Repeated Moral Hazard with Worker Mobility via Directed On-the-Job Search

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Abstract

This paper studies a repeated moral hazard problem in a general equilibrium framework. I develop a model of dynamic employment contracts by integrating an optimal contracting problem into an equilibrium search framework. The proposed framework enables us to analyze the interaction between the contracting problem and the endogenously evolving outside environment via worker mobility, and I characterize the optimal long-term wage contract as well as the optimal incentive compatible effort-tenure profile. The optimal contract exhibits an increasing wage-tenure profile for two reasons: 1) it induces the workers to be more likely to stay in their current contracts, and 2) it can induce the workers to exert efforts when the current up-front wages cannot. The optimal incentive-compatible effort also has an increasing profile due to an interaction between 1) the workers' fear of losing their jobs, and 2) their incentive to obtain better outside offers. I then show the existence of an equilibrium in which individuals' optimal decisions and optimal contracts are independent of the distribution of workers. Lastly, I provide a general equilibrium implication of the model to illustrate the interaction between the microstructure and macro-behavior of this framework.

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1 Introduction

Structures of long-term contractual relationship in the economy are intrinsically related to the mobility of the contracting party. For example, worker motivation inside a firm changes over time depending on alternative employment opportunities outside the firm. Workers may find outside opportunities more appealing and choose to leave their current employer. This mobility, in turn, affects long-term contractual structure inside the firm. Understanding this interaction between the structure of long-term contracts and the mobility of workers within a dynamic environment is the central issue of this paper.

I study a repeated moral hazard problem in a dynamic general equilibrium framework. I develop a model of long-term employment contracting by integrating a moral hazard problem into an equilibrium search framework. The proposed framework enables me to examine properties of the optimal long-term contract, especially a wage-tenure profile, in relation to worker mobility via directed on-the-job search and an endogenously evolving outside environment. It is also possible to investigate how workers' efforts evolve over tenure; i.e., do senior workers exert more effort or less than junior workers? Moreover, due to a useful property of the equilibrium, I can investigate the interaction between the microstructure and macro-behavior of this framework.

In this model, there are risk-neutral ex-ante homogeneous firms and risk-averse ex-ante homogeneous workers in a labor market. Firms offer contracts with different *values* to attract workers, and each value forms a submarket. Workers are allowed to search for a new contract both on- and off-the-job. They direct their search for a job, i.e., choose which submarket to enter, depending on the value of the contract offered in each submarket and the probability of obtaining a job there. The number of firms and workers entering a submarket determine how tight the submarket is (*tightness of submarket*), and a free-entry condition adjusts the tightness so that firms are indifferent between submarkets when considering entry (*zero-profit condition*).

Once a firm and a worker form a match and sign a contract, they perform a series of projects: one in each period. Each project results in one of two possible outcomes: success or failure; the probability of success depends stochastically on the worker's unobservable effort. The series of projects continues as long as the project keeps succeeding, but if a project fails at any time, the match separates and the worker needs to search for a new job.

Firms can commit to a long-term contract, while employed workers can quit their jobs in any period if they find a new and better contract through on-the-job search. A long-term contract can depends only on the tenure length of the worker on the current contract. This implies that a firm cannot respond to a worker's outside offer and that the period wage cannot depend on the current outcome of the project. Therefore, firms need to design a long-term contract so that it induces the worker to stay on the contract as well as to exert an appropriate level of effort.

I firstly show that an equilibrium with these features indeed exists. The equilibrium inherits the "block recursivity" developed by Shi (2008) and Menzio and Shi (2008); that is, individuals' optimal decisions and optimal contracts are independent of the distribution of workers.

I find that the optimal long-term contract exhibits an increasing wage-tenure profile. This is so for two reasons: 1) it can induce workers to be more likely to stay in their current contracts, and 2) it can induces workers to exert efforts when the current up-front wages cannot. The former is a standard result in the labor search literature with on-the-job search (e.g., Burdett and Coles (2003) and Shi (2008)). Given worker mobility, a firm has an incentive to backload wages to entice the worker to stay; as wages rise with tenure, it is more difficult for the worker to find a better offer elsewhere, and so the worker's quit rate falls. On the other hand, the latter is new in the literature. Since the current wage does not depend on outcomes of the current project, it cannot give the worker incentives to exert effort for the current period project. The firm, however, can promise to give a higher wage in the next period, and thus higher continuation value, if the current project succeeds. Hence, the worker's motivation to stay employed to receive a higher continuation value in the firm and obtain a better outside offer can provide incentive for the worker to exert effort for the current project. Overall, an increasing wage profile is less costly to the firm than a constant profile that promises the same value to the worker over time.

I find that the optimal incentive-compatible effort also has an increasing profile. This is due to a combined effect between 1) the workers' fear of losing their jobs and 2) their incentive to obtain better outside offers. Given an increasing wage-tenure profile, a worker's value of the current contract increases, and thus the cost of losing the job after a failure increases over time. Also, given the increasing value of the current contract, the worker's value of optimal search also increases. They both increase the expected benefit of making effort over time, and the worker keeps making more and more effort over the tenure. It implies that the market forces ameliorate the moral hazard problem inside a firm.

Due to these characterizations of the optimal contract and the independence property of the equilibrium, the theory provides the following macroeconomic implications. First of all, it shows that the labor market is divided into a segment of competitive markets and a segment of monopsonistic markets. In the competitive segment, firms enter the market competitively and workers are mobile depending on their search opportunity. Therefore, all of the above incentive mechanisms arise. On the other hand, in the monopsonistic segment, workers' continuation value is so high that no new firm offers a higher value to the worker, but the incumbent firm that has been employing the worker still wants to induce a higher effort by the worker. Therefore, even though there is no search opportunity for the worker, the firms optimal contract still exhibits an increasing wage profile.

Secondly, it shows that a temporary shock to the contractual structure yields a persistent shock to the economy's average productivity. Suppose a shock to the worker's cost of effort that makes effort making less costly. It increases all the workers' effort provision for any given terms of contract. If we take the probability of success as a measure of productivity of the match, this shock increases the current productivity. However, this increase in productivity shift the distribution of workers to the right (first-order stochastically dominating the old one). Therefore, even if a shock to each of the worker's incentive mechanism is temporary, it affect the average productivity through the shift in the distribution until it gets back to the old distribution. The independence property makes it possible to analyze this mechanism clearly.

This paper makes contributions to such literature as dynamic contracting, contracting in general equilibrium, and equilibrium labor search. The literature on dynamic contracting is enormous, but this paper is especially related to Rogerson (1985), Holmstrom and Milgorm (1987), Spear and Srivastava (1987), and Sannikov (2008). These papers focus on contracting between a single firm and a single worker, and the outside environment is kept constant over time. Since contracting agents are not allowed to look for other contracts in the market, the theory does not address the issue of mobility, which is the key aspect of this paper.

Key papers in the literature on contracting in general equilibrium theory are Prescott and Townsend (1984) and Phelan (1995). They consider contracting problems in which agents can move across contracts in competitive equilibrium frameworks. A common feature of these models is that contracting agents are perfectly mobile in the markets. On the other hand, the present paper focuses on frictional mobility in a market and shows how it affects the resulting optimal contract.

Closely related papers in the equilibrium labor search literature are Burdett and Coles (2003), Menzio and Shi (2009), and Shi (2009). They consider labor market matching models and shed new light on interactions between employment contracting problems and frictional mobility via on-the-job search in labor markets. However, since their focus is on worker behavior in the market, and since they fail to incorporate issues of asymmetric information within the contractual relationship, worker behavior in the firm due to an incentive problem is abstracted from these analysis. This lack of consideration motivated the investigation in this paper.

