Efficient FFT-based Computation of Option Greeks

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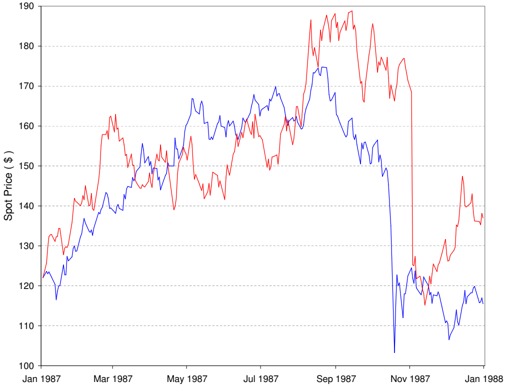
CAIMS 2009

Models in Equity Markets



- Merton/Kou jump-diffusion
- Exponential Lévy models - Variance Gamma, Carr-Geman-Madan-Yor

Numerical Methods for Option Pricing

Monte-Carlo methods

Tree methods

Finite difference methods

- Alternating Direction Implicit-FFT - Andersen and Andreasen (2000)
- Implicit-Explicit (IMEX) - Cont and Tankov (2004)
- IMEX Runge-Kutta - Briani, Natalini, and Russo (2004)
- Fixed Point Iteration - d'Halluin, Forsyth, and Vetzal (2005)

Quadrature methods

- Reiner (2001)
- QUAD - Andricopoulos, Widdicks, Duck, and Newton (2003)
- Q-FFT - O'Sullivan (2005)

Transform-based methods

- Carr and Madan (1999)
- Raible (2000)
- Lewis (2001)

Motivation for Research

Develop a framework for numerical pricing of financial derivatives in various markets that ranks high in

- Precision, speed and convergence
- Efficient handling of path-independent and discretely monitored derivatives
- Ability to handle path-dependent and multi-asset derivatives
- Generic handling of various spot-price models
- Utilization of multi-core architectures

The Approach

- Derive the PIDE for the option price
- Transform the PIDE into ODE in Fourier space and solve the ODE
- Utilize FFT to efficiently compute Fourier transforms

Pricing Framework

The Model

$$\mathbf{S}(t) = \mathbf{S}(0)\, e^{\mathbf{X}(t)}$$

where $\mathbf{X}(t)$ is a Lévy process with characteristic triplet $(\gamma, \mathbf{\Sigma}, \boldsymbol{\nu})$.

- The discount-adjusted and log-transformed price process $v(t, \mathbf{X}(t)) \triangleq e^{-r(T-t)} V(t, \mathbf{S}(0) e^{\mathbf{X}(t)})$ satisfies a PIDE

$$\begin{cases} (\partial_t + \mathcal{L})\, v(t, \mathbf{x}) & = \mathbf{0}\,, \\ v(T, \mathbf{x}) & = \varphi(\mathbf{S}(0)\, e^{\mathbf{x}})\,, \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator of the Lévy process:

$$\mathcal{L}g(\mathbf{x}) = \left(\gamma' \partial_{\mathbf{x}} + \tfrac{1}{2}\, \partial_{\mathbf{x}}' \mathbf{\Sigma} \partial_{\mathbf{x}}\right) g(\mathbf{x}) + \int \left(g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}) - \mathbb{1}_{\{|\mathbf{y}|<1\}} \mathbf{y}' \partial_{\mathbf{x}} g(\mathbf{x})\right) \nu(d\mathbf{y}).$$

Pricing Framework in Fourier Space

- Applying the Fourier transform to the infinitesimal generator $\mathcal{L}$ of $\mathbf{X}(t)$ allows the characteristic exponent $\Psi(\omega)$ to be factored out:

$$\mathcal{F}[\mathcal{L}v](t,\omega) = \Psi(\omega)\mathcal{F}[v](t,\omega)\,,$$

  where

$$\Psi(\omega) = i\gamma'\omega - \tfrac{1}{2}\,\omega'\mathbf{\Sigma}\omega + \int \left(e^{i\omega'\mathbf{y}} - 1 - i\mathbb{1}_{\{|\mathbf{y}|<1\}}\omega'\mathbf{y}\right)\nu(d\mathbf{y})\,.$$

