, it highlights a mechanism by which incentives and search frictions generate workers' career concerns and productivity dispersion among ex ante identical agents. Moreover, it shows that a temporary reduction in workers' cost to exerting effort propagates through equilibrium dynamics and yields persistent effects on the economy's average productivity.

Keywords: Employment Contract, Repeated Moral Hazard, Directed On-the-Job Search, Worker Mobility, Career Concern,

JEL Codes: D8, E24, J3, J6.

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1 Introduction

Workers typically change jobs many times in their lifetimes. Even if already working for a firm, they can look for a better employer in the labor market. Indeed, in the US, job-to-job transitions constitute 49% of all separation from employers over the past decade (Nagypál 2008). When workers are mobile, their incentives at their current jobs depend not only on the current contract but also on their career concerns–concerns about the effects of current performance on future labor market outcomes, such as a probability of getting a desirable job. The firm needs to take the mobility and resulting career concerns into account when offering a long-term wage contract. On the other hand, a worker's outside options in the market are the contracts offered by other firms that face similar considerations. Therefore, the contracts offered by individual firms and workers' outside options influence each other and should be determined in equilibrium. Addressing these interactions in a consistent equilibrium framework is the primary goal of this paper. Moreover, the proposed model enables to analyze how workers' incentives affect important macroeconomic variables, such as wage and productivity dispersions.

For this purpose, I develop a model of dynamic model which integrates an optimal contracting problem into an equilibrium search framework. In my model, firms are risk-neutral and workers are risk-averse, and both are ex ante homogeneous. The firms enter the labor market competitively and offer long-term wage contracts to attract workers. A contract specifies a wage profile that depends on the worker's tenure in the contract. Workers, both employed and unemployed, observe these offers and choose which contract to try to obtain. Once a firm and a worker form a match and sign a contract, they perform a series of projects. In each period, the firm pays the promised wage at the beginning, and the worker provides unobservable and costly effort on the project. A project results in either success or failure, with the probability of success being increasing in the worker's effort. The match continues as long as its projects keep succeeding. A failure, however, physically destroys the match, and in that case the worker needs to search for a new job in the market.

There are two types of frictions in this economy: search frictions and informational frictions due to moral hazard. On the one hand, search frictions create restricted mobility in the labor market, and because of on-the-job search, a worker's current contact determines the set of contracts that are desirable for the worker. Restricted mobility, combined with workers' on-the-job search, is the key source of workers' career concerns. On the other hand, given moral hazard, match productivity and thus job destruction probabilities are determined by the incentive structure. Therefore, firms need to design a long-term contract that has the optimal chance of retaining the worker and, at the same time, induces the desired amount of effort.

In equilibrium, there is a non-degenerate distribution of contract offers in the market, due to search frictions and workers' on-the-job search. I show the existence of an equilibrium in which the workers' optimal decisions and the firms' optimal contracts are independent of the distribution of workers over different contracts. This property allows a very simple characterization of the optimal contract. First of all, the firm's incentives to induce effort as well as to retain the worker generate an optimal long-term wage contract that has an increasing wage-tenure profile. Moreover, the optimal incentive compatible effort level also increases with tenure, which implies that the match productivity is increasing over time. This productivity increase is driven solely by the evolution of incentives, and not by human capital accumulation or learning on the job.

The model also illustrates how workers' incentives–the implicit incentives from career concerns and the explicit incentives from the wage contract–evolve over time. As wages increase with tenure, his explicit incentives increase. However, his implicit incentives from career concerns decrease as he finds more difficult to get a better job in the market. Nonetheless, the result suggests that his total incentives increase with tenure. This evolution of incentives implies job-turnover dynamics that is decreasing with tenure in the contract. First, due to the increasing explicit incentives within the firm and decreasing market opportunities, workers' job-to-job transition rate is decreasing. Second, due to the increasing effort provision and thus increasing productivities, the job destruction probability is decreasing with tenure. This, in turn, implies that their job-to-unemployment turnover is also decreasing.

Moreover, in this frictional labor market with workers' on-the-job search, ex ante identical workers endogenously sort themselves into different contracts and, thus, face different incentives. Therefore, non-degenerate productivity distributions could arise naturally as an equilibrium outcome through the incentive mechanisms. The independence property of the equilibrium enables to show that a temporary reduction in workers' cost to exerting effort propagates through equilibrium dynamics and yields persistent effects on the economy's average productivity.

This paper builds on models in the literature of dynamic contracting (See, for example, Rogerson 1985, Holmstrom and Milgorm 1987, Spear and Srivastava 1987, Sannikov 2008) and equilibrium labor search (See, for example, Burdett and Mortensen 1998, Burdett and Coles 2003, and Shi 2009). How I formulate the repeated moral hazard problem within a match is closely related to Spear and Srivastava (1987). They analyze an infinitely repeated agency model and show that there is a recursive representation of the optimal contract with the agents' conditional discounted expected utility as a state variable. However, analyzing dynamic incentive problems in a consistent market equilibrium context is a difficult task even with this simplified recursive approach. The difficulty comes from the fact that workers' mobility is endogenously determined in equilibrium and, therefore, the workers' distribution is an important state variable for the optimal contracting problem. The useful framework introduced in Shi (2009) and Menzio and Shi (2009) makes it possible for me to analyze such a recursive contracting problem in an equilibrium search framework. By modelling search as a directed process, they introduce the class of equilibria in which the workers' optimal decisions and the firms' optimal contracts are independent of the distribution of workers over different contracts. This independence property, called *block recursivity*, is the key to find an equilibrium in this unified framework.

Recent research on equilibrium labor search also studies the interactions between employment contracting problems and frictional mobility via on-the-job search. However, they focus on worker behavior in the market and do not consider issues of asymmetric information within the contractual relationship. The unified framework of this paper addresses these issues of the dynamic contracting and the labor search literature jointly and provides useful implications that each cannot explain independently.

Many papers place a contracting problem in a market equilibrium context. Early such attempts include Harris and Holmstrom (1982) and Holmstrom (1983) which characterize equilibrium long-term wage contracts. Shapiro and Stigliz (1984) show that unemployment can act as a worker discipline device when it is costly to observe their on-the-job effort. The literature on career concerns also studies incentive contracts in a market context (Gibbons and Murphy 1992 and Holmstrom 1999). These papers assume competitive labor market without trading friction, but the present paper shows that search frictions provide new insights into the incentive mechanism in an equilibrium framework.

More recently, the literature on relational contracts (See, for example, MacLeod and Malcomson 1998, Board 2007, and Fuchs 2007) analyzes situations where neither the firm nor the worker can commit to a contractual relationship and can walk away from it in any period. These studies show that a long-term relationship acts as a self-enforcing mechanism, and even an informal agreement on wages can provide incentives to the worker. Board (2007) incorporates on-the-job search into the model and derives implications that are similar to mine.

Many studies address the relationship between wages and productivity. Using a large longitudinal data of French workers and firms, Abowd, Kramarz, and Margolis (1999) find a positive correlation between wages and productivity after controlling for observable variation. Even though they do not consider the issue of incentive provision, their finding is consistent with the implication of this paper. Lazear and Moore (1984) analyze how the shape of an age-earnings profile affects the productivity of workers and argue that most of the increasing age-earnings profile is accounted for by the firm's desire to provide its employee with incentives.

The equilibrium search literature offers useful frameworks to study wage distributions. Theoretically, for example, Burdett and Coles (2003) and Shi (2009) study workers' on-the-job search and generate nondegenerate equilibrium wage distributions among identical firms and workers. Acemoglu and Shimer (2000) argue that search frictions induce the firms to choose heterogeneous technologies to attract workers. This firm heterogeneity, in turn, generates heterogeneous wage offers in the market in equilibrium.