There are several recent papers that integrate these areas of research to obtain novel insights. Shimer and Wright (2004) study a static employment contracting problem with search frictions and bilateral asymmetric information. Manoli and Sannikov (2005) study a dynamic employment contracts in an environment with competition between firms and with informational frictions. Board (2007) characterizes the optimal self-enforcing contracts in a large anonymous labour market that allows for on-the-job search by workers. They all try to examine further implications of contract theory by placing it in more realistic contexts, and the current paper shares this spirit.

This paper, however, presents some distinct features. First, in my model, a worker's outside environment interacts with his behavior inside the firm and evolves endogenously via worker mobility as an equilibrium outcome. Then, the directed search framework enables me to characterize the equilibrium contract without referring to the distribution of workers in the market. Because of this property, I can obtain very clear predictions on the optimal contract and macroeconomic implications.

2 A Model of Labor Market with Search Friction and Moral Hazard

2.1 Physical Environment

I consider a labor market with a continuum of infinitely lived workers with measure 1, and a continuum of firms whose measure is determined by competitive entry. All workers and firms are ex-ante homogeneous. Time is discrete and continues forever. Each worker has a utility function $u(w)$ where w is income in a period. I assume that $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, weakly concave, and the first derivative is bounded, i.e., $u'(w) \in [\underline{u}', \bar{u}']$ for all w. When employed, each worker exerts costly effort, $e \in \mathbb{R}_+$, for the project of the firm in each period. Worker's effort is unobservable to the employer. I assume that the cost function $c : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, and weakly convex. Each worker maximizes the expected sum of lifetime utilities minus costs of effort discounted at the rate $\beta \in (0,1)$.

Each firm is endowed with a series of projects. One project is executed in each period if the firm hires a worker, and it results in one of two possible outcome: $\{0, y\}$. When outcome is y it is called a "success," and when 0 a "failure." The probability of success in each period depends on an effort level by the worker employed in the firm and is given by $r(e)$. I assume that $r : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and weakly concave in e. I also assume that $r'(0) < \infty$ and $\lim_{e\to\infty} r(e) = 1$. Each firm maximizes the expected sum of profits discounted at the rate β.

There is a continuum of labor markets indexed by x, where x denotes the value of contract offered to a worker in that submarket, i.e., whenever a vacant firm meets a worker in that submarket, it has to offer him an employment contract that gives him the lifetime utility x. I assume that $x \in X \subseteq \mathbb{R}_+$. Let G be a cumulative distribution of workers over X and u a fraction of unemployed workers. The ratio of vacant firms to searching workers in submarket x is denoted by $\theta(x)$ and is referred to as the tightness of submarket x.

In each period, there are two stages: search and matching, and production. In a search and matching stage, firms post a vacancy at a flow cost $k > 0$ and offer a long-term contract to recruit a worker. Firms offer contracts with different values to attract workers, and each value will form a submarket. Workers are allowed to search for a new contract both on- and off-the job. If employed, a worker receives the opportunity of searching for a new job with probability $\lambda_e \in [0,1]$, and with probability $\lambda_u \in [0,1]$ if the worker is unemployed. I assume a standard matching technology as in the job-search literature. If a worker receives the opportunity of searching and chooses to visit submarket x, he meets a vacant firm with probability $p(\theta(x))$, where $p : \mathbb{R}_+ \to [0,1]$ is twice continuously differentiable, strictly increasing and strictly concave and such that $p(0) = 0, p'(0) < \infty$. On the other hand, if a vacant firm enters a submarket x, it finds a worker with probability $q(\theta(x))$, where $q : \mathbb{R}_+ \to [0,1]$ is twice continuously differentiable, strictly decreasing and strictly convex and such that $\theta^{-1}p(\theta) = q(\theta), q(0) = 1$, and $p(q^{-1}(\cdot))$ is concave. If an employed worker receives the opportunity of searching and matches with a firm and accepts the offer, he must leave his previous employment position before entering the production stage with a new firm. If he rejects the offer, he enters the production stage with his current employer.

In the production stage, each unemployed worker receives and consumes unemployment benefit, b. I assume that each employed worker receives the current period wage at the beginning of the period before the production takes place. Therefore, the wage cannot depend on the current project outcome as in standard moral hazard literature. This assumption is made to clarify the effect of a structure of a long-term contract on worker behavior. At the end of this stage, the project outcome, which stochastically depends on the worker's effort, is publicly realized. If project fails, the firm fires the worker, and the worker looses his employment position and becomes unemployed[1]. This new unemployed worker will not receive an opportunity of searching in the following period and needs to stay unemployed for one period[2]. If the project succeeds, the match stays together and the worker keeps his current employment position in the next period. Projects continue forever as long as they are succeeding. There is no exogenous separation of the match, and a failure is the only reason of separation.

2.2 Contractual Environment

I assume that firms can commit to a long-term contract. On the other hand, workers can quit a job at any time. An employment contract specifies the worker's wage as a function of his tenure in the firm. In particular, when an employed worker receives an opportunity to search for an outside contract, the firm does not respond to the employee's outside offers. As mentioned, a wage in each period does not depend on the realization of the current project outcome because the worker receives up-front payments at the beginning of each production stage. Therefore, firms need to design a long-term contract so that it induces the worker to stay on the contract as well as to exert a desired level of effort.

2.3 Equilibrium Conditions

In the following, I describe the equilibrium conditions for this economy. Following the approach taken by Spear and Srivastava (1987), I will set up the optimal contracting problem as a recursive problem.

First, I will explain a market tightness from firm's free entry condition. Second, given the market tightness function, I will illustrate worker's problems in terms of optimal job search and optimal effort making when employed. Third, I will give the value of unemployed worker.

[1] The following results are qualitatively unchanged as long as the firm commit to a firing rule that fires the worker with strictly positive probability after observing a project failure. I assume this particular firing policy for expositional simplicity.

[2] This assumption is, again, for the simplicity of exposition.

Bringing all these elements together, I will present the firm's optimal contracting problem.

2.3.1 Market Tightness and Free Entry Condition

During the search stage, firms choose how many vacancies to create and where to locate them. Let $J(x)$ denote a firm's value of employing a worker in submarket x. Then, the firm's expected benefit of creating a vacancy in submarket x is given by $q(\theta(x))J(x)$, the product of the probability and the value of meeting a worker in the submarket. Given the market tightness function $\theta(x)$, if the cost k of creating a vacancy is strictly greater than the expected benefit, then firms do not create any vacancies in submarket x. If k is strictly smaller than the expected benefit, then firms create infinitely many vacancy in x. When they are equal, then firms expected profits are zero and independent from the number of vacancies they create in submarket x. Therefore, in any x, $\theta(x)$ is consistent with firms' profit maximization if

$$q(\theta(x))J(x) - k \leq 0, \tag{1}$$

and $\theta(x) \geq 0$, with complementary slackness. The complementary slackness condition ensures that if the expected benefit is less than the cost of creating vacancy, then no firms create vacancy in submarket x; that is $\theta(x) = 0$. Also, free entry and exit condition drives the maximized expected profit to zero.

2.3.2 Worker's Problems

Optimal Search of the Worker

Suppose a worker's current value of employment position is V. If he receives the opportunity to search for a new contract and visits submarket x, he find an employer with probability $p(\theta(x))$ and yields the additional value of $x - V$. The worker chooses which submarket to visit to maximize the expected value of search. I will denote the worker's value of search as

$$D(V) = \max_{x \in \mathbb{R}} p(\theta(x))(x - V), \tag{2}$$

given the current value of employment position V. I denote with $m(V)$ the worker's optimal search policy of this problem, and denote the composite function $\hat{p}(V) = p(\theta(m(V)))$.