- The PIDE is therefore transformed into a d-parameter family of ODEs parameterized by ω:

$$\begin{cases} \partial_t\mathcal{F}[v](t,\omega) + \Psi(\omega)\mathcal{F}[v](t,\omega) & = 0\,, \\ \mathcal{F}[v](T,\omega) & = \mathcal{F}[\varphi](\omega)\,. \end{cases}$$

Pricing Framework in Fourier Space

- Given the value of $\mathcal{F}[v](t,\omega)$ at time $t_2 \leq T$, the system is easily solved to find the value at time $t_1 < t_2$:

$$\mathcal{F}[v](t_1,\omega) = \mathcal{F}[v](t_2,\omega) \cdot e^{\Psi(\omega)(t_2-t_1)} .$$

- Taking the inverse transform leads to the final result

$$v(t_1,\mathbf{x}) = \mathcal{F}^{-1}\left[\mathcal{F}[v](t_2,\omega) \cdot e^{\Psi(\omega)(t_2-t_1)}\right](\mathbf{x}) .$$

- In discrete space, a step backwards is computed via

FST Method - Jackson, Jaimungal, Surkov 2008

$$\mathbf{v}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}[\mathbf{v}_m] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]$$

American Put Option Results with Penalty Method

N	M	Value	Change	$\log_2$(Ratio)	Time (s)
2048	128	9.22478538			0.027
4096	256	9.22523484	0.0004495		0.109
8192	512	9.22538196	0.0001471	1.6114	0.451
16384	1024	9.22542478	0.0000428	1.7808	1.869
32768	2048	9.22543516	0.0000104	2.0444	8.195

- *Option:* American put option $S = 90.0$, $K = 98.0$, $T = 0.25$
- *Model:* CGMY model $C = 0.42$, $G = 4.37$, $M = 191.2$, $Y = 1.0102$, $r = 0.06$
- *Convergence:* 2 in space and 2 in time
- *Reference price:* 9.2254803 from Forsyth, Wan and Wang 2007

FST Methods Characteristics

- All exponential Lévy models, and path-dependent and multi-asset options handled are generically
- Two FFTs per time-step are required
- No time-stepping for European options or between monitoring dates of discretely monitored options
- Second order convergence in space and second order convergence in time for American options with penalty method
- Extendable to computation of the Greeks

The Greeks & Sensitivities

- The Greeks (option price sensitivities) play a paramount role in risk management
- Options can be immunized from gains or losses by taking offsetting position in the underlying asset, with the position determined by the Greeks

Function	f	1^{st} Derivative	$\frac{\partial f}{\partial S}$	2^{nd} Derivative	$\frac{\partial^2 f}{\partial S^2}$
Price	V	Delta	$\frac{\partial V}{\partial S}$	Gamma	$\frac{\partial^2 V}{\partial S^2}$
Vega	$\frac{\partial V}{\partial \sigma}$	Vanna	$\frac{\partial^2 V}{\partial \sigma \partial S}$	Zomma	$\frac{\partial^3 V}{\partial \sigma \partial S^2}$
Theta	$\frac{\partial V}{\partial t}$	Charm	$-\frac{\partial^2 V}{\partial t \partial S}$	Color	$-\frac{\partial^3 V}{\partial t \partial S^2}$
Rho	$\frac{\partial V}{\partial r}$	Rhonna*	$\frac{\partial^2 V}{\partial r \partial S}$	Rhomma*	$\frac{\partial^3 V}{\partial r \partial S^2}$
Volga	$\frac{\partial^2 V}{\partial \sigma^2}$	Ultima	$\frac{\partial^3 V}{\partial \sigma^2 \partial S}$	Volamma*	$\frac{\partial^4 V}{\partial \sigma^2 \partial S^2}$