There are also empirical studies that analyze wage distributions in an equilibrium search framework (See, for example, Van den Berg and Ridder 1998 and Postel-Vinay and Robin 2002). Van den Berg and Ridder show that heterogeneous productivity among workers is important to generate an empirically reasonable wage distribution with their model. A recent paper by Bagger and Lentz (2008) assume workers with permanent skill differences and firms with productivity differences and generates wage dispersion through an equilibrium sorting in a frictional labor market. These papers assume exogenously given heterogeneity among workers and/or firms. and they do not show explicitly how such heterogeneity could arise in a model with identical workers and firms.

2 A Model of Labor Market with Search Friction and Moral Hazard

2.1 Physical Environment

I consider a labor market with a continuum of infinitely lived workers with measure 1 and a continuum of firms whose measure is determined by competitive entry. All workers and firms are ex-ante homogeneous. Time is discrete and continues forever. Each worker has a utility function $u(w)$ where w is income in a period. I assume that $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, weakly concave. I further assume that the first derivative is bounded, i.e., $u'(w) \in [\underline{u}', \bar{u}']$ for all w. When employed, each worker exerts costly effort, $e \in \mathbb{R}_+$, for the project of the firm in each period. Worker's effort is unobservable to the employer. I assume that the cost of effort by a worker is given by a function $c : \mathbb{R} \to \mathbb{R}$ that is twice continuously differentiable, strictly increasing, and weakly convex. Each worker maximizes the expected sum of lifetime utilities minus costs of effort discounted at the rate $\beta \in (0, 1)$.

Each firm is endowed with a series of projects. One project is executed in each period if the firm hires a worker, and it results in one of two possible outcome: $\{0, y\}$. When outcome is y it is called a "success," and when 0 a "failure." The probability of success in each period depends on an effort level by the worker employed in the firm and is given by $r(e)$. I assume that $r : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and weakly concave in e. I also assume that $r'(0) < \infty$ and $\lim_{e\to\infty} r(e) = 1$. Each firm maximizes the expected sum of profits discounted at the rate β.

2.2 Contractual Environment and the Labor market

I assume that firms commit to a long-term contract while workers can quit a job at any time. An employment contract specifies the worker's wage as a function of his tenure in the firm[1]. In particular, when an employed worker receives a better offer in the market, the current firm does not respond to the employee's outside offers.

There is a continuum of labor markets indexed by x, where x denotes the value of contract offered to a worker in that submarket, i.e., whenever a vacant firm meets a worker in that submarket, it has to offer him a long-term wage contract that gives him the lifetime utility of x given that the worker stays on the job. As mentioned, a wage in each period does not depend on the realization of the current project outcome because the worker receives up-front period payments at the beginning of each production stage. Therefore, firms need to design a long-term contract so that it induces the worker to stay on the contract as well as to exert a desired level of effort. I assume that $x \in X \subseteq \mathbb{R}_+$. Let G be a cumulative distribution of workers over X and u a fraction of unemployed workers. The ratio of vacant firms to searching workers in submarket x is denoted by $\theta(x)$ and is referred to as the tightness of submarket x.

[1]Therefore, a period wage does not depend on the outcome of that period. There are several interpretations about this assumption in the literature, such as unverifiable or subjective outcomes. However, I simply assume that the wage is paid at the beginning of each period.

2.3 Job Search and Employment Relationship

In each period, there are two stages: search and matching, and production. In a search and matching stage, firms post a vacancy at a flow cost $k > 0$ and offer a long-term contract to recruit a worker. Firms offer contracts with different values to attract workers, and each value will form a submarket. Workers are allowed to search for a new contract both on- and off-the job. If employed, a worker receives the opportunity of searching for a new job with probability $\lambda_e \in [0,1]$, and with probability $\lambda_u \in [0,1]$ if the worker is unemployed. I assume a standard matching technology as in the job-search literature. If a worker receives the opportunity of searching and chooses to visit submarket x, he meets a vacant firm with probability $p(\theta(x))$, where $p : \mathbb{R}_+ \to [0,1]$ is twice continuously differentiable, strictly increasing and strictly concave and such that $p(0) = 0, p'(0) < \infty$. On the other hand, if a vacant firm enters a submarket x, it finds a worker with probability $q(\theta(x))$, where $q : \mathbb{R}_+ \to [0,1]$ is twice continuously differentiable, strictly decreasing and strictly convex and such that $\theta^{-1}p(\theta) = q(\theta), q(0) = 1$, and $p(q^{-1}(\cdot))$ is concave. If an employed worker receives the opportunity of searching and matches with a firm and accepts the offer, he must leave his previous employment position before entering the production stage with a new firm. If he rejects the offer, he enters the production stage with his current employer.

In the production stage, each unemployed worker receives and consumes unemployment benefit, b. I assume that each employed worker receives the current period wage at the beginning of the period before the production takes place. Therefore, the wage cannot depend on the current project outcome as in standard moral hazard literature. This assumption is made to clarify the effect of a structure of a long-term contract on worker behavior. At the end of this stage, the project outcome, which stochastically depends on the worker's effort, is publicly realized. If the project succeeds, the match stays together and the worker keeps his current employment position in the next period. Projects continue forever as long as they are succeeding. However, a project failure destroy the match, and the worker looses his employment position and becomes unemployed. This new unemployed worker will not receive an opportunity of searching in the following period and needs to stay unemployed for one period. There is no exogenous separation of the match, and a failure is the only reason of separation[2].

[2]The following results are qualitatively unchanged as long as the firm commit to a firing rule under which the firm fires the worker with strictly positive probability after observing a project failure. I assume this particular separation rule for expositional simplicity.

2.4 Equilibrium Conditions

In the following, I describe the equilibrium conditions for this economy. Following the approach taken by Spear and Srivastava (1987), I will set up the optimal contracting problem as a recursive problem. First, I will explain a market tightness from firm's free entry condition. Second, given the market tightness function, I will illustrate worker's problems in terms of optimal job search and optimal effort making when employed. Third, I will give the value of unemployed worker. Bringing all these elements together, I will present the firm's optimal contracting problem.

2.4.1 Market Tightness and Free Entry Condition

During the search stage, firms choose how many vacancies to create and where to locate them. Let $J(x)$ denote a firm's value of employing a worker in submarket x. Then, the firm's expected benefit of creating a vacancy in submarket x is given by $q(\theta(x))J(x)$, the product of the probability and the value of meeting a worker in the submarket. Given the market tightness function $\theta(x)$, if the cost k of creating a vacancy is strictly greater than the expected benefit, then firms do not create any vacancies in submarket x. If k is strictly smaller than the expected benefit, then firms create infinitely many vacancy in x. When they are equal, then firms expected profits are zero and independent from the number of vacancies they create in submarket x. Therefore, in any x, $\theta(x)$ is consistent with firms' profit maximization if

$$q(\theta(x))J(x) - k \leq 0, \tag{1}$$

and $\theta(x) \geq 0$, with complementary slackness. The complementary slackness condition ensures that if the expected benefit is less than the cost of creating vacancy, then no firms create vacancy in submarket x; that is $\theta(x) = 0$. Also, free entry and exit condition drives the maximized expected profit to zero.