Optimal Effort Choice of the Worker

Consider a worker in the production stage. He must choose how much effort to provide for the current project. If the project succeeds, the current contract will provide a continuation value to the worker, W, and it will be the worker's reservation value from the current contract. In the next search stage, if he receives the opportunity to search with probability λ_e, he will obtain the expected benefit of $W + D(W)$ through the optimal search explained above. If he does not receive the opportunity to search, he keeps the reservation value W. On the other hand, if the project fails, the worker will loose the job in the next period and spend as an unemployed receiving the value of unemployed U as explained below. Hence, the net expected benefit of success with the effort level e is

$$r(e)(\lambda_e(W + D(W)) + (1 - \lambda_e)W) + (1 - r(e))U.$$

The cost of effort level e is $c(e)$. Therefore, given the continuation value W, the worker will choose his effort level to solve:

$$\max_{e \in \mathbb{R}} \Big(- c(e) + \beta\big(r(e)(W + \lambda_e D(W)) + (1 - r(e))U\big)\Big).$$

Given the structure of the contract, current period wage is independent of the project outcome and does not affect the worker's optimal effort choice. Note that the expected benefit from the effort is given in the next period and the worker discount its value at β.

Worker's Value of Unemployment

Finally, consider an unemployed worker at the beginning of the production stage. The worker obtains $u(b)$ from the unemployment benefit. Let U be the value of unemployed. If he receives an opportunity to search for a job in the next period, he will obtain the expected benefit of $U + D(U)$ through the optimal search. If he does not receive an opportunity to search, he will stay unemployed and receive U again. Therefore, the value of unemployed is expressed recursively as

$$U = u(b) + \beta\big(U + \lambda_u D(U)\big). \tag{3}$$

2.3.3 The Firm's Optimal Contracting Problem

Consider a firm that promises to provide a continuation value V in this period. Let $J(V)$ denote the current value of a contract for the firm. The firm chooses: i) w, how much wage to pay in this period, ii) e, how much effort to induce, and iii) W, how much continuation value to provide to the worker in the next period conditional on the survival of the relationship. I also allow for randomization over these choices, that is, the firm offers two sets of subcontract and a probability distribution, $\{\pi_i\}_{i=1,2}$, over these subcontracts. Denote by $\xi = (\{w_i, e_i, W_i, \pi_i\}_{i=1,2})$ the contract offered by the firm at the beginning of a period.

The firm's optimal contracting problem is given by

$$J(V) = \max_{\xi} \sum_{i=1,2} \pi_i \left\{ r(e_i)y - w_i + \beta r(e_i)(1 - \lambda_e \hat{p}(W_i))J(W_i) \right\} \tag{4}$$

subject to

$$\begin{aligned}
\xi \in \Xi = \Big\{ &\{w_i, e_i, W_i, \pi_i\}_{i=1,2} : \quad W_i \in X \quad \text{for } i = 1, 2 \\
&V = \sum_{i=1,2} \pi_i \left\{ u(w_i) - c(e_i) + \beta[r(e_i)(W_i + \lambda_e D(W_i)) + (1 - r(e_i))U] \right\} \\
&e_i \in \arg\max_{e \in \mathbb{R}} \Big(-c(e) + \beta\big(r(e)(W + \lambda_e D(W)) + (1 - r(e))U\big)\Big), \quad \text{for } i = 1, 2 \\
&\pi_1 + \pi_2 = 1, \quad \pi_i \in [0, 1] \quad \text{for } i = 1, 2 \Big\}.
\end{aligned}$$

By inducing an effort level e and paying w, the firm's expected current period payoff is $r(e)y - w$. In the next period, by offering continuation value W, the current worker will stay in the current firm with probability $r(e)(1 - \lambda_e \hat{p}(W))$: product between a probability the project succeeds and a probability the worker will not leave for the outside option, and the firm will enjoy the value $J(W)$ of remaining contract.

To design the contract, the firm faces three constraints. The first is the promise-keeping or consistency constraint, that is, the contract has to provide the worker with the promised value V. Since offer of the contract is evaluated ex-ante, only the expected value over the values of subcontracts must provide the promised continuation value; either subcontract may fail to provide the promised value. Then, the effort that the firm wants to induce must be incentive compatible, that is, given the contract offered the worker voluntarily choose to exert that level of effort. Since realization of subcontract occurs before the worker chooses his effort level, both subcontracts the firm prepares need to meet the incentive compatibility constraint. Finally, when the firm randomizes the contracts, the probabilities assigned to each subcontract must sum to one. I denote with $\xi(V)$ the optimal policy functions given V associated with this contracting problem.

2.4 Block (Distribution-free) Recursive Equilibrium

Definition: A Recursive Equilibrium *is a set of functions* $\{J^*, \theta^*, D^*, m^*, U^*, \xi^*\}$ *and a distribution of workers* (G^*, u^*) *such that*

1. θ^* *satisfies condition (1),*

2. D^* *and* m^* *satisfies the functional equation (2),*

3. U^* *satisfies the functional equation (3),*

4. J^* *and* ξ^* *satisfies the functional equation (10), and*

5. (G^*, u^*) *is consistent with* m^* *and* ξ^**.*

Definition: A Block (Distribution-free) Recursive Equilibrium *is a recursive equilibrium such that functions* $\{J^*, \theta^*, D^*, m^*, U^*, \xi^*\}$ *are independent from the distribution of workers* (G^*, u^*)*.*

3 General Properties and Existence of an Equilibrium

In this section, I first describe general properties of an equilibrium. I will then establish the existence of an equilibrium with these properties. Proofs of each lemma is almost identical to that in Menzio and Shi (2009) or requires only minor modifications for taking worker's effort choice into account. I collect all the proofs for those requiring extra steps or modifications in the appendix.

We start with specifying a set of functions with certain properties. Then, I take an arbitrary function from the set as a firm's value function and characterize the equilibrium objects given the properties of functions. With all these results, I construct an operator defined over the set of functions and show that any fixed point of the operator is a Block Recursive Equilibrium.

3.1 General Properties

We define the set $\mathcal{J}(X)$ of functions $J : X \to \mathbb{R}$ such that:

(i) $J(V)$ is strictly decreasing and Lipschitz continuous with respect to V,

(ii) $J(V)$ is bounded both from below and above, and

(iii) $J(V)$ is concave.

We can show that the set $\mathcal{J}(X)$ is non-empty, bounded, closed, and convex subset of the space of bounded, continuous functions on X with the sup norm.

3.1.1 Market Tightness and Free Entry Condition

First, I take an arbitrary firm's value function $J \in \mathcal{J}(X)$ and solve the equilibrium condition (1) with respect to the market tightness function θ. We get

$$\theta(x) = \begin{cases} q^{-1}(k/J(x)) & \text{if } \; J(x) \geq k \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Because q is a probability, q^{-1} is defined only on $[0,1]$, that is $J(x) \geq k$. Since $J(x)$ is strictly decreasing, there exists a unique $\bar{x} \in \mathbb{R}$ such that $J(x) > k$ for all $x < \bar{x}$ and $J(x) < k$ for all $x > \bar{x}$. That is, offering any contract with the value more than $\bar{x}$ provide negative expected profits even if the firm can hire a worker with probability one. Therefore, no firm will enter submarkets with $x > \bar{x}$, and the market tightness takes nonnegative value only if $x \leq \bar{x}$. We call this threshold value a *bound of market* given J. Since J is bounded from above, θ is also bounded from above. Now I have the following properties of market tightness functions.