The Greeks & Sensitivities

- Other parameter sensitivities are valuable in model calibration
  Jump arrival rate $\frac{\partial V}{\partial \lambda}$, jump mean $\frac{\partial V}{\partial \tilde{\boldsymbol{\mu}}}$, and jump volatility $\frac{\partial V}{\partial \tilde{\boldsymbol{\Sigma}}}$

The Greeks & Sensitivities

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The Approach

- Derive the PIDE for the Greeks / sensitivities
- Solve the PIDE by transforming into ODE in Fourier space
- Use FFT to efficiently compute Fourier transforms

Notes:

- Fourier transforms of 1^{st} and 2^{nd} derivatives are easily computed from the Fourier transform of the function (without solving a new PIDE)
- For calculation of jump paramer sensitivities, it is convenient to transform the option pricing PIDE into ODE first and then differentiate to obtain an ODE for the Fourier transform of sensitivities.

Delta and Gamma

The Fourier transform of Delta (or any 1^{st} derivative) and Gammma (or any 2^{nd} derivative) can be computed from the Fourier transform of option (function) values via scaling:

- Delta

$$\partial_{\mathbf{S}_k} v(t,\mathbf{x}) = \partial_{\mathbf{x}_k} v(t,\mathbf{x})\, e^{-\mathbf{x}_k} = \mathcal{F}^{-1}\left[(i\omega_k)\mathcal{F}\left[v\right](t,\boldsymbol{\omega})\right](\mathbf{x})\, e^{-\mathbf{x}_k}$$

$$\boldsymbol{\Delta}_{k,m-1} = \mathrm{FFT}^{-1}\left[i\omega_k \cdot \mathrm{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right] \cdot e^{-\mathbf{x}_k}$$

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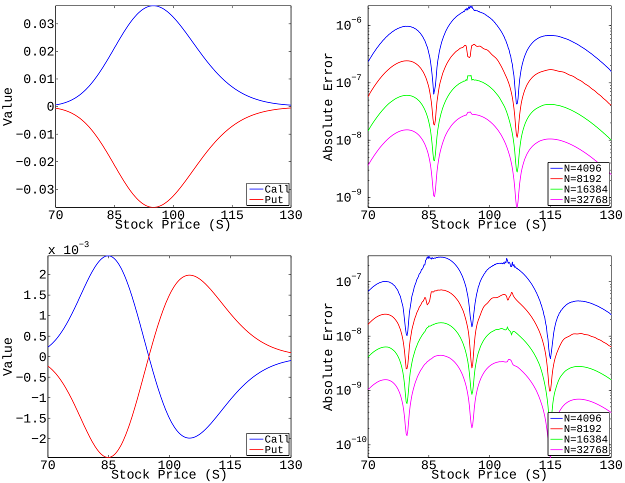
- Gamma

$$\begin{aligned}\partial^2_{S_k^2} v(t, \mathbf{x}) &= \left(\partial_{\mathbf{x}_k^2} - \partial_{\mathbf{x}_k}\right) v(t, \mathbf{x})\, e^{-2\mathbf{x}_k} \\ &= \mathcal{F}^{-1}\left[-(i\omega_k + \omega_k^2)\mathcal{F}\left[v\right](t, \boldsymbol{\omega})\right](\mathbf{x})\, e^{-2\mathbf{x}_k}\end{aligned}$$

$$\boldsymbol{\Gamma}_{k,m-1} = \text{FFT}^{-1}\left[-(i\omega_k + \omega_k^2) \cdot \text{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right] \cdot e^{-2\mathbf{x}_k}$$