2.4.2 Worker's Problems

Optimal Search of the Worker

Suppose a worker's current value of employment position is V. If he receives the opportunity to search for a new contract and visits submarket x, he find an employer with probability $p(\theta(x))$ and yields the additional value of $x - V$. The worker chooses which submarket to visit to maximize the expected value of search. I will denote the worker's value of search as

$$D(V) = \max_{x \in \mathbb{R}} p(\theta(x))(x - V), \tag{2}$$

given the current value of employment position V. I denote with $m(V)$ the worker's optimal search policy of this problem, and denote the composite function $\hat{p}(V) = p(\theta(m(V)))$.

Optimal Effort Choice of the Worker

Consider a worker in the production stage. He must choose how much effort to provide for the current project. If the project succeeds, the current contract will provide a continuation value to the worker, W, and it will be the worker's reservation value from the current contract. In the next search stage, if he receives the opportunity to search with probability λ_e, he will obtain the expected value of $W + D(W)$ through the optimal search explained above. If he does not receive the opportunity to search, he keeps the reservation value W. On the other hand, if the project fails, the worker will loose the job in the next period and spend as an unemployed receiving the value of unemployed U as explained below. Hence, the net expected value of success with the effort level e is

$$r(e)(\lambda_e(W + D(W)) + (1 - \lambda_e)W) + (1 - r(e))U.$$

The cost of effort level e is $c(e)$. Given the structure of the contract, current period wage is independent of the project outcome and does not affect the worker's optimal effort choice. Therefore, given the continuation value W, the worker will choose his effort level to solve:

$$\max_{e \in \mathbb{R}} \Big(-c(e) + \beta\big(r(e)(W + \lambda_e D(W)) + (1 - r(e))U\big)\Big).$$

Note that the expected benefit from the effort is given in the next period and the worker discount its value at β.

Worker's Value of Unemployment

Finally, consider an unemployed worker at the beginning of the production stage. The worker obtains $u(b)$ from the unemployment benefit. Let U be the value of unemployed. If he receives an opportunity to search for a job in the next period, he will obtain the expected benefit of $U + D(U)$ through the optimal search. If he does not receive an opportunity to search, he will stay unemployed and receive U again. Therefore, the value of unemployed is expressed recursively as

$$U = u(b) + \beta\big(U + \lambda_u D(U)\big). \tag{3}$$

2.4.3 The Firm's Optimal Contracting Problem

Consider a firm that promises to provide a continuation value V in this period. Let $J(V)$ denote the current value of a contract for the firm. The firm chooses: i) w, how much wage to pay in this period, ii) e, how much effort to induce, and iii) W, how much continuation value to provide to the worker in the next period conditional on the survival of the relationship. I also allow for randomization over these choices, that is, the firm offers two sets of subcontract and a probability distribution, $\{\pi_i\}_{i=1,2}$, over these subcontracts. Denote by $\xi = (\{w_i, e_i, W_i, \pi_i\}_{i=1,2})$ the contract offered by the firm at the beginning of a period.

The firm's optimal contracting problem is given by

$$J(V) = \max_{\xi} \sum_{i=1,2} \pi_i \left\{ r(e_i)y - w_i + \beta r(e_i)(1 - \lambda_e \hat{p}(W_i))J(W_i) \right\} \tag{4}$$

subject to

$$\begin{aligned}
\xi \in \Xi = \Big\{ & \{w_i, e_i, W_i, \pi_i\}_{i=1,2} : \quad W_i \in X \quad \text{for } i = 1,2 \\
& V = \sum_{i=1,2} \pi_i \left\{ u(w_i) - c(e_i) + \beta[r(e_i)(W_i + \lambda_e D(W_i)) + (1 - r(e_i))U] \right\} \\
& e_i \in \arg\max_{e \in \mathbb{R}} \Big(- c(e) + \beta\big(r(e)(W + \lambda_e D(W)) + (1 - r(e))U\big)\Big), \quad \text{for } i = 1,2 \\
& \pi_1 + \pi_2 = 1, \quad \pi_i \in [0,1] \quad \text{for } i = 1,2 \Big\}.
\end{aligned}$$

By inducing an effort level e and paying w, the firm's expected current period payoff is $r(e)y - w$. In the next period, by offering continuation value W, the current worker will stay in the current firm with probability $r(e)(1 - \lambda_e \hat{p}(W))$: product between a probability the project succeeds and a probability the worker will not leave for the outside option, and the firm will enjoy the value $J(W)$ of remaining contract.

To design the contract, the firm faces three constraints. The first is the promise-keeping or consistency constraint; that is, the contract has to provide the worker with the promised value V. Since the offer of the contract is evaluated ex-ante, only the expected value of subcontracts must provide the promised continuation value; either subcontract may fail to provide the promised value. The other constraint is that the effort that the firm wants to induce must be incentive compatible; that is, given the contract offered the worker voluntarily choose to exert that level of effort. Since realization of subcontract occurs before the worker chooses his effort level, both subcontracts the firm prepares need to meet the incentive compatibility constraint. Finally, when the firm randomizes the contracts, the probabilities assigned to each subcontract must sum to one. I denote with $\xi(V)$ the optimal policy functions given V associated with this contracting problem.

2.5 Block Recursive Equilibrium

Definition: A Recursive Equilibrium *is a set of functions* $\{J^*, \theta^*, D^*, m^*, U^*, \xi^*\}$ *and a distribution of workers* $\{(G_t^*, u_t^*)\}_{t\geq 0}$ *such that*

1. θ^* *satisfies condition (1),*
2. D^* *and* m^* *satisfies the functional equation (2),*
3. U^* *satisfies the functional equation (3),*
4. J^* *and* ξ^* *satisfies the functional equation (4), and*
5. $\{(G_t^*, u_t^*)\}_{t\geq 0}$ *is consistent with* m^* *and* ξ^*.

Since the definition is not limited to the stationary equilibrium, the distribution of workers over the value of contract changes over time depending on workers' mobility on the market as well as on the job. The last condition, therefore, requires that the evolution of distributions over time needs to be consistent with the optimal policy functions.

Lastly, I will define a useful class of equilibrium as follows.

Definition: A Block (Distribution-free) Recursive Equilibrium *is a recursive equilibrium such that functions* $\{J^*, \theta^*, D^*, m^*, U^*, \xi^*\}$ *are independent from the distribution of workers* (G_t^*, u_t^*) *in any period* $t \geq 0$.

3 General Properties and Existence of an Equilibrium

In this section, I first describe general properties of an equilibrium. I will then establish the existence of an equilibrium with these properties. Proofs of each lemma are almost identical to that in Menzio and Shi (2009) or require only minor modifications for taking worker's effort choice into account. I collect all the proofs extra steps or modifications in the appendix.

We start with specifying a set of functions with certain properties. Then, I take an arbitrary function from the set as a firm's value function and characterize the equilibrium objects given the properties of functions. With all these results, I construct an operator defined over the set of functions and show that any fixed point of the operator is a Block Recursive Equilibrium.

3.1 General Properties

We define the set $\mathcal{J}(X)$ of functions $J : X \to \mathbb{R}$ such that:

(i) $J(V)$ is strictly decreasing and Lipschitz continuous with respect to V,

(ii) $J(V)$ is bounded both from below and above, and

(iii) $J(V)$ is concave.

We can show that the set $\mathcal{J}(X)$ is non-empty, bounded, closed, and convex subset of the space of bounded, continuous functions on X with the sup norm.