Lemma 3.1. *If $x < \bar{x}$, the market tightness function $\theta(x)$ is strictly positive and strictly decreasing, and Lipschitz continuous.$\theta(x) = 0$ for all $x \geq \bar{x}$*

3.1.2 Worker's Problem

Optimal Search of the Worker

In this section, I insert the market tightness function, $\theta(x)$, obtained from an arbitrary firm's value function $J \in \mathcal{J}(X)$ into the worker's search problem:

$$D(V) = \max_{x \geq V} p(\theta(x))(x - V).$$

Now, I have the following series of properties which enable to compute the optimal contracts.

Lemma 3.2. *For all $V \in X$, the worker's objective function $f(x;V) = p(\theta(x))(x - V)$ is strictly concave with respect to x.*

The concavity of the problem implies that, given a continuation value of the current contract, the worker will find unique submarket that he optimally visits.

Lemma 3.3. *The worker's optimal search strategy* $m(V) \in \arg\max p(\theta(x))(x - V)$ *is unique, weakly increasing and Lipshitz continuous.*

The lemma also implies that a worker searches in a submarket which offers a higher value the higher your continuation value from the current contract. This can be seen as a version of the single crossing property.

Lemma 3.4. *If* $V < \bar{x}$*, the worker's value of searching* $D(V)$ *is strictly positive and weakly decreasing and Lipschitz continuous. If* $V \geq \bar{x}$*,* $D(V) = 0$*.*

As long as a worker's current continuation value is less than $\bar{x}$, there are submarkets that offer higher values and they provide positive expected value of search. However, once his continuation value hits the bound, there are no outside firms offering better offer and the market tightness becomes zero, $D(V) = 0$ since $p(0) = 0$.

Lemma 3.5. $\hat{p}(V)$ *is weakly decreasing and Lipschitz continuous.Moreover* $\hat{p}(\bar{x}) = 0$

Remember that the composite function $\bar{p}(V)$ is a probability that a worker with a continuation value V from the current contract meets a firm in the optimally targeted submarket. Since a worker with higher V applies to a submarket with higher tightness, he is less likely to meet a firm in an optimally targeted submarket.

Optimal Effort Choice of the Worker

We now turn to a worker's optimal effort choice problem:

$$\max_{e \in \mathbb{R}} \Big(-c(e) + \beta\big(r(e)(W + \lambda_e D(W)) + (1 - r(e))U\big)\Big).$$

Under the assumption on $c(\cdot)$ and $r(\cdot)$, it is a concave problem with respect to e. Therefore, the first-order condition sufficiently characterizes the optimal effort.

Lemma 3.6. *Given a continuation value* W*, there is a unique level of optimal effort. The worker's optimal effort function is increasing in the continuation value of contract,* W*.*

Proof: First, for a given continuation value W, the worker's optimal effort is implicitly and given by

$$-c'(e) + \beta r'(e)(W + \lambda_e D(W) - U) = 0.$$

Define $\Omega(W) = W + \lambda_e D(W) - U$. Then I can write $\frac{c'(e)}{\beta r'(e)} = \Omega(W)$. Under the assumptions ($c(\cdot)$: convex, continuous, $r(\cdot)$: concave continuous, and $r' > 0$ everywhere), $\frac{c'}{\beta r'}$ is continuous and monotonically increasing, and thus invertible. Therefore, there exists a unique level of e that satisfies the equality.

Moreover the inverse function is also continuous. Hence, I write $e(W) = g(\Omega(W))$ where g is the inverse function of $\frac{c'}{\beta r'}$. Differentiating the composite function,

$$\begin{aligned}\frac{\partial e(W)}{\partial W} &= g'(\cdot)\Omega'(W) \\ &= \left(\left(\frac{c'}{\beta r'}\right)^{-1}\right)'(1-\lambda_e\hat{p}(W)) \\ &= \left(\frac{c''r' - c'r''}{\beta r'^2}\right)^{-1}(1-\lambda_e\hat{p}(W)) \\ &= \frac{\beta r'^2}{c''r' - c'r''}(1-\lambda_e\hat{p}(W)),\end{aligned}$$

where the first equality is by the chain rule and the third by the inverse function theorem. Since the denominator of the right hand side is strictly positive by assumption, and thus the derivative is positive, the worker's optimal effort is increasing in W. □

3.2 Existence of a Block Recursive Equilibrium

So far, I have examined properties of equilibrium objects given an arbitrary firm's value function $J \in \mathcal{J}(X)$. In this subsection, I establish that a Block Recursive Equilibrium exists.

Proposition 3.1. *A Block Recursive Equilibrium exists.*

Detailed proof is given in the appendix. I will only outline the argument in the proof. First, by inserting all the previous equilibrium objects, given an arbitrary value function $J \in \mathcal{J}(X)$, into a firm's optimal contracting problem, I construct the following operator that maps from $\mathcal{J}(X)$ to some space of functions:

$$(TJ)(V) = \max_{\xi\in\Xi} \sum_{i=1,2} \pi_i \left\{r(e_i)y - w_i + \beta r(e_i)(1-\lambda_e\hat{p}(W_i))J(W_i)\right\}.$$

We show that T is a self-map, that is, T maps from $\mathcal{J}(X)$ to itself (Lemma 7.8). Then, I show that T is a continuous map (Lemma 7.9). These suffice to show that the operator T satisfies the assumptions of Schauder Fixed Point Theorem (Stokey and Lucas with Prescott, 1989, Theorem 17.4); that is, (i) T is continuous, (ii) the family of functions $T(\mathcal{J})$ is equicontinuous, and (iii)

T maps the set $\mathcal{J}(X)$ into itself. Therefore, there exists a firm's value function $J^*(V) \in \mathcal{J}(X)$ such that $TJ^* = J^*$. Denote with $\{\theta^*, D^*, m^*, U^*, e^*, \xi^*\}$ the respective functions associated to J^*. By construction, the functions $\{J^*, \theta^*, D^*, m^*, U^*, e^*, \xi^*\}$ satisfy conditions in the definition of a Recursive Equilibrium and do not depend on the distribution of workers. Therefore, they constitute a Block Recursive Equilibrium.

4 Characterization of the Optimal Long-Term Contracts

We now characterize the optimal contract. Using the first-order approach, the constraint set is modified as follows:

$$
\begin{aligned}
\xi \in \Xi' = \Big\{ & \{w_i, e_i, W_i, \pi_i\}_{i=1,2} : \quad W_i \in X \quad \text{for } i = 1, 2 \\
& V = \sum_{i=1,2} \pi_i \left\{ u(w_i) - c(e_i) + \beta[r(e_i)(W_i + \lambda_e D(W_i)) + (1 - r(e_i))U] \right\} \\
& - c'(e_i) + \beta r'(e_i)(W_i + \lambda_e D(W_i) - U) = 0, \quad \text{for } i = 1, 2 \\
& \pi_1 + \pi_2 = 1, \quad \pi_i \in [0, 1] \quad \text{for } i = 1, 2 \Big\}.
\end{aligned}
$$

First, we consider only the promise-keeping constraint and the incentive compatibility constraint to obtain the following result.

Proposition 4.1. *Under the optimal contract, the current period wage is independent of the realization of lottery.*

Proof. Let $\eta(V)$ be the Lagrange multiplier for the promise-keeping constraint. The first order condition of the maximization problem with respect to w_i implies

$$\eta(V) = \frac{1}{u'(w_i)} \quad \text{for} \quad i = 1, 2.$$

This implies that $w_1 = w_2$. Therefore, the current wage, w, does not depend on the realization of the lottery. □

Given the above proposition, the firm's problem can be greatly simplified.