Delta and Gamma Values and Errors



- Digital option $K = 100$, $T = 0.5$
- Merton jump-diffusion model $\sigma = 0.15$, $r = 0.05$, $q = 0.02$, $\lambda = 0.1$, $\tilde{\mu} = -1.08$, $\tilde{\sigma} = 0.4$

Vega

- Applying derivative with respect to σ_k to the pricing PIDE, a PIDE satisfied by Vega is obtained:

$$\partial_{\sigma_k} \{(\partial_t + \mathcal{L})\, v(t,\mathbf{x})\} = (\partial_t + \mathcal{L})\, \partial_{\sigma_k} v(t,\mathbf{x}) + \mathcal{H}_{\sigma_k} v(t,\mathbf{x}) = 0\,,$$

where $\mathcal{H}_{\sigma_k} = (\partial_{\sigma_k}\gamma)'\partial_{\mathbf{x}} + \partial_{\mathbf{x}}'(\partial_{\sigma_k}\mathbf{\Sigma})\partial_{\mathbf{x}}$

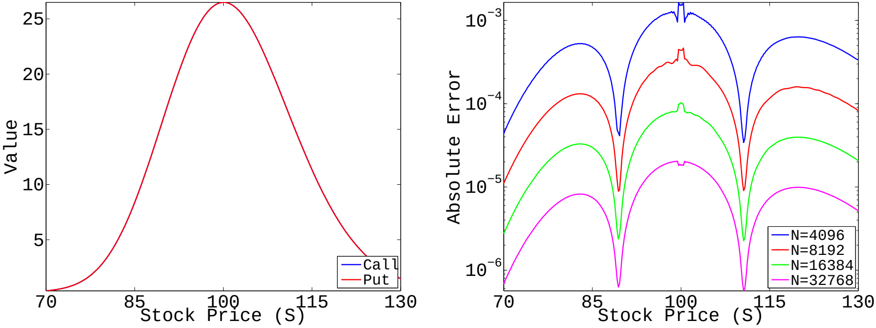
- Applying the Fourier transform to the Vega PIDE, an ODE with a source term is obtained. It is solved explicitly:

$$\partial_{\sigma_k} v(t,\mathbf{x}) = (T-t)\,\mathcal{F}^{-1}\left[\mathcal{F}[\mathcal{H}_{\sigma_k}](\boldsymbol{\omega}) \cdot \mathcal{F}[v](t,\boldsymbol{\omega})\right](\mathbf{x})$$

$$\nabla_{k,m-1} = \Delta t_m \mathrm{FFT}^{-1}\left[\mathcal{F}[\mathcal{H}_{\sigma_k}](\boldsymbol{\omega}) \cdot \mathrm{FFT}[\mathbf{v}_m] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]$$

- For example, in the one dimensional case, $\mathcal{F}[\mathcal{H}_{\sigma_k}](\boldsymbol{\omega}) = -(i\omega + \omega^2)\sigma$

Vega Values and Errors



- European option $K = 100$, $T = 0.5$
- Merton jump-diffusion model $\sigma = 0.15, r = 0.05, q = 0.02$, $\lambda = 0.1$, $\tilde{\mu} = 0.1, \tilde{\sigma} = 0.4$

Volga

- Applying the mixed derivative with respect to σ_k and σ_l to the pricing PIDE, a PIDE satisfied by the mixed Volga is obtained:

$$\partial^2_{\sigma_k\sigma_l}\{(\partial_t+\mathcal{L})\,v(t,\mathbf{x})\} = (\partial_t+\mathcal{L})\,\partial^2_{\sigma_k}v(t,\mathbf{x}) + \mathcal{H}_{\sigma_k\sigma_l}v(t,\mathbf{x}) = 0\,,$$

  where $\mathcal{H}_{\sigma_k\sigma_l} = \partial'_{\mathbf{x}}(\partial^2_{\sigma_k\sigma_l}\boldsymbol{\Sigma})\partial_{\mathbf{x}}$.
- Applying the Fourier transform to the mixed Volga PIDE yields an ODE with a source term, which can be solved explicitly:

$$\partial_{\sigma_k\sigma_l}v(t,\mathbf{x}) = (T-t)\,\mathcal{F}^{-1}\left[\mathcal{F}\left[\mathcal{H}_{\sigma_k\sigma_l}\right](\boldsymbol{\omega})\cdot\mathcal{F}\left[v\right](t,\boldsymbol{\omega})\right](\mathbf{x})\,.$$

$$\nabla_{k,l,m-1} = \Delta t_m \mathrm{FFT}^{-1}\left[\mathcal{F}\left[\mathcal{H}_{\sigma_k\sigma_l}\right](\boldsymbol{\omega})\cdot\mathrm{FFT}\left[\mathbf{v}_m\right]\cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]\,,$$

  where the $\mathcal{F}\left[\mathcal{H}_{\sigma_k\sigma_l}\right](\boldsymbol{\omega})$ term can be computed analytically.
- For example, in the case of 2D BSM model, $\mathcal{F}\left[\mathcal{H}_{\sigma_k\sigma_l}\right](\boldsymbol{\omega}) = -\rho\omega_1\omega_2$.

Volga

- Applying the mixed derivative with respect to σ_k and σ_l to the pricing PIDE, a PIDE satisfied by the mixed Volga is obtained:

$$\partial^2_{\sigma_k \sigma_l} \{(\partial_t + \mathcal{L})\, v(t,\mathbf{x})\} = (\partial_t + \mathcal{L})\, \partial^2_{\sigma_k} v(t,\mathbf{x}) + \mathcal{H}_{\sigma_k \sigma_l} v(t,\mathbf{x}) = 0\,,$$

  where $\mathcal{H}_{\sigma_k \sigma_l} = \partial'_{\mathbf{x}}(\partial^2_{\sigma_k \sigma_l} \boldsymbol{\Sigma})\partial_{\mathbf{x}}$.
- Applying the Fourier transform to the mixed Volga PIDE yields an ODE with a source term, which can be solved explicitly:

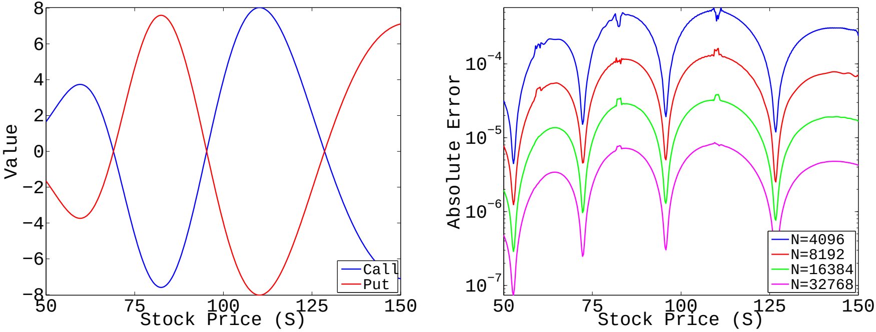
$$\partial_{\sigma_k \sigma_l} v(t,\mathbf{x}) = (T - t)\, \mathcal{F}^{-1} \left[\mathcal{F}\left[\mathcal{H}_{\sigma_k \sigma_l}\right](\boldsymbol{\omega}) \cdot \mathcal{F}\left[v\right](t,\boldsymbol{\omega})\right](\mathbf{x})\,.$$

$$\nabla_{k,l,m-1} = \Delta t_m \mathrm{FFT}^{-1} \left[\mathcal{F}\left[\mathcal{H}_{\sigma_k \sigma_l}\right](\boldsymbol{\omega}) \cdot \mathrm{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]\,,$$

  where the $\mathcal{F}\left[\mathcal{H}_{\sigma_k \sigma_l}\right](\boldsymbol{\omega})$ term can be computed analytically.
- For example, in the case of 2D BSM model, $\mathcal{F}\left[\mathcal{H}_{\sigma_k \sigma_l}\right](\boldsymbol{\omega}) = -\rho\omega_1\omega_2$.