3.1.1 Market Tightness and Free Entry Condition

First, I take an arbitrary firm's value function $J \in \mathcal{J}(X)$ and solve the equilibrium condition (1) with respect to the market tightness function θ. We get

$$\theta(x) = \begin{cases} q^{-1}(k/J(x)) & \text{if } \; J(x) \geq k \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Because q is a probability, q^{-1} is defined only on $[0, 1]$, that is $J(x) \geq k$. Since $J(x)$ is strictly decreasing, there exists a unique $\bar{x} \in \mathbb{R}$ such that $J(x) > k$ for all $x < \bar{x}$ and $J(x) < k$ for all $x > \bar{x}$. That is, offering any contract with the value more than $\bar{x}$ provide negative expected profits even if the firm can hire a worker with probability one. Therefore, no firm will enter submarkets with $x > \bar{x}$, and the market tightness takes nonnegative value only if $x \leq \bar{x}$. We call this threshold value a *bound of market* given J. Since J is bounded from above, θ is also bounded from above. Now I have the following properties of market tightness functions.

Lemma 3.1. *If $x < \bar{x}$, the market tightness function $\theta(x)$ is strictly positive and strictly decreasing, and Lipschitz continuous. $\theta(x) = 0$ for all $x \geq \bar{x}$*

3.1.2 Worker's Problem

Optimal Search of the Worker

In this section, I insert the market tightness function, $\theta(x)$, obtained from an arbitrary firm's value function $J \in \mathcal{J}(X)$ into the worker's search problem:

$$D(V) = \max_{x \geq V} p(\theta(x))(x - V).$$

Now, I have the following series of properties which enable to compute the optimal contracts.

Lemma 3.2. *For all $V \in X$, the worker's objective function $f(x; V) = p(\theta(x))(x - V)$ is strictly concave with respect to x.*

The concavity of the problem implies that, given a continuation value of the current contract, the worker will find unique submarket that he optimally visits.

Lemma 3.3. *The worker's optimal search strategy $m(V) \in \arg\max p(\theta(x))(x - V)$ is unique, weakly increasing and Lipshitz continuous.*

The lemma also implies that a worker searches in a submarket which offers a higher value the higher your continuation value from the current contract. This can be seen as a version of the single crossing property.

Lemma 3.4. *If $V < \bar{x}$, the worker's value of searching $D(V)$ is strictly positive and weakly decreasing and Lipschitz continuous. If $V \geq \bar{x}$, $D(V) = 0$.*

As long as a worker's current continuation value is less than $\bar{x}$, there are submarkets that offer higher values and they provide positive expected value of search. However, once his continuation value hits the bound, there are no outside firms offering better offer and the market tightness becomes zero, $D(V) = 0$ since $p(0) = 0$.

Lemma 3.5. *$\hat{p}(V)$ is weakly decreasing and Lipschitz continuous.Moreover $\hat{p}(\bar{x}) = 0$*

Remember that the composite function $\bar{p}(V)$ is a probability that a worker with a continuation value V from the current contract meets a firm in the optimally targeted submarket. Since a worker with higher V applies to a submarket with higher tightness, he is less likely to meet a firm in an optimally targeted submarket.

Optimal Effort Choice of the Worker

We now turn to a worker's optimal effort choice problem:

$$\max_{e\in\mathbb{R}} \Big(- c(e) + \beta\big(r(e)(W + \lambda_e D(W)) + (1 - r(e))U\big)\Big).$$

Under the assumption on $c(\cdot)$ and $r(\cdot)$, it is a concave problem with respect to e. Therefore, the first-order condition sufficiently characterizes the optimal effort.

Lemma 3.6. *Given a continuation value W, there is a unique level of optimal effort. The worker's optimal effort function is increasing in the continuation value of contract, W.*

Proof: First, for a given continuation value W, the worker's optimal effort is implicitly and given by

$$-c'(e) + \beta r'(e)(W + \lambda_e D(W) - U) = 0.$$

Define $\Omega(W) = W + \lambda_e D(W) - U$. Then I can write $\frac{c'(e)}{\beta r'(e)} = \Omega(W)$. Under the assumptions ($c(\cdot)$: convex, continuous, $r(\cdot)$: concave continuous, and $r' > 0$ everywhere), $\frac{c'}{\beta r'}$ is continuous and monotonically increasing, and thus invertible. Therefore, there exists a unique level of e that satisfies the equality.

Moreover the inverse function is also continuous. Hence, I write $e(W) = g(\Omega(W))$ where g is the inverse function of $\frac{c'}{\beta r'}$. Differentiating the composite function,

$$\begin{aligned}
\frac{\partial e(W)}{\partial W} &= g'(\cdot)\Omega'(W) \\
&= \left(\left(\frac{c'}{\beta r'}\right)^{-1}\right)' (1 - \lambda_e \hat{p}(W)) \\
&= \left(\frac{c''r' - c'r''}{\beta r'^2}\right)^{-1} (1 - \lambda_e \hat{p}(W)) \\
&= \frac{\beta r'^2}{c''r' - c'r''}(1 - \lambda_e \hat{p}(W)),
\end{aligned}$$

where the first equality is by the chain rule and the third by the inverse function theorem. Since the denominator of the right hand side is strictly positive by assumption, and thus the derivative is positive, the worker's optimal effort is increasing in W. □

3.2 Existence of a Block Recursive Equilibrium

So far, I have examined properties of equilibrium objects given an arbitrary firm's value function $J \in \mathcal{J}(X)$. In this subsection, I establish that a Block Recursive Equilibrium exists.

Proposition 3.1. *A Block Recursive Equilibrium exists.*

Detailed proof is given in the appendix. I will only outline the argument in the proof. First, by inserting all the previous equilibrium objects, given an arbitrary value function $J \in \mathcal{J}(X)$, into a firm's optimal contracting problem, I construct the following operator that maps from $\mathcal{J}(X)$ to some space of functions:

$$(TJ)(V) = \max_{\xi \in \Xi} \sum_{i=1,2} \pi_i \left\{ r(e_i)y - w_i + \beta r(e_i)(1 - \lambda_e \hat{p}(W_i))J(W_i) \right\}.$$

We show that T is a self-map, that is, T maps from $\mathcal{J}(X)$ to itself (Lemma 7.8). Then, I show that T is a continuous map (Lemma 7.9). These suffice to show that the operator T satisfies the assumptions of Schauder Fixed Point Theorem (Stokey and Lucas with Prescott, 1989, Theorem 17.4); that is, (i) T is continuous, (ii) the family of functions $T(\mathcal{J})$ is equicontinuous, and (iii) T maps the set $\mathcal{J}(X)$ into itself. Therefore, there exists a firm's value function $J^*(V) \in \mathcal{J}(X)$ such that $TJ^* = J^*$. Denote with $\{\theta^*, D^*, m^*, U^*, e^*, \xi^*\}$ the respective functions associated to J^*. By construction, the functions $\{J^*, \theta^*, D^*, m^*, U^*, e^*, \xi^*\}$ satisfy conditions in the definition of a Recursive Equilibrium and do not depend on the distribution of workers. Therefore, they constitute a Block Recursive Equilibrium.

4 Characterization of the Optimal Long-Term Contracts

We now characterize the optimal contract. Using the first-order approach, the constraint set is modified as follows:

$$\begin{aligned}
\xi \in \Xi' = \Big\{ & \{w_i, e_i, W_i, \pi_i\}_{i=1,2} : \quad W_i \in X \quad \text{for } i = 1,2 \\
& V = \sum_{i=1,2} \pi_i \left\{ u(w_i) - c(e_i) + \beta[r(e_i)(W_i + \lambda_e D(W_i)) + (1 - r(e_i))U] \right\} \\
& - c'(e_i) + \beta r'(e_i)(W_i + \lambda_e D(W_i) - U) = 0, \quad \text{for } i = 1,2 \\
& \pi_1 + \pi_2 = 1, \quad \pi_i \in [0,1] \quad \text{for } i = 1,2 \Big\}.
\end{aligned}$$

First, we consider only the promise-keeping constraint and the incentive compatibility constraint to obtain the following result.