Lemma 4.1. (Reduction of the Problem) *Let $\phi = (\{W_i, \pi_i\}_{i=1,2})$. The optimal contracting problem can be reduced to the following optimization problem with respect to ϕ.*

$$J(V) = \max_{\phi} \left\{ -w(V,\phi) + \sum_i \pi_i r(e(W_i)) \Big[y + \beta(1-\lambda_e \hat{p}(W_i)) J(W_i) \Big] \right\} \tag{6}$$

where

$$w(V,\phi) = u^{-1} \left(V - \beta U - \sum_i \pi_i \Big[-c(e(W_i)) + \beta r(e(W_i))[W_i + \lambda_e D(W_i) - U] \Big] \right)$$

and

$$\phi \in \Phi = \Big\{ \{W_i, \pi_i\}_{i=1,2} : \ W_i \in X, \quad \pi_i \in [0,1] \ \text{ for } i = 1,2 \ \text{ and } \pi_1 + \pi_2 = 1 \Big\}$$

Proof. Substituting the optimal effort function into both the objective function and the promise-keeping constraint eliminate the incentive compatibility constraint. Then, using the result that $w_1 = w_2$ makes the promise-keeping constraint as

$$V = u(w) + \beta U + \sum_i \pi_i \Big[-c(e(W_i)) + \beta r(e(W_i))[W_i + \lambda D(W_i) - U] \Big].$$

Since u is strictly concave and thus invertible, this constraint can be solved for w. Substituting resulting expression for w into the objective function gives the desired form of unconstrained optimization problem. □

4.1 The Optimal Wage and Continuation Value Prfiles

We are interested in a shape of long-term wage-tenure profile in this environment. Let $\hat{x} \in X$ denote a value such that $J(x) < 0$ for all $x > \hat{x}$ and $J(x) > 0$ for all $x < \hat{x}$. Since J is strictly decreasing, there is a unique such value. As the bound for the market, $\bar{x}$, determines the firms' entry into the market, the threshold value $\hat{x}$ characterizes both the optimal wage-tenure profile and the market structure.

Proposition 4.2. *Under the optimal contract, wages are nondecreasing over the tenure if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$ they are strictly increasing, and they stay constant once the continuation value hits the value $\hat{x}$.*

Proof: First, ignoring the feasibility constraints for W_i and π_i, the first order condition of the reduced form of the problem (6) with respect to W_i is

$$\begin{aligned}
&-\frac{1}{u'(w)}\cdot(-\pi_i\beta r(e(W_i))(1-\lambda_e\hat{p}(W_i)))\\
&+\pi_i r'(e(W_i))[y+\beta(1-\lambda_e\hat{p}(W_i))J(W_i)]g'(\Omega(W_i))(1-\lambda_e\hat{p}(W_i))\\
&+\pi_i r(e(W_i))\beta\big[(1-\lambda_e\hat{p}(W_i))J'(W_i)-\lambda_e J(W_i)\hat{p}'(W_i)\big]=0
\end{aligned}$$

where we use that $\frac{\partial\Omega(W)}{\partial W}=1-\lambda_e\hat{p}(W)$ and $e'(W)=g'(\Omega(W_i))\Omega'(W)$. Then, dividing through by $\beta,\pi_i,r(e),(1-\lambda_e\hat{p}(W_i))$ gives

$$-\frac{1}{u'(w)}+\frac{r'(e(W_i))}{r(e(W_i))}\left(\frac{y}{\beta}+(1-\lambda_e\hat{p}(W_i))J(W_i)\right)g'(\Omega(W_i))+J'(W_i)-\frac{\lambda_e J(W_i)\hat{p}'(W_i)}{1-\lambda_e\hat{p}(W_i)}=0.$$

Now, from the first order condition of the constrained problem, we have $\eta(V)=\frac{1}{u'(w(V))}$. The theorem of Lagrangian multiplier implies that $J'(V)=-\eta(V)$, so we have

$$J'(V)=-\frac{1}{u'(w(V))}. \tag{7}$$

Shifting one period forward gives $J'(W_i)=-\frac{1}{u'(w(W_i))}$ where $w(W_i)$ is the wage in the next period when the current lottery realization is i.

Substituting it into the previous equation and rearranging the terms yield

$$\frac{1}{u'(w(W_i))}-\frac{1}{u'(w)}=-\frac{\lambda_e J(W_i)\hat{p}'(W_i)}{1-\lambda_e\hat{p}(W_i)}+\frac{r'(e(W_i))}{r(e(W_i))}\left(\frac{y}{\beta}+(1-\lambda_e\hat{p}(W_i))J(W_i)\right)g'(\Omega(W_i)). \tag{8}$$

The right hand side is positive since $\hat{p}'(W_i)$ is non-positive. Therefore, we have

$$\frac{1}{u'(w(W_i))}-\frac{1}{u'(w)}>0. \tag{9}$$

Since u is concave function, this implies $w(W_i)>w$ for $i=1,2$. Hence, the next period wage is higher than the current wage.

However, ever increasing wage profile is not optimal; after some point, the firm is paying to the worker more than the project produce. The above equation is not sufficient for the complete characterization. Once a worker's value of contract reaches the value $\bar{x}$, in equilibrium, there is no other submarkets offering higher value, i.e., $\bar{p}(\bar{x})=0$, so the worker has no incentive to search on-the-job. Therefore, the firm does not need to induce the worker to stay in the contract. At this point, firm's optimal contracting problem becomes as if there is no on-the-job search on the worker's side. As can be easily shown, optimal wage profile is still increasing until the firms value becomes negative to induce higher effort. (see Appendix B) However, if the worker's value reaches $\hat{x}$, inducing further effort by paying higher wage yields negative value for the firm. So, at this point, the firm does not have any further incentive to induce a higher effort, and thus it maintains the constant wage just to provide the value $\hat{x}$ to the worker as follows:

$$\hat{w} = u^{-1}\big(\hat{x} - \beta U + c(e(\hat{x})) - \beta r(e(\hat{x}))(\hat{x} - U)\big).$$

□

The equation (11) in the proof gives explicit mechanics of backloaded wage-tenure profile; the first term considers the worker's mobility whereas the second term considers the moral hazard problem. Given the worker's mobility, a firm has an incentive to backload wages to entice the worker to stay; as wages rise with tenure, it is more difficult for the worker to find a better offer elsewhere, and so the worker's quit rate falls (see Shi 2008). On the other hand, the second term is new in the literature. Since the current wage does not depend on outcome of the current project, the current period wage cannot give a worker incentives to exert effort for the current period project. The firm, however, gives incentives to promise higher continuation value in the next period. Overall, a rising wage profile is less costly to the firm than a constant profile that promises the same value to the worker.

Corollary 4.1. *Under the optimal contract, the next period promised continuation values are nondecreasing over the tenure if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$, they are strictly increasing, and they stay constant once they hit the value $\hat{x}$.*

Proof: If $V < \hat{x}$, with (7), the inequality (12) implies $J'(W_i) - J'(V) < 0$. Then, concavity of J implies $W_i > V$ for $i = 1, 2$. If $V \geq \hat{x}$, optimal offer stays constant at $\hat{x}$. □

4.2 The Optimal Incentive Compatible Effort Profile

Proposition 4.3. *Under the optimal contract, the worker's optimal efforts are nondecreasing if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$, they are strictly increasing, and they stay constant once the continuation value hits the value $\hat{x}$.*

Proof: As shown in lemma 3.6, the optimal effort function is increasing in the continuation value of the contract. Since the continuation value of the contract is higher than the current value of the contract (Corollary (4.1)), the worker will provide higher effort next period than the current period effort if stays on the contract. □

This is due to a combined effect between 1) a worker's fear of loosing the job and 2) his incentive to obtain the better outside offer. In this environment, when a project fails, the worker looses not only his current job but also his better market environment. Given increasing wages, a worker's value of the current contract increases over time and cost of loosing the job is increasing. Moreover, the worker's perception of the labor market depends on his current contract because searching for a high value contract makes sense only when he is employed at sufficiently high value. Therefore, having a high value contract provides a better market environment, and loosing the current job implies loosing this improved market environment. Both of these factors give the worker incentives to exert higher and higher efforts.

