Efficient Fourier Transform-based Pricing of Interest Rate Derivatives

Vladimir Surkov
vladimir.surkov@utoronto.ca

Department of Statistical and Actuarial Sciences, University of Western Ontario
The Fields Institute, University of Toronto

Young Researchers Workshop on Finance 2010
March 10, 2010

1. Numerical Methods for Option Pricing
2. Multi-factor Jump-diffusion Model for the Short Rate
3. Fourier Space Time-stepping Method
4. Option Greeks and Static Hedging

Numerical Option Pricing

Option Pricing Problem

$$V(t,\mathbf{x}) = \mathbb{E}^{\mathbb{Q}}_{t,\mathbf{x}}\left[e^{-\int_t^T r(s)ds}\Phi(\mathbf{X}(T))\right]$$

Numerical Option Pricing

Option Pricing Problem

$$V(t, \mathbf{x}) = \mathbb{E}^{\mathbb{Q}}_{t,\mathbf{x}}\left[e^{-\int_t^T r(s)ds}\Phi(\mathbf{X}(T))\right]$$

- Closed-form solutions exist only in a limited number of cases
- Must resort to numerical methods for valuation of path-dependent, early-exercise and exotic options, or under models with jumps

Numerical Option Pricing

Option Pricing Problem

$$V(t, \mathbf{x}) = \mathbb{E}^{\mathbb{Q}}_{t,\mathbf{x}} \left[e^{-\int_t^T r(s)ds} \Phi(\mathbf{X}(T)) \right]$$

- Closed-form solutions exist only in a limited number of cases
- Must resort to numerical methods for valuation of path-dependent, early-exercise and exotic options, or under models with jumps
- A wide range of methods have been developed
  - Monte Carlo and tree methods
  - Finite difference methods
    Andersen, Andreasen (2000), Cont, Tankov (2004), Briani, Natalini, Russo (2004), d'Halluin, Forsyth, Vetzal (2005)
  - Transform-based methods
    Carr, Madan (1999), Raible (2000), Lewis (2001), Lord, Fang, Bervoets, Oosterlee (2008), Jackson, Jaimungal, Surkov (2008)
  - Quadrature methods, Wiener-Hopf factorization, Hilbert and Laplace transforms, Hermite and cosine expansions, etc.

The Fourier Transform

- A function in the space domain $g(t,\mathbf{x})$ can be transformed to a function in the frequency domain $\hat{g}(t,\boldsymbol{\omega})$ and vice-versa using the continuous Fourier transform (CFT):

$$\mathcal{F}[g](t,\boldsymbol{\omega}) \triangleq \int_{-\infty}^{\infty} g(t,\mathbf{x})e^{-i\boldsymbol{\omega}'\mathbf{x}}d\mathbf{x}$$

$$\mathcal{F}^{-1}[\hat{g}](t,\mathbf{x}) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(t,\boldsymbol{\omega})e^{i\boldsymbol{\omega}'\mathbf{x}}d\boldsymbol{\omega}$$

The Fourier Transform

- A function in the space domain $g(t, \mathbf{x})$ can be transformed to a function in the frequency domain $\hat{g}(t, \boldsymbol{\omega})$ and vice-versa using the continuous Fourier transform (CFT):

$$\mathcal{F}[g](t, \boldsymbol{\omega}) \triangleq \int_{-\infty}^{\infty} g(t, \mathbf{x}) e^{-i\boldsymbol{\omega}'\mathbf{x}} d\mathbf{x}$$

$$\mathcal{F}^{-1}[\hat{g}](t, \mathbf{x}) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(t, \boldsymbol{\omega}) e^{i\boldsymbol{\omega}'\mathbf{x}} d\boldsymbol{\omega}$$