- If $k = l$ the solution of the Volga ODE depends on the solution to the Vega ODE. $\mathcal{F}\left[\mathcal{H}_{\sigma_k^2}\right](\boldsymbol{\omega})$ can still be computed analytically. For example, in the case of the BSM model $\mathcal{F}\left[\mathcal{H}_{\sigma_k^2}\right](\boldsymbol{\omega}) = -(i\omega + \omega^2)\Delta t_m + (i\omega + \omega^2)^2\sigma^2\Delta t_m^2$.

Volga Values and Errors



- Digital option $K = 100$, $T = 1.2$
- Black-Scholes-Merton model $\sigma = 0.2, q = 0.01$

Theta and Rho

- Rearranging the pricing PIDE

$$\partial_t v(t, \mathbf{x}) = -\mathcal{L} v(t, \mathbf{x}) = \mathcal{F}^{-1}\left[-\Psi(\omega) \cdot \mathcal{F}[v](t, \omega)\right](\mathbf{x})$$

$$\boldsymbol{\Theta}_{m-1} = \mathrm{FFT}^{-1}\left[-\Psi(\omega) \cdot \mathrm{FFT}\left[\mathbf{v}_m\right] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]$$

Theta and Rho

- Rearranging the pricing PIDE

$$\partial_t v(t,\mathbf{x}) = -\mathcal{L}v(t,\mathbf{x}) = \mathcal{F}^{-1}\left[-\Psi(\omega)\cdot\mathcal{F}[v](t,\omega)\right](\mathbf{x})$$

$$\mathbf{\Theta}_{m-1} = \mathrm{FFT}^{-1}\left[-\Psi(\omega)\cdot\mathrm{FFT}[\mathbf{v}_m]\cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]$$

- Applying derivative with respect to r to the pricing PIDE:

$$\partial_r\left\{(\partial_t+\mathcal{L})\,v(t,\mathbf{x})\right\} = (\partial_t+\mathcal{L})\,\partial_r v(t,\mathbf{x}) + \mathcal{H}_r v(t,\mathbf{x}) = 0\,,$$

where $\mathcal{H}_r = (\partial_r\gamma)'\partial_{\mathbf{x}}$.

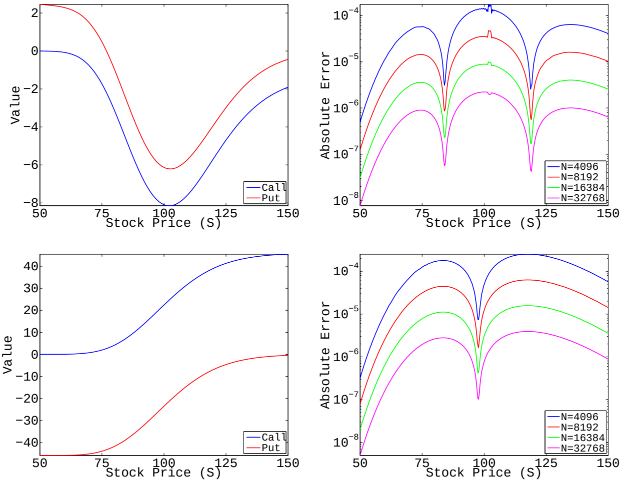
- The Rho PIDE can be solved explicitly in Fourier space:

$$\partial_r v(t,\mathbf{x}) = (T-t)\,\mathcal{F}^{-1}\left[\mathcal{F}[\mathcal{H}_r](\omega)\cdot\mathcal{F}[v](t,\omega)\right](\mathbf{x})$$

$$\mathbf{P}_{m-1} = \Delta t_m \mathrm{FFT}^{-1}\left[\mathcal{F}[\mathcal{H}_r](\omega)\cdot\mathrm{FFT}[\mathbf{v}_m]\cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]$$

- For example, in the case of the BSM model $\mathcal{F}[\mathcal{H}_r](\omega) = i\omega$.