Note that, as shown in the appendix, the worker's optimal incentive compatible effort profile is increasing even though there is no on-the-job search. There, optimal wage profile is still increasing. So, the worker's cost of loosing the job is also increasing over tenure, and the worker exerts more and more effort solely due to his fear of loosing the current job. However, what makes different is that, with on-the-job search, a firm can induce a higher level of effort for a given promised continuation value than the firm can single-handedly induce because of the additional source of incentive from the outside option. In this sense, the market forces due to the worker mobility ameliorates the moral hazard problem inside the firm.

4.3 Discussion of the Results: Market Structure

Above analysis shows that the labor market is divided into a segment of competitive markets and a segment of monopsonistic markets.

As long as the promised value of contract x is below $\bar{x}$, $J(x) \geq k$ and firms rationally enter the markets to employ a worker. Here, workers move across firms depending on their search opportunity. In this competitive segment, all of the above incentive mechanisms arise to generate the increasing wage profile and effort profile.

However, the promised value also increases within the contract if the projects keep succeeding. Once it reaches $\bar{x}$, the worker's continuation value is so high that no outside firm offers a higher value to recruit him, i.e., $J(x) \leq k$. If there is no incentive issue for the effort provision, the incumbent firm that has been employing the worker also does not have an incentive to offer any higher value to the worker. However, in this model, the firm still wants to induce a higher effort by the worker by offering a higher continuation value. This continues until any further increase yields negative value to the firm, i.e., $J(x) < 0$. Therefore, even though there is no search opportunity for the worker, the firms optimal contract still exhibits an increasing wage profile. In $[\bar{x}, \hat{x}]$, the current employer is the sole buyer of the worker's labor, and this range of promised value constitutes a monopsonistic segment of the labor market.

5 General Equilibrium Implication: Temporary Technology Shock and Average Productivity

In this section, I will examine how a temporary shock to the contractual structure affects the average productivity of the economy. This is one example that illustrates the interaction between the microstructure of the economy and its aggregate behavior. I will use the probability that project succeeds for given effort as a measure of productivity. Therefore, the average productivity of this economy is the integration of the success probability with respect to the distribution of workers.

Before starting the analysis, I will explain the distribution of workers. For any $V \in X$, $G_t(V)$ is the fraction of worker having a value V, including unemployed worker. Given the optimal contract, the distribution of workers over the set of values evolve as follows.

$$G_{t+1}(V) = G_t(V) - \left[\int_{m^{-1}(V)}^{H^{-1}(V)} r(e(H(x)))\lambda\hat{p}(x)dG_t(x) + \int_{H^{-1}(V)}^{V} r(e(H(x)))dG_t(x) \right] \\ + \int_{V}^{\bar{x}} (1 - r(e(H(x))))dG_t(x)$$

Here H is a function that specifies the next period value under the current contract and is implicitly defined by the equation (11). Note that, given the current value x, the current period effort is determined by the next period continuation value $H(x)$.

The whole bracket in the first line is the outflow from $G_t(V)$. Workers in the interval $[H^{-1}(V), V]$ will be promoted to a value higher than V after a success of their current project with probability $r(e(H(x)))$. Workers in the interval $[m^{-1}, H^{-1}(V)]$, though not be promoted that much, will search for a new contract that offers more than V after the success and will successfully find a firm with probability $r(e(H(x)))\lambda\hat{p}(x)$. On the other hand, workers whose current value is more than V will fail in their current project with probability (1-r(e(H(x)))) and will have a value of unemployed worker. This is the inflow into $G_t(V)$.

Given this set up, consider, at any stationary state of distribution G^*, a temporary (one period) technology shock that decreases the cost of effort for workers. To clarify the point, suppose the shock is private to the worker and the firms do not modify the contract. The decrease in the cost of effort will increase the optimal effort choice by workers for any continuation value, that is, $e(H(x))$ increases for any $x \in X$. Therefore, worker outflow from $G_t(V)$ increases while worker inflow into $G_t(V)$ decreases for any V. This implies that $G_{t+1}(V) < G_t^*(V)$ for all V, and thus, the next period distribution first-order stochastically dominates the current stationary distribution. It is equivalent to $\int_x r(e(H(x)))dG_{t+1}(x) > \int_x r(e(H(x)))dG_t^*(x)$. Here, I use the property of Block recursive equilibrium that the optimal contract is independent of the distribution of workers to calculate this integration. Therefore, average productivity is higher after the technology shock. Since the distribution of workers does not return to the stationary distribution immediately and follows the above law of motion, a temporary shock to the contractual structure will yield a persistent aggregate shock to the economy.

6 Conclusion

In this paper, I developed a search theoretic general equilibrium model of an employment contracting with repeated moral hazard. The optimal contract exhibits an increasing wage-tenure profile as well as an increasing incentive-compatible effort profile. I showed that an equilibrium exists and is independent of the distribution of workers. Lastly, I provide a general equilibrium implication of the model to illustrate the interaction between the microstructure and macro-behavior of this framework.

I am currently analyzing how the presence of a moral hazard problem affects the distribution of workers. Since the equilibrium is independent of distributions, it is possible to further analyze the evolution of the distribution along the equilibrium out of the steady state. I am also examining the impact of policies such as a minimum wage law or changes in unemployment benefits on the optimal contract and on the distribution of workers. While, in this paper, I focused exclusively on the positive aspects of the benchmark model, it may also be interesting to investigate the efficiency or normative properties of the equilibrium.

There are several potential extensions and future research areas. First, it is important to investigate the worker's saving behavior since, in my model, workers always risk losing their jobs.

Allowing them to access a credit market is an important extension. Other more challenging extensions include 1) introducing heterogeneous firms (sectors) to examine the worker's inter-sectorial mobility, 2) allowing firms to choose to either fire or retain a worker after a project fails, 3) introducing a costly search on the worker's side, and 4) considering the delayed effects of the worker's effort on the project outcome.

Lastly, as a companion part to my ongoing research project, I am studying a case of repeated adverse selection in which workers repeatedly receive unobservable productivity shocks or consumers repeatedly receive unobservable endowment shocks as in Phelan (1995). Incorporating other cases of asymmetric information seems to be an important research area.

7 Appendix A

7.1 Other Properties of Equilibrium

Note that, in the text, all the objects in the characterizations depend on a specific $J \in \mathcal{J}(X)$. To show the existence of a fixed point of the operator $TJ(V)$, we need to show that they are continuous with respect to $J \in \mathcal{J}(X)$. The following series of lemmas imply the continuity of them. Almost all are shown in Menzio and Shi (2008) and proofs are omitted.