- CFT is a linear operator that maps spatial derivatives ∂_x into multiplications in the frequency domain:

$$\mathcal{F}[\partial_{\mathbf{x}}^n g](t, \boldsymbol{\omega}) = i\boldsymbol{\omega}\, \mathcal{F}\left[\partial_{\mathbf{x}}^{n-1} g\right](t, \boldsymbol{\omega}) = \cdots = (i\boldsymbol{\omega})^n \mathcal{F}[g](t, \boldsymbol{\omega})$$

Fourier Space Time-stepping Method

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

Fourier Space Time-stepping Method

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- The option price satisfied a PIDE

$$\left(\partial_t + \mathcal{L}\right) V(t, \mathbf{x}) = \ \mathbf{0}$$

Fourier Space Time-stepping Method

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- The option price satisfied a PIDE

$$\left(\partial_t + \mathcal{L}\right) V(t, \mathbf{x}) = \mathbf{0}$$

- Applying Fourier transform obtain an ODE in time parameterized by ω

$$\left(\partial_t + \Psi(\omega)\right) \mathcal{F}[V](t, \omega) = \mathbf{0}$$

Fourier Space Time-stepping Method

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- The option price satisfied a PIDE

$$\big(\partial_t + \mathcal{L}\big)\, V(t, \mathbf{x}) = \mathbf{0}$$

- Applying Fourier transform obtain an ODE in time parameterized by ω

$$\big(\partial_t + \Psi(\omega)\big)\, \mathcal{F}[V](t, \omega) = \mathbf{0}$$

- The ODE can be solved analytically

$$\mathcal{F}[V](t, \omega) = \mathcal{F}[\Phi](\omega) \cdot e^{\Psi(\omega)\Delta t}$$

FST Method - Convolution Formulation

- Write expectation as a convolution of payoff and density

$$V(t, \mathbf{x}) = \int_{-\infty}^{\infty} \Phi(\mathbf{x} + \mathbf{y}) \, f_{\mathbf{X}(T-t)}(\mathbf{y}) \, d\mathbf{y}$$

FST Method - Convolution Formulation

- Write expectation as a convolution of payoff and density

$$V(t,\mathbf{x}) = \int_{-\infty}^{\infty} \Phi(\mathbf{x}+\mathbf{y})\, f_{\mathbf{X}(T-t)}(\mathbf{y})\, d\mathbf{y}$$

- Convolution in real space corresponds to multiplication in Fourier space

$$\mathcal{F}[V](t,\boldsymbol{\omega}) = \mathcal{F}[\Phi](\boldsymbol{\omega}) \cdot \mathcal{F}[f_{\mathbf{X}(\Delta t)}](-\boldsymbol{\omega}) \ ,$$

  where the characteristic function of the density is known analytically $\mathcal{F}[f_{\mathbf{X}(\Delta t)}](-\boldsymbol{\omega}) = e^{\Psi(\boldsymbol{\omega})\Delta t}$

FST Method - Convolution Formulation

- Write expectation as a convolution of payoff and density

$$V(t,\mathbf{x}) = \int_{-\infty}^{\infty} \Phi(\mathbf{x}+\mathbf{y})\ f_{\mathbf{X}(T-t)}(\mathbf{y})\ d\mathbf{y}$$

- Convolution in real space corresponds to multiplication in Fourier space

$$\mathcal{F}[V](t,\boldsymbol{\omega}) = \mathcal{F}[\Phi](\boldsymbol{\omega}) \cdot \mathcal{F}[f_{\mathbf{X}(\Delta t)}](-\boldsymbol{\omega})\ ,$$

  where the characteristic function of the density is known analytically $\mathcal{F}[f_{\mathbf{X}(\Delta t)}](-\boldsymbol{\omega}) = e^{\Psi(\boldsymbol{\omega})\Delta t}$

- PIDE formulation allows to extend results to early-exercise problems and regime-switching models.