Proposition 4.1. *Under the optimal contract, the current period wage is independent of the realization of lottery.*

Proof. Let $\eta(V)$ be the Lagrange multiplier for the promise-keeping constraint. The first order condition of the maximization problem with respect to w_i implies

$$\eta(V) = \frac{1}{u'(w_i)} \quad \text{for} \quad i = 1, 2.$$

This implies that $w_1 = w_2$. Therefore, the current wage, w, does not depend on the realization of the lottery. □

Given the above proposition, the firm's problem can be greatly simplified.

Lemma 4.1. (Reduction of the Problem) *Let* $\phi = (\{W_i, \pi_i\}_{i=1,2})$. *The optimal contracting problem can be reduced to the following optimization problem with respect to* ϕ.

$$J(V) = \max_{\phi} \left\{ -w(V, \phi) + \sum_i \pi_i r(e(W_i)) \Big[y + \beta(1 - \lambda_e \hat{p}(W_i)) J(W_i) \Big] \right\} \tag{6}$$

where

$$w(V, \phi) = u^{-1} \left(V - \beta U - \sum_i \pi_i \Big[-c(e(W_i)) + \beta r(e(W_i)) [W_i + \lambda_e D(W_i) - U] \Big] \right)$$

and

$$\phi \in \Phi = \Big\{ \{W_i, \pi_i\}_{i=1,2} : \; W_i \in X, \quad \pi_i \in [0,1] \; \text{ for } i = 1, 2 \; \text{ and } \pi_1 + \pi_2 = 1 \Big\}$$

Proof. Substituting the optimal effort function into both the objective function and the promise-keeping constraint eliminate the incentive compatibility constraint. Then, using the result that $w_1 = w_2$ makes the promise-keeping constraint as

$$V = u(w) + \beta U + \sum_i \pi_i \Big[-c(e(W_i)) + \beta r(e(W_i)) [W_i + \lambda D(W_i) - U] \Big].$$

Since u is strictly concave and thus invertible, this constraint can be solved for w. Substituting resulting expression for w into the objective function gives the desired form of unconstrained optimization problem. □

4.1 The Optimal Wage and Continuation Value Prfiles

We are interested in a shape of long-term wage-tenure profile in this environment. Let $\hat{x} \in X$ denote a value such that $J(x) < 0$ for all $x > \hat{x}$ and $J(x) > 0$ for all $x < \hat{x}$. Since J is strictly decreasing, there is a unique such value. As the bound for the market, $\bar{x}$, determines the firms' entry into the market, the threshold value $\hat{x}$ characterizes both the optimal wage-tenure profile and the market structure.

Proposition 4.2. *Under the optimal contract, wages are nondecreasing over the tenure if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$ they are strictly increasing, and they stay constant once the continuation value hits the value $\hat{x}$.*

Proof: First, ignoring the feasibility constraints for W_i and π_i, the first order condition of the reduced form of the problem (6) with respect to W_i is

$$\begin{aligned}
&\frac{1}{u'(w)}\pi_i \beta r(e(W_i))(1-\lambda_e \hat{p}(W_i)) \\
&+ \pi_i r'(e(W_i))[y + \beta(1-\lambda_e \hat{p}(W_i))J(W_i)]g'(\Omega(W_i))(1-\lambda_e\hat{p}(W_i)) \\
&+ \pi_i r(e(W_i))\beta\big[(1-\lambda_e\hat{p}(W_i))J'(W_i) - \lambda_e J(W_i)\hat{p}'(W_i)\big] = 0
\end{aligned}$$

where we use that $\frac{\partial \Omega(W)}{\partial W} = 1 - \lambda_e \hat{p}(W)$ and $e'(W) = g'(\Omega(W_i))\Omega'(W)$. Then, dividing through by $\beta, \pi_i, r(e), (1-\lambda_e\hat{p}(W_i))$ gives

$$\frac{1}{u'(w)} + \frac{r'(e(W_i))}{r(e(W_i))}\left(\frac{y}{\beta} + (1-\lambda_e\hat{p}(W_i))J(W_i)\right)g'(\Omega(W_i)) + J'(W_i) - \frac{\lambda_e J(W_i)\hat{p}'(W_i)}{1-\lambda_e\hat{p}(W_i)} = 0.$$

Now, from the first order condition of the constrained problem, we have $\eta(V) = \frac{1}{u'(w(V))}$. The theorem of Lagrangian multiplier implies that $J'(V) = -\eta(V)$, so we have

$$J'(V) = -\frac{1}{u'(w(V))}. \tag{7}$$

Shifting one period forward gives $J'(W_i) = -\frac{1}{u'(w(W_i))}$ where $w(W_i)$ is the wage in the next period when the current lottery realization is i.

Substituting it into the previous equation and rearranging the terms yield

$$\frac{1}{u'(w(W_i))} - \frac{1}{u'(w)} = -\frac{\lambda_e J(W_i)\hat{p}'(W_i)}{1-\lambda_e\hat{p}(W_i)} + \frac{r'(e(W_i))}{r(e(W_i))}\left(\frac{y}{\beta} + (1-\lambda_e\hat{p}(W_i))J(W_i)\right)g'(\Omega(W_i)). \tag{8}$$

The right hand side is positive since $\hat{p}'(W_i)$ is non-positive. Therefore, we have

$$\frac{1}{u'(w(W_i))} - \frac{1}{u'(w)} > 0. \tag{9}$$

Since u is concave function, this implies $w(W_i) > w$ for $i = 1, 2$. Hence, the next period wage is higher than the current wage.

However, ever increasing wage profile is not optimal; after some point, the firm is paying to the worker more than the project produce. The above equation is not sufficient for the complete characterization. Once a worker's value of contract reaches the value $\bar{x}$, in equilibrium, there is no other submarkets offering higher value, i.e., $\bar{p}(\bar{x}) = 0$, so the worker has no incentive to search on-the-job. Therefore, the firm does not need to induce the worker to stay in the contract. At this point, firm's optimal contracting problem becomes as if there is no on-the-job search on the worker's side. As can be easily shown, optimal wage profile is still increasing until the firms value becomes negative to induce higher effort (see Appendix B). However, if the worker's value reaches $\hat{x}$, inducing further effort by paying higher wage yields negative value for the firm. So, at this point, the firm does not have any further incentive to induce a higher effort, and thus it maintains the constant wage just to provide the value $\hat{x}$ to the worker as follows:

$$\hat{w} = u^{-1}\big(\hat{x} - \beta U + c(e(\hat{x})) - \beta r(e(\hat{x}))(\hat{x} - U)\big).$$

□

Corollary 4.1. *Under the optimal contract, the next period promised continuation values are nondecreasing over the tenure if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$, they are strictly increasing, and they stay constant once they hit the value $\hat{x}$.*

Proof: If $V < \hat{x}$, with (7), the inequality (9) implies $J'(W_i) - J'(V) < 0$. Then, concavity of J implies $W_i > V$ for $i = 1, 2$. If $V \geq \hat{x}$, optimal offer stays constant at $\hat{x}$. □

4.2 The Optimal Incentive Compatible Effort Profile

Proposition 4.3. *Under the optimal contract, the worker's optimal efforts are nondecreasing if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$, they are strictly increasing, and they stay constant once the continuation value hits the value $\hat{x}$.*

Proof: As shown in lemma 3.6, the optimal effort function is increasing in the continuation value of the contract. Since the continuation value of the contract is higher than the current value of the contract (Corollary (4.1)), the worker will provide higher effort next period than the current period effort if stays on the contract. □

5 Implications of the Results

The above section characterized the optimal long-term contract in this environment. I will discuss key features of the optimal contract and derive some important implications of these results.