7.1.1 Free Entry Condition and Market Tightness

Lemma 7.1. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\theta_j(x)$ be the market tightness function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\theta_m - \theta_n|| < \varepsilon_\theta \rho$.*

7.1.2 Worker's Search Problem

Lemma 7.2. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $D_j(V)$ be the worker's value of searching implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||D_m - D_n|| < \varepsilon_D \rho$.*

Lemma 7.3. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $m_j(V)$ be the optimal search strategy implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||m_m - m_n|| < \varepsilon_m \rho$.*

Lemma 7.4. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\hat{p}_j(V) = p(\theta_j(m_j(V)))$ be the conposite function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\hat{p}_m - \hat{p}_n|| < \varepsilon_p \rho$.*

7.1.3 Worker's Value of Unemployment

Lemma 7.5. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let U_j be the worker's unemployment value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||U_m - U_n|| < \varepsilon_U \rho$.*

7.1.4 Worker's Optimal Effort

Lemma 7.6. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\Omega_j(W) = W + \lambda D_j(W) - U_j$ be the worker's net continuation value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\Omega_m - \Omega_n|| < \varepsilon_\Omega \rho$.*

Proof of Lemma 7.6:

$$\begin{aligned}
&|\Omega_m(W) - \Omega_n(W)| \\
&= |\lambda(D_m(W) - D_n(W)) - (U_m - U_n)| \\
&\le |\lambda\varepsilon_D - \varepsilon_U|\rho.
\end{aligned}$$

Lemma 7.7. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $e_j(W) = g(\Omega_i(W))$ be the worker's optimal effort function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||e_m - e_n|| < \varepsilon_e\rho$.*

Proof of Lemma 7.7:

Let $\bar{g'} = |\sup g'(\cdot)|$. Given the assumptions about $c(\cdot)$ and $r(\cdot)$, $\bar{g'} < \infty$. Then,

$$\begin{aligned}
|e_m(W) - e_n(W)| &= |g(\Omega_m(W)) - g(\Omega_n(W))| \\
&\le \bar{g'}|\Omega_m(W) - \Omega_n(W)| \\
&\le \bar{g'}\varepsilon_\Omega\rho.
\end{aligned}$$

7.2 Omitted Lemmas and Proofs

These lemmas extend the results shown in Menzio and Shi (2008) to show the existence of a Block Recursive Equilibrium when there is a moral hazard problem. Let $\hat{J}(V)$ be a firm's updated value function by the operator T, i.e., $\hat{J}(V) = (TJ)(V)$.

Lemma 7.8. *The firm's value function $\hat{J}(V)$ belongs to the set $\mathcal{J}(X)$. That is,*

(i) $\hat{J}(V)$ is strictly decreasing, and Lipschitz continuous with respect to V.

(ii) $\hat{J}(V)$ is bounded both from below and above.

(iii) $\hat{J}(V)$ is concave.

Proof of Lemma 7.8: The operator is a self-mapping.

(i) From the characterization result, let F be the objective function of the reduced problem, i.e.,

$$\begin{aligned}
F(V,\gamma) = \Bigg\{ &-u^{-1}\left(V - \beta U - \sum_i \pi_i \Big[-c(e(W_i)) + \beta r(e(W_i))[W_i + \lambda D(W_i) - U]\Big]\right) \\
&+ \sum_i \pi_i r(e(W_i))\Big[y + \beta(1 - \lambda\hat{p}(W_i)J(W_i)\Big]\Bigg\}
\end{aligned}$$

where γ is a contract. Let Γ be the set of feasible contracts as defined in the text. By the Inverse Function Theorem,

$$F'(V,\gamma) = -\frac{1}{u'(w)} \in \left[-\frac{1}{\underline{u}'}, -\frac{1}{\bar{u}'}\right]$$

Now, for any $V_a, V_b \in X$, such that $V_a \leq V_b$, we have

$$\begin{aligned}
|\hat{J}(V_b) - \hat{J}(V_b)| &\leq \max_{\gamma\in\Gamma} |F(V_b,\gamma) - F(V_a,\gamma)| \\
&= \max_{\gamma\in\Gamma} \left| \int_{V_a}^{V_b} F'(V,\gamma) dV \right| \\
&\leq \max_{\gamma\in\Gamma} \int_{V_a}^{V_b} \left| F'(V,\gamma) \right| dV \\
&\leq \frac{1}{\underline{u}'} |V_b - V_a|
\end{aligned}$$

Therefore $\hat{J}(V)$ is Lipschitz continuous in V. From this result, $\hat{J}(V)$ is absolutely continuous and thus almost everywhere differentiable (Folland, 1999). Moreover, at any point of differentiability, we have $\hat{J}'(V) = F'(V, \gamma(V))$ where $\gamma(V)$ is the optimal contract given V (Milgrom and Segal, 2002). Then,

$$\hat{J}(V_b) - \hat{J}(V_a) = \int_{V_a}^{V_b} F'(V,\gamma(V)) dV \in \left[-\frac{1}{\underline{u}'}(V_b - V_a), -\frac{1}{\bar{u}'}(V_b - V_a)\right]$$

Hence, $\hat{J}(V)$ is strictly decreasing and the difference is bounded.

(ii) Next, we estimate the bounds of $\hat{J}(V)$. Let $\underline{w}$ be the lowest possible wage under the feasible contract. That is

$$\underline{w} = \min_{\gamma\in\Gamma} u^{-1}\left(V - \sum_{i=1,2} \pi_i(-c(e(W_i)) + \beta[r(e_i)(W_i + \lambda D(W_i)) + (1 - r(e_i))U]) \right)$$

Since u' is increasing function and the expected continuation value for the worker is bounded by $\bar{x} = \sup_{J\in\mathcal{J}} \hat{x}_J$, which is finite, we have $\underline{w} \geq u'(\underline{x} + c(\underline{e}) - \beta\bar{x})$. Using the fact that $\hat{J}(V)$ is strictly decreasing in V, we have

$$\begin{aligned}
\hat{J}(V) &< \hat{J}(\underline{x}) \\
&\leq r(\bar{e})y - u'(\underline{x} + c(\underline{e}) - \beta\bar{x}) + \beta\bar{J} \equiv \bar{J}.
\end{aligned}$$

Then, $\hat{J}(V) \leq \bar{J} = \frac{y - u'(\underline{x} + c(\underline{e}) - \beta\bar{x})}{1-\beta}$.

Similarly to the previous argument,

$$\begin{aligned}
\hat{J}(V) &> \hat{J}(\bar{x}) \\
&\geq r(\underline{e})y - u'(\bar{x} + c(\bar{e}) - \beta U) + \beta\underline{J} \equiv \underline{J}.
\end{aligned}$$

Then, $\hat{J}(V) \geq \underline{J} = \frac{r(\underline{e})y - u'(\bar{x}+c(\bar{e})-\beta U)}{1-\beta}$. Hence, $\hat{J}(V)$ is bounded both from below and above.

(iii) Concavity of $\hat{J}(V)$ can be shown with two-point convexification result developed by Menzio and Shi (2008) and omitted in this paper. $\square$

Lemma 7.9. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\hat{J}_j(W)$ be the firm's value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\hat{J}_m - \hat{J}_n|| < \varepsilon_T \rho$.*

Continuity of the operator with respect to $J \in \mathcal{J}(X)$.

Proof of Lemma 7.9: Continuity of the operator.

Let F_j be the objective function of the firms optimal contracting problem implied by J_j: $F_j : \Gamma \times X \to \mathbb{R}$. Consider $J_m, J_n \in \mathcal{J}(X)$ such that $||J_m - J_n|| < \rho$. Take $V \in X$ such that $\hat{J}_m(V) - \hat{J}_n(V) > 0$. Let γ_j be the maximizer of F_j and $w_j(\gamma)$ be the wage function given by J_j. Then, we have

$$
\begin{aligned}
0 &\leq |\hat{J}_m(V) - \hat{J}_n(V)| \\
&= |F_m(\gamma_m, V) - F_n(\gamma_n, V)| \\
&\leq |F_m(\gamma_m, V) - F_n(\gamma_m, V)| \\
&\leq \Big| - w_m(\gamma_m) + \sum_i \pi_{i,m} r(e_m(W_{i,m}))\,[y + \beta(1 - \lambda \hat{p}_m(W_{i,m}))J_m(W_{i,m})] \\
&\quad + w_n(\gamma_m) - \sum_i \pi_{i,m} r(e_n(W_{i,m}))\,[y + \beta(1 - \lambda \hat{p}_n(W_{i,m}))J_n(W_{i,m})] \Big| \\
&\leq |w_m(\gamma_m) - w_n(\gamma_m)| \\
&\quad + \sum_i \pi_{i,m} \Big| r(e_m(W_{i,m}))\,[y + \beta(1 - \lambda \hat{p}_m(W_{i,m}))J_m(W_{i,m})] \\
&\qquad\qquad - r(e_n(W_{i,m}))\,[y + \beta(1 - \lambda \hat{p}_n(W_{i,m}))J_n(W_{i,m})] \Big|.
\end{aligned}
$$

We want to estimate a bound for $|\hat{J}_m(V) - \hat{J}_n(V)|$. We will consider a bound for each part of the last expression separately as follows.