FST Method - Convolution Formulation

- Write expectation as a convolution of payoff and density

$$V(t,\mathbf{x}) = \int_{-\infty}^{\infty} \Phi(\mathbf{x}+\mathbf{y})\, f_{\mathbf{X}(T-t)}(\mathbf{y})\, d\mathbf{y}$$

- Convolution in real space corresponds to multiplication in Fourier space

$$\mathcal{F}[V](t,\boldsymbol{\omega}) = \mathcal{F}[\Phi](\boldsymbol{\omega}) \cdot \mathcal{F}[f_{\mathbf{X}(\Delta t)}](-\boldsymbol{\omega}) \ ,$$

  where the characteristic function of the density is known analytically $\mathcal{F}[f_{\mathbf{X}(\Delta t)}](-\boldsymbol{\omega}) = e^{\Psi(\boldsymbol{\omega})\Delta t}$

- PIDE formulation allows to extend results to early-exercise problems and regime-switching models.
- In discrete space, a step backwards is computed via

FST Method

$$\mathbf{V}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\mathbf{V}_m\right] \cdot e^{\Psi(\boldsymbol{\omega})\Delta t_m}\right]$$

The Model under $\mathbb{Q}$

- The short rate $r(t)$ is driven by d market factors $X_j(t)$

$$r(t) = X_1(t) + \ldots + X_d(t) + \gamma(t),$$

where $\gamma(t)$ is a deterministic function chosen to fit the currently-observed term structure of bond prices.

The Model under $\mathbb{Q}$

- The short rate $r(t)$ is driven by d market factors $X_j(t)$

$$r(t) = X_1(t) + \ldots + X_d(t) + \gamma(t),$$

where $\gamma(t)$ is a deterministic function chosen to fit the currently-observed term structure of bond prices.

- The dynamics of $X_j(t)$ satisfies

$$dX_j(t) = -\kappa_j X_j(t)dt + \sigma_j \, dW_j(t) + \int_{-\infty}^{\infty} z \left(\mu_j(dz, dt) - \nu_j(dz, dt)\right)$$

with $X_j(0) = 0$.

The Model under $\mathbb{Q}$

- The short rate $r(t)$ is driven by d market factors $X_j(t)$

$$r(t) = X_1(t) + \ldots + X_d(t) + \gamma(t),$$

  where $\gamma(t)$ is a deterministic function chosen to fit the currently-observed term structure of bond prices.

- The dynamics of $X_j(t)$ satisfies

$$dX_j(t) = -\kappa_j X_j(t)dt + \sigma_j \, dW_j(t) + \int_{-\infty}^{\infty} z \left(\mu_j(dz, dt) - \nu_j(dz, dt)\right)$$

  with $X_j(0) = 0$.

- Generalizes the Hull-White model and it's multi-factor and jump-extensions

Option Pricing via PIDEs

- The price of any traded asset satisfies the PIDE

$$\begin{cases} \big(\partial_t + \mathcal{H} - (\bar{\mathbf{x}} + \gamma(t))\big) V(t, \mathbf{x}) & = \mathbf{0}\,, \\ V(T, \mathbf{x}) & = \Phi(\mathbf{x}; T)\,, \end{cases}$$

where $\bar{\mathbf{x}} = x_1 + \ldots + x_d$

Option Pricing via PIDEs

- The price of any traded asset satisfies the PIDE

$$\begin{cases} \left(\partial_t + \mathcal{H} - (\bar{\mathbf{x}} + \gamma(t))\right) V(t, \mathbf{x}) & = & \mathbf{0}\,, \\ V(T, \mathbf{x}) & = & \Phi(\mathbf{x}; T)\,, \end{cases}$$

where $\bar{\mathbf{x}} = x_1 + \ldots + x_d$

- The $\mathbb{Q}$-infinitesimal generator $\mathcal{H}$ of the joint processes $\mathbf{X}(t)$ acts on $f(t, \mathbf{x})$