Theta and Rho Values and Errors



- European option $K = 100$, $T = 0.46575$
- Black-Scholes-Merton model $\sigma = 0.25$, $q = 0.01$

Jump Sensitivities

- The pricing ODE in Fourier space can be written as

$$\left(\partial_t + \hat{\mathcal{D}}(\omega) + \hat{\mathcal{J}}(\omega)\right) v(t,\omega) = \mathbf{0}\,.$$

- Differentiating the ODE w.r.t. each model parameter $\star$, an ODEs with a source term is obtained. It is solved explicitly:

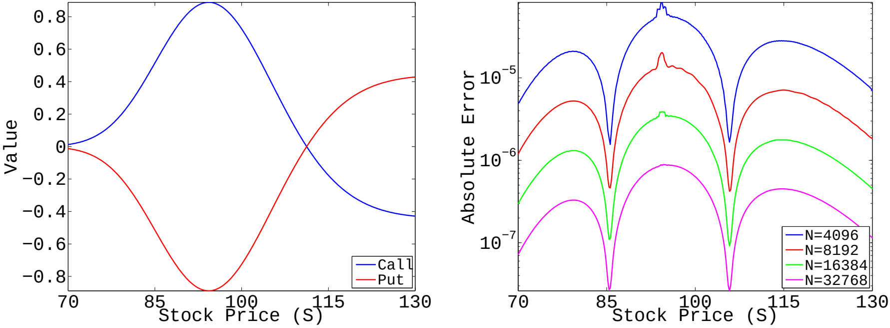
$$\left(\partial_t + \hat{\mathcal{D}}(\omega) + \hat{\mathcal{J}}(\omega)\right) \mathcal{F}[v_\star](t,\omega) + (\hat{\mathcal{D}}_\star(\omega) + \hat{\mathcal{J}}_\star(\omega))\mathcal{F}[v](t,\omega) = \mathbf{0}$$

- The greekFST method is

$$\mathbf{v}_{\star,m-1} = \Delta t_m \mathrm{FFT}^{-1}\left[(\hat{\mathcal{D}}_\star(\omega) + \hat{\mathcal{J}}_\star(\omega)) \cdot \mathrm{FFT}[\mathbf{v}_m] \cdot e^{\Psi(\,\cdot\,)\Delta t_m}\right]$$

- For example, under the Merton jump-diffusion model $\hat{\mathcal{D}}(\omega) = i\gamma'\omega - \frac{1}{2}\omega'\boldsymbol{\Sigma}\omega$ and $\hat{\mathcal{J}}(\omega) = \lambda(e^{i\tilde{\boldsymbol{\mu}}'\omega - \frac{1}{2}\omega'\tilde{\boldsymbol{\Sigma}}\omega} - 1)$

Jump Lambda Values and Errors



- Digital option $K = 100$, $T = 0.5$
- Merton jump-diffusion model $\sigma = 0.15, r = 0.05, q = 0.02, \lambda = 0.1, \tilde{\mu} = -1.08, \tilde{\sigma} = 0.4$

Fourier Space Time-stepping Framework Summary

- Stable and robust, even for options with discontinuous payoffs
- Easily extendable to various stochastic processes and no loss of performance for infinite activity processes
- Can be applied to multi-dimensional mean-reverting and regime-switching problems in a natural manner
- Computationally efficient
  - Computational cost is $O(MNlogN)$ while the error is $O(\Delta x^2 + \Delta t^2)$
  - European options priced in a single time-step
  - Bermudan style options do not require time-stepping between monitoring dates
- Option Greeks can be readily computed
- Price and k Greeks require $k + 2$ FFT evaluations
- American options do not require solving new PIDE

Thank You