1. $|w_m(\gamma_m) - w_n(\gamma_m)|$:

Since u is concave function, for any w_1 and w_2, $|w_1 - w_2|u' < |u(w_1) - u(w_2)|$. Also,

$$u(w_m(\gamma_m)) = V - \beta U_m - \sum_i \pi_{i,m}[-c(e_m(W_{i,m})) + \beta r(e_m(W_{i,m}))\Omega_m(W_{i,m})$$

$$u(w_n(\gamma_m)) = V - \beta U_n - \sum_i \pi_{i,m}[-c(e_n(W_{i,m})) + \beta r(e_n(W_{i,m}))\Omega_n(W_{i,m})$$

Then

$$
\begin{aligned}
&|u(w_m(\gamma_m)) - u(w_n(\gamma_m))| \\
\leq\ & \beta|U_m - U_n| + \sum_i \pi_{i,m}\Big\{|c(e_m(W_{i,m})) - c(e_m(W_{i,m}))| \\
&\qquad + \beta|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})|\Big\}
\end{aligned}
$$

Now, consider the last part: $|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})|$.

$$
\begin{aligned}
&|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})| \\
\leq\ & |r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})| \\
=\ & |r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|\bar{x} + r(e_n(W_{i,m}))|\Omega_m(W_{i,m}) - \Omega_n(W_{i,m})| \\
\leq\ & r'(\underline{e})|e_m(W_{i,m}) - e_n(W_{i,m})|\bar{x} + |\Omega_m(W_{i,m}) - \Omega_n(W_{i,m})| \\
\leq & (r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega)\rho
\end{aligned}
$$

I use the fact that $\Omega(\cdot)$ is bounded by $\bar{x}$. Collecting them together, we have

$$
\begin{aligned}
&|u(w_m(\gamma_m)) - u(w_n(\gamma_m))| \\
\leq\ & (\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega))\rho.
\end{aligned}
$$

Hence

$$
|w_m(\gamma_m) - w_n(\gamma_m)| \leq u'^{-1} \cdot (\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega))\rho.
$$

2.

$$
\begin{aligned}
\sum_i \pi_{i,m}\Big| & r(e_m(W_{i,m}))\left[y + \beta(1 - \lambda\hat{p}_m(W_{i,m}))J_m(W_{i,m})\right] \\
& - r(e_n(W_{i,m}))\left[y + \beta(1 - \lambda\hat{p}_n(W_{i,m}))J_n(W_{i,m})\right]\Big| :
\end{aligned}
$$

This expression can still be divided into subcomponents after expanding the brackets and collecting terms:

(ii) $|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})|$:

$$\begin{aligned}
&|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})| \\
\leq\ &|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})| \\
=\ &|r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|J_m(W_{i,m}) \\
&+ r(e_n(W_{i,m}))|J_m(W_{i,m}) - J_n(W_{i,m})| \\
\leq\ &r'(\underline{e})|e_m(W_{i,m}) - e_n(W_{i,m})|\bar{J} \\
&+ |J_m(W_{i,m}) - J_m(W_{i,m})| \\
\leq\ &(r'(\underline{e})\varepsilon_e\bar{J} + 1)\rho.
\end{aligned}$$

(iii) $|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})|$:

$$\begin{aligned}
&|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_m(W_{i,m})| \\
&+\ |\hat{p}_n(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
=\ &|\hat{p}_m(W_{i,m}) - \hat{p}_n(W_{i,m})|J_m(W_{i,m}) \\
&+\ \hat{p}_n(W_{i,m})|J_m(W_{i,m}) - J_n(W_{i,m})| \\
\leq &(\varepsilon_p\bar{J} + 1)\rho.
\end{aligned}$$

(iv) $|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})|$:

$$\begin{aligned}
&|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
=\ &|r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|\hat{p}_m(W_{i,m})J_m(W_{i,m}) \\
&+ r(e_n(W_{i,m}))|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &((r'(\underline{e})\varepsilon_e\bar{J} + 1)\bar{J} + (\varepsilon_p\bar{J} + 1))\rho
\end{aligned}$$

Finally, putting everything together gives that

$$\begin{aligned}
&|\hat{J}_m(V) - \hat{J}_n(V)| \\
\leq & u'^{-1} \cdot \big(\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega)\big)\,\rho \\
&+ \big\{(y + \beta + \beta\lambda\bar{J})(r'(\underline{e})\varepsilon_e\bar{J} + 1) + r(\bar{e})(\varepsilon_p\bar{J} + 1)\big\}\,\rho \\
\equiv & \varepsilon_T\rho.
\end{aligned}$$

□

Proof of Proposition 3.1: Existence of a Block Recursive Equilibrium.

Given Lemma 7.8 and 7.9, we can show that the operator T satisfies the assumptions of Schauder's Fixed Point Theorem (Stokey and Lucas with Prescott, 1989, Theorem 17.4); that is, (i) T is continuous, (ii) the family of functions $T(\mathcal{J})$ is equicontinuous, and (iii) T maps the set $\mathcal{J}(X)$ into itself. Therefore, there exists a firm's value function $J^*(V) \in \mathcal{J}(X)$ such that $TJ^* = J^*$. Denote with $\{\theta^*, D^*, m^*, U^*, e^*\}$ the respective functions associated to J^*. By construction,the functions $\{J^*, \theta^*, D^*, m^*, U^*, e^*\}$ satisfy conditions in the definition of a Recursive Equilibrium and do not depend on the distribution of workers. Therefore, they constitute a Block Recursive Equilibrium.

8 Appendix B: The Case of No On-the-Job Search

This appendix examines the case where there is no on-the-job search. This corresponds to the case where $\lambda_e = 0$. Then, the optimal contracting problem becomes:

$$J(V) = \max_{\xi} \{r(e)y - w_i + \beta r(e)J(W)\} \tag{10}$$

subject to

$$\begin{aligned}\xi \in \Xi'' = \Big\{ \{w, e, W\}_{i=1,2} : \quad & W \in X \\ & V = u(w) - c(e) + \beta\big(r(e)W + (1 - r(e))U\big) \\ & - c'(e) + \beta r'(e)(W - U) = 0 \Big\}.\end{aligned}$$

where I simplify it by ignoring the lottery to clarify the point. By reducing the problem following the method in the paper, it becomes:

$$J(V) = \max_{W} \Big\{ r(g((W - U))\big(y + \beta J(W)\big) - w \Big\}$$

where

$$w = u^{-1}\big(V + c(g((W - U))) - \beta r(g((W - U)))(W - U) - \beta U\big).$$

Taking the first order condition with respect to W and rearranging term give:

$$\frac{1}{u'(w(W_i))} - \frac{1}{u'(w)} = \frac{r'(e(W))}{r(e(W))} \left(\frac{y}{\beta} + J(W)\right) g'((W - U)). \tag{11}$$

The right hand side is clearly positive. Therefore, we have

$$\frac{1}{u'(w(W))} - \frac{1}{u'(w)} > 0. \tag{12}$$

Therefore, $w(W) \geq w$ by the concavity of u. This implies that, even though there is no on-the-job search, it is still optimal for the firm to pay increasing wages over tenure for inducing worker's effort.

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