5.1 Wage-Tenure Wages and Career Concerns

The equation (8) in the proof for the optimal wage schedule gives explicit mechanics of back-loaded wage-tenure profile. The first term considers the worker's mobility. Given the worker's mobility, a firm has an incentive to backload wages to entice the worker to stay; as wages rise with tenure, it is more difficult for the worker to find a better offer elsewhere, and so the worker's quit rate falls (see Shi 2008). On the other hand, the second term captures the worker's moral hazard problem. The mechanism is similar to deferred compensation scheme discussed in Lazear and Moore (1986); since the firm gives incentive only through a long-term wage streams and the current wage does not affect the worker's current period incentive, the firm has incentive to put higher weight on the later period wages so as to provide a promised value to the worker.

This increasing wage-tenure wages has an important implication for the worker's career concerns. Workers' career concerns can be seen as their desire to obtain a higher value of contract. As shown in Corollary 4.1, the increasing wages corresponds to the worker's value of contract that is also increasing over tenure. In this environment, the value of contract improves both on the job through the increasing wages and on the market through the on-the-job search. The worker's incentive responds to both of them: explicit wage-driven incentive and implicit market incentive. His expected value from the effort choice is given by $W + \lambda_e D(W)$. Corollary 4.1 implies that W is increasing, and we know that the whole term is increasing in W. However, it is important to realize that the term $D(W)$ that captures the gain from search and thus his market incentive is indeed decreasing in W (Lemma 3.4). This implies that the worker's explicit incentive increases over tenure whereas his implicit market incentive decreases over tenure. This is a consistent feature found in Gibbons and Murphy (1992).

5.2 Worker's Productivity and Productivity Dispersion

Workers' endogenous effort choice provide further implications for their productivity and economy's productivity dispersion. Here I will identify the probability of project success with the productivity of the match. Then, the increasing effort profile implies that the worker's productivity increases over tenure on the contract. Also, this mechanism implies the following features. First, if a worker moves across jobs without a period of unemployment, his productivity will increase discontinuously. Second, if a worker experiences a period of unemployment, his productivity after getting a new job will be substantially lower than his previous productivity. Previous literature assumes human capital accumulation, learning on the job, or match specific productivity gain to address this feature, but the present theory suggests that the worker's incentive itself generates increasing productivity profile over tenure. The above implication contrast to the case with other mechanisms. First, if general human capital that is compatible with any firms is important for productivity increase over time, these two cases of job change will result in increasing productivity. Second, if firm specific human capital or match specific productivity is important for productivity increase, these both job changes will result in decreasing productivity after all. These are qualitatively stark differences among these theories and they would help disentangle each mechanism in quantitative studies.

Job-search literature has long addressed wage dispersion across workers. Having endogenous productivities given the incentive issue, the theory provide a simple mechanism that will generate not only wage dispersion but also productivity dispersion across workers. First of all, the market friction and workers' on-the-job search endogenously assign identical workers to different contracts. Different contracts, then, provide workers with different incentive structures, and workers exert different levels of effort on each project. This leads to heterogeneous productivities among identical workers and thus to a productivity dispersion. The market friction is crucial in this mechanism. If the market works without search friction and all the workers are ex-ante identical, there is a unique market value of employment for all the workers as in Phelan (1995). Then, even workers are allowed to search a new contract on-the-job, there is no gains from doing so as long as the current job keeps paying increasing wages. Previous job-search literature has identified the mechanism up to this endogenous allocation of homogeneous workers to different contracts. A simple but important step to take is to realize that the workers incentives on the contract differ in such an environment and endogenous productivity dispersion arises naturally because of these differences.

5.3 Temporary Technology Shock and Average Productivity

Finally, to illustrate how the microstructure of the economy interacts with its macroeconomic behavior, I will examine how a temporary shock to the contractual structure affects the average productivity of the economy. As in above analysis, I will use the probability that project succeeds for given effort as a measure of productivity. Therefore, the average productivity of this economy is an integration of success probabilities with respect to the distribution of workers over the values of contracts, i.e., $\int r(\cdot)dG$.

Before starting the analysis, I will explain the distributions of workers evolve over time. At any time t and for any $V \in X$, $G_t(V)$ is the fraction of worker having a value less than V, including unemployed worker. Given the optimal contract, the distributions of workers evolve as follows:

$$G_{t+1}(V) = G_t(V) - \left[\int_{m^{-1}(V)}^{H^{-1}(V)} r(e(H(x)))\lambda\hat{p}(x)dG_t(x) + \int_{H^{-1}(V)}^{V} r(e(H(x)))dG_t(x) \right] + \int_{V}^{\bar{x}} (1 - r(e(H(x))))dG_t(x).$$

Here H is a function that specifies the next period value under the current contract and is implicitly defined by the equation (8). Note that, given the current value x, the current period effort is determined by the next period continuation value $H(x)$.

The whole bracket in the first line is the outflow from $G_t(V)$. Workers in the interval $[H^{-1}(V), V]$ will be promoted to a value higher than V after a success of their current project with probability $r(e(H(x)))$. Workers in the interval $[m^{-1}, H^{-1}(V)]$, though not be promoted that much, will search for a new contract that offers more than V after the success and will successfully find a firm with probability $r(e(H(x)))\lambda\hat{p}(x)$. On the other hand, workers whose current value is more than V will fail in their current project with probability (1-r(e(H(x)))) and will have a value of unemployed worker. This is the inflow into $G_t(V)$.

Given this set up, consider, at any stationary state of distribution G^*, a temporary (one period) technology shock that decreases the cost of effort for workers. To clarify the point, suppose the shock is private to the worker and the firms do not modify the contract. The decrease in the cost of effort will increase the optimal effort choice by workers for any continuation value, that is, $e(H(x))$ increases for any $x \in X$. Therefore, worker outflow from $G_t(V)$ increases while worker inflow into $G_t(V)$ decreases for any V. This implies that $G_{t+1}(V) < G_t^*(V)$ for all V, and thus, the next period distribution first-order stochastically dominates the current stationary distribution. It is equivalent to $\int_x r(e(H(x)))dG_{t+1}(x) > \int_x r(e(H(x)))dG_t^*(x)$. Here, I use the property of Block recursive equilibrium that the optimal contract is independent of the distribution of workers to calculate this integration. Therefore, average productivity is higher after the technology shock. Since the distribution of workers does not return to the stationary distribution immediately and follows the above law of motion, a temporary shock to the contractual structure will yield a persistent aggregate shock to the economy.

6 Conclusion

The purpose of this paper has been to explore how workers' incentives inside a firm interact with their mobility in the labor market. To this end, I have developed a search theoretic model of employment contracts with repeated moral hazard. I find that in equilibrium the optimal long-term contract is characterized by an increasing wage-tenure profile. The optimal incentive compatible effort level also increases with tenure. These results provide implications for the workers' career concerns, productivity profile, and job turnover in equilibrium. Moreover, the theory makes predictions about how the microstructures of the economy interact with macroeconomic behavior.

The framework is another step toward further understanding of agency problems in a market equilibrium context. Examining moral hazard problems in a frictional labor market not only adds one more layer of microfoundation but also provides rich implications for important macroeconomic issues. For example, the current framework can be applied for some policy analyses, such as evaluating the effects of unemployment insurance policies on aggregate productivity as well as wage inequality. In my model, such policies affect not only unemployed workers but also currently employed workers through incentive mechanisms on the job. Such exercises will require quantitative analyses of the model. I believe that pursuing these lines of research is an important next step.

7 Appendix A

7.1 Other Properties of Equilibrium

Note that, in the text, all the objects in the characterizations depend on a specific $J \in \mathcal{J}(X)$. To show the existence of a fixed point of the operator $TJ(V)$, we need to show that they are continuous with respect to $J \in \mathcal{J}(X)$. The following series of lemmas imply the continuity of them. Almost all are shown in Menzio and Shi (2008) and proofs are omitted.

7.1.1 Free Entry Condition and Market Tightness

Lemma 7.1. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\theta_j(x)$ be the market tightness function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\theta_m - \theta_n|| < \varepsilon_\theta \rho$.*

7.1.2 Worker's Search Problem

Lemma 7.2. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $D_j(V)$ be the worker's value of searching implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||D_m - D_n|| < \varepsilon_D \rho$.*

Lemma 7.3. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $m_j(V)$ be the optimal search strategy implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||m_m - m_n|| < \varepsilon_m \rho$.*

Lemma 7.4. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\hat{p}_j(V) = p(\theta_j(m_j(V)))$ be the conposite function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\hat{p}_m - \hat{p}_n|| < \varepsilon_p \rho$.*

7.1.3 Worker's Value of Unemployment

Lemma 7.5. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let U_j be the worker's unemployment value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||U_m - U_n|| < \varepsilon_U \rho$.*

7.1.4 Worker's Optimal Effort

Lemma 7.6. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\Omega_j(W) = W + \lambda D_j(W) - U_j$ be the worker's net continuation value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\Omega_m - \Omega_n|| < \varepsilon_\Omega \rho$.*

Proof of Lemma 7.6:

$$
\begin{aligned}
&|\Omega_m(W) - \Omega_n(W)| \\
&= |\lambda(D_m(W) - D_n(W)) - (U_m - U_n)| \\
&\leq |\lambda\varepsilon_D - \varepsilon_U|\rho.
\end{aligned}
$$

Lemma 7.7. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $e_j(W) = g(\Omega_i(W))$ be the worker's optimal effort function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||e_m - e_n|| < \varepsilon_e\rho$.*

Proof of Lemma 7.7:

Let $\bar{g}' = |\sup g'(\cdot)|$. Given the assumptions about $c(\cdot)$ and $r(\cdot)$, $\bar{g}' < \infty$. Then,

$$
\begin{aligned}
|e_m(W) - e_n(W)| &= |g(\Omega_m(W)) - g(\Omega_n(W))| \\
&\leq \bar{g}'|\Omega_m(W) - \Omega_n(W)| \\
&\leq \bar{g}'\varepsilon_\Omega\rho.
\end{aligned}
$$

7.2 Omitted Lemmas and Proofs

These lemmas extend the results shown in Menzio and Shi (2008) to show the existence of a Block Recursive Equilibrium when there is a moral hazard problem. Let $\hat{J}(V)$ be a firm's updated value function by the operator T, i.e., $\hat{J}(V) = (TJ)(V)$.

Lemma 7.8. *The firm's value function $\hat{J}(V)$ belongs to the set $\mathcal{J}(X)$. That is,*

(i) $\hat{J}(V)$ is strictly decreasing, and Lipschitz continuous with respect to V.

(ii) $\hat{J}(V)$ is bounded both from below and above.

(iii) $\hat{J}(V)$ is concave.

Proof of Lemma 7.8: The operator is a self-mapping.

(i) From the characterization result, let F be the objective function of the reduced problem, i.e.,

$$
F(V,\gamma) = \Bigg\{ -u^{-1}\Bigg(V - \beta U - \sum_i \pi_i\Big[-c(e(W_i)) + \beta r(e(W_i))[W_i + \lambda D(W_i) - U]\Big]\Bigg) + \sum_i \pi_i r(e(W_i))\Big[y + \beta(1 - \lambda\hat{p}(W_i)J(W_i)\Big]\Bigg\}
$$

where γ is a contract. Let Γ be the set of feasible contracts as defined in the text. By the Inverse Function Theorem,

$$F'(V,\gamma) = -\frac{1}{u'(w)} \in \left[-\frac{1}{\underline{u}'}, -\frac{1}{\bar{u}'}\right]$$

Now, for any $V_a, V_b \in X$, such that $V_a \leq V_b$, we have

$$\begin{aligned}
|\hat{J}(V_b) - \hat{J}(V_b)| &\leq \max_{\gamma\in\Gamma} |F(V_b,\gamma) - F(V_a,\gamma)| \\
&= \max_{\gamma\in\Gamma} \left| \int_{V_a}^{V_b} F'(V,\gamma) dV \right| \\
&\leq \max_{\gamma\in\Gamma} \int_{V_a}^{V_b} \left| F'(V,\gamma) \right| dV \\
&\leq \frac{1}{\underline{u}'} |V_b - V_a|
\end{aligned}$$

Therefore $\hat{J}(V)$ is Lipschitz continuous in V. From this result, $\hat{J}(V)$ is absolutely continuous and thus almost everywhere differentiable (Folland, 1999). Moreover, at any point of differentiability, we have $\hat{J}'(V) = F'(V,\gamma(V))$ where $\gamma(V)$ is the optimal contract given V (Milgrom and Segal, 2002). Then,

$$\hat{J}(V_b) - \hat{J}(V_a) = \int_{V_a}^{V_b} F'(V,\gamma(V))dV \in \left[-\frac{1}{\underline{u}'}(V_b - V_a), -\frac{1}{\bar{u}'}(V_b - V_a)\right]$$

Hence, $\hat{J}(V)$ is strictly decreasing and the difference is bounded.

(ii) Next, we estimate the bounds of $\hat{J}(V)$. Let $\underline{w}$ be the lowest possible wage under the feasible contract. That is

$$\underline{w} = \min_{\gamma\in\Gamma} u^{-1}\left(V - \sum_{i=1,2} \pi_i(-c(e(W_i)) + \beta[r(e_i)(W_i + \lambda D(W_i)) + (1 - r(e_i))U]) \right)$$

Since u' is increasing function and the expected continuation value for the worker is bounded by $\bar{x} = \sup_{J\in\mathcal{J}} \hat{x}_J$, which is finite, we have $\underline{w} \geq u'(\underline{x} + c(\underline{e}) - \beta\bar{x})$. Using the fact that $\hat{J}(V)$ is strictly decreasing in V, we have

$$\begin{aligned}
\hat{J}(V) &< \hat{J}(\underline{x}) \\
&\leq r(\bar{e})y - u'(\underline{x} + c(\underline{e}) - \beta\bar{x}) + \beta\bar{J} \equiv \bar{J}.
\end{aligned}$$

Then, $\hat{J}(V) \leq \bar{J} = \frac{y - u'(\underline{x} + c(\underline{e}) - \beta\bar{x})}{1-\beta}$.

Similarly to the previous argument,

$$\begin{aligned}
\hat{J}(V) &> \hat{J}(\bar{x}) \\
&\geq r(\underline{e})y - u'(\bar{x} + c(\bar{e}) - \beta U) + \beta\underline{J} \equiv \underline{J}.
\end{aligned}$$

Then, $\hat{J}(V) \geq \underline{J} = \frac{r(\underline{e})y - u'(\bar{x}+c(\bar{e})-\beta U)}{1-\beta}$. Hence, $\hat{J}(V)$ is bounded both from below and above.

(iii) Concavity of $\hat{J}(V)$ can be shown with two-point convexification result developed by Menzio and Shi (2008) and omitted in this paper. □

Lemma 7.9. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\hat{J}_j(W)$ be the firm's value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\hat{J}_m - \hat{J}_n|| < \varepsilon_T \rho$.*

Continuity of the operator with respect to $J \in \mathcal{J}(X)$.

Proof of Lemma 7.9: Continuity of the operator.

Let F_j be the objective function of the firms optimal contracting problem implied by J_j: $F_j : \Gamma \times X \to \mathbb{R}$. Consider $J_m, J_n \in \mathcal{J}(X)$ such that $||J_m - J_n|| < \rho$. Take $V \in X$ such that $\hat{J}_m(V) - \hat{J}_n(V) > 0$. Let γ_j be the maximizer of F_j and $w_j(\gamma)$ be the wage function given by J_j. Then, we have

$$\begin{aligned}
0 &\leq |\hat{J}_m(V) - \hat{J}_n(V)| \\
&= |F_m(\gamma_m, V) - F_n(\gamma_n, V)| \\
&\leq |F_m(\gamma_m, V) - F_n(\gamma_m, V)| \\
&\leq \Big| - w_m(\gamma_m) + \sum_i \pi_{i,m} r(e_m(W_{i,m})) \left[y + \beta(1 - \lambda \hat{p}_m(W_{i,m})) J_m(W_{i,m})\right] \\
&\quad + w_n(\gamma_m) - \sum_i \pi_{i,m} r(e_n(W_{i,m})) \left[y + \beta(1 - \lambda \hat{p}_n(W_{i,m})) J_n(W_{i,m})\right] \Big| \\
&\leq |w_m(\gamma_m) - w_n(\gamma_m)| \\
&\quad + \sum_i \pi_{i,m} \Big| r(e_m(W_{i,m})) \left[y + \beta(1 - \lambda \hat{p}_m(W_{i,m})) J_m(W_{i,m})\right] \\
&\qquad - r(e_n(W_{i,m})) \left[y + \beta(1 - \lambda \hat{p}_n(W_{i,m})) J_n(W_{i,m})\right] \Big|.
\end{aligned}$$

We want to estimate a bound for $|\hat{J}_m(V) - \hat{J}_n(V)|$. We will consider a bound for each part of the last expression separately as follows.

1. $|w_m(\gamma_m) - w_n(\gamma_m)|$:

Since u is concave function, for any w_1 and w_2, $|w_1 - w_2|u' < |u(w_1) - u(w_2)|$. Also,

$$u(w_m(\gamma_m)) = V - \beta U_m - \sum_i \pi_{i,m}[-c(e_m(W_{i,m})) + \beta r(e_m(W_{i,m}))\Omega_m(W_{i,m})$$

$$u(w_n(\gamma_m)) = V - \beta U_n - \sum_i \pi_{i,m}[-c(e_n(W_{i,m})) + \beta r(e_n(W_{i,m}))\Omega_n(W_{i,m})$$

Then

$$\begin{aligned}
&|u(w_m(\gamma_m)) - u(w_n(\gamma_m))| \\
\leq\ & \beta|U_m - U_n| + \sum_i \pi_{i,m}\Big\{|c(e_m(W_{i,m})) - c(e_m(W_{i,m}))| \\
&\qquad + \beta|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})|\Big\}
\end{aligned}$$

Now, consider the last part: $|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})|$.

$$\begin{aligned}
&|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})| \\
\leq\ & |r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})| \\
=\ & |r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|\bar{x} + r(e_n(W_{i,m}))|\Omega_m(W_{i,m}) - \Omega_n(W_{i,m})| \\
\leq\ & r'(\underline{e})|e_m(W_{i,m}) - e_n(W_{i,m})|\bar{x} + |\Omega_m(W_{i,m}) - \Omega_n(W_{i,m})| \\
\leq & (r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega)\rho
\end{aligned}$$

I use the fact that $\Omega(\cdot)$ is bounded by $\bar{x}$. Collecting them together, we have

$$\begin{aligned}
&|u(w_m(\gamma_m)) - u(w_n(\gamma_m))| \\
\leq\ & (\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega))\rho.
\end{aligned}$$

Hence

$$|w_m(\gamma_m) - w_n(\gamma_m)| \leq u'^{-1} \cdot (\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega))\rho.$$

2.

$$\begin{aligned}
\sum_i \pi_{i,m}\Big| & r(e_m(W_{i,m}))\,[y + \beta(1 - \lambda\hat{p}_m(W_{i,m}))J_m(W_{i,m})] \\
& - r(e_n(W_{i,m}))\,[y + \beta(1 - \lambda\hat{p}_n(W_{i,m}))J_n(W_{i,m})]\Big| :
\end{aligned}$$

This expression can still be divided into subcomponents after expanding the brackets and collecting terms:

(ii) $|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})|$:

$$\begin{aligned}
&|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})| \\
\leq\ &|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})| \\
=\ &|r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|J_m(W_{i,m}) \\
&+ r(e_n(W_{i,m}))|J_m(W_{i,m}) - J_n(W_{i,m})| \\
\leq\ &r'(\underline{e})|e_m(W_{i,m}) - e_n(W_{i,m})|\bar{J} \\
&+ |J_m(W_{i,m}) - J_m(W_{i,m})| \\
\leq\ &(r'(\underline{e})\varepsilon_e\bar{J} + 1)\rho.
\end{aligned}$$

(iii) $|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})|$:

$$\begin{aligned}
&|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_m(W_{i,m})| \\
&+\ |\hat{p}_n(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
=\ &|\hat{p}_m(W_{i,m}) - \hat{p}_n(W_{i,m})|J_m(W_{i,m}) \\
&+\ \hat{p}_n(W_{i,m})|J_m(W_{i,m}) - J_n(W_{i,m})| \\
\leq\ &(\varepsilon_p\bar{J} + 1)\rho.
\end{aligned}$$

(iv) $|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})|$:

$$\begin{aligned}
&|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
=\ &|r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|\hat{p}_m(W_{i,m})J_m(W_{i,m}) \\
&+ r(e_n(W_{i,m}))|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &((r'(\underline{e})\varepsilon_e\bar{J} + 1)\bar{J} + (\varepsilon_p\bar{J} + 1))\rho
\end{aligned}$$

Finally, putting everything together gives that

$$
\begin{aligned}
& |\hat{J}_m(V) - \hat{J}_n(V)| \\
\leq & \, u'^{-1} \cdot \left(\beta \varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e \bar{x} + \varepsilon_\Omega)\right) \rho \\
& + \left\{ (y + \beta + \beta\lambda\bar{J})(r'(\underline{e})\varepsilon_e \bar{J} + 1) + r(\bar{e})(\varepsilon_p \bar{J} + 1) \right\} \rho \\
\equiv & \, \varepsilon_T \rho.
\end{aligned}
$$

□

Proof of Proposition 3.1: Existence of a Block Recursive Equilibrium.

Given Lemma 7.8 and 7.9, we can show that the operator T satisfies the assumptions of Schauder's Fixed Point Theorem (Stokey and Lucas with Prescott, 1989, Theorem 17.4); that is, (i) T is continuous, (ii) the family of functions $T(\mathcal{J})$ is equicontinuous, and (iii) T maps the set $\mathcal{J}(X)$ into itself. Therefore, there exists a firm's value function $J^*(V) \in \mathcal{J}(X)$ such that $TJ^* = J^*$. Denote with $\{\theta^*, D^*, m^*, U^*, e^*\}$ the respective functions associated to J^*. By construction,the functions $\{J^*, \theta^*, D^*, m^*, U^*, e^*\}$ satisfy conditions in the definition of a Recursive Equilibrium and do not depend on the distribution of workers. Therefore, they constitute a Block Recursive Equilibrium.

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