$$\mathcal{H}f(\mathbf{x}) = -(\boldsymbol{\mu} + \boldsymbol{\kappa}\mathbf{x})'\,\partial_{\mathbf{x}} f(\mathbf{x}) + \tfrac{1}{2}\,\partial_{\mathbf{x}} f(\mathbf{x})'\boldsymbol{\Sigma}\,\partial_{\mathbf{x}} f(\mathbf{x}) + \int_{\mathbb{R}^d} \left(f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x})\right)\nu(d\mathbf{z})\,,$$

where $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_d]'$, $\boldsymbol{\kappa} = \mathrm{diag}[\kappa_1, \ldots, \kappa_d]$, $\boldsymbol{\Sigma}_{jk} = \sigma_j \sigma_k \rho_{jk}$, $\partial_{\mathbf{x}} f(\mathbf{x}) = [\partial_{x_1} f(\mathbf{x}), \ldots, \partial_{x_d} f(\mathbf{x})]'$, and $\nu(d\mathbf{z})$ is the multi-dimensional jump measure.

Pricing under $\mathbb{Q}_T$

- Let $U(t,\mathbf{x}) = \mathbb{E}_{t,\mathbf{x}}^{\mathbb{Q}_T}\left[\Phi(\mathbf{X}(T);T)\right]$ be the option price under the T-forward measure. Then

$$V(t,\mathbf{x}) = P(t,\mathbf{x};T)\cdot U(t,\mathbf{x})$$

Pricing under $\mathbb{Q}_T$

- Let $U(t, \mathbf{x}) = \mathbb{E}^{\mathbb{Q}_T}_{t,\mathbf{x}}\left[\Phi(\mathbf{X}(T); T)\right]$ be the option price under the T-forward measure. Then

$$V(t, \mathbf{x}) = P(t, \mathbf{x}; T) \cdot U(t, \mathbf{x})$$

- $U(t, \mathbf{x})$ satisfies the PIDE

$$\begin{cases} \big(\partial_t + \mathcal{L}(t)\big)\, U(t, \mathbf{x}) & = \; 0\,, \\ U(T, \mathbf{x}) & = \; \Phi(\mathbf{x}; T)\,. \end{cases}$$

Pricing under $\mathbb{Q}_T$

- Let $U(t,\mathbf{x}) = \mathbb{E}^{\mathbb{Q}_T}_{t,\mathbf{x}}\left[\Phi(\mathbf{X}(T);T)\right]$ be the option price under the T-forward measure. Then
$$V(t,\mathbf{x}) = P(t,\mathbf{x};T)\cdot U(t,\mathbf{x})$$

- $U(t,\mathbf{x})$ satisfies the PIDE
$$\begin{cases} \left(\partial_t + \mathcal{L}(t)\right)U(t,\mathbf{x}) &= 0\,, \\ U(T,\mathbf{x}) &= \Phi(\mathbf{x};T)\,. \end{cases}$$

- The $\mathbb{Q}_T$-infinitesimal generator $\mathcal{L}(t)$ of the joint processes $\mathbf{X}(t)$ acts on functions $f(\mathbf{x})$
$$\mathcal{L}(t)f(t,\mathbf{x}) = -\left(\boldsymbol{\mu} + \boldsymbol{\kappa}\mathbf{x} + \boldsymbol{\Sigma}\,\mathbf{B}(t;T)\right)'\partial_{\mathbf{x}}f(t,\mathbf{x}) + \frac{1}{2}\partial_{\mathbf{x}}f(t,\mathbf{x})'\,\boldsymbol{\Sigma}\,\partial_{\mathbf{x}}f(t,\mathbf{x}) + \int_{\mathbb{R}^d} e^{-\mathbf{B}(t;T)'\mathbf{z}}\left(f(t,\mathbf{x}+\mathbf{z}) - f(t,\mathbf{x})\right)\nu(d\mathbf{z})\,.$$

The PIDE in Frequency Space

- Applying the Fourier transform to the pricing PIDE, obtain a PDE in frequency space

$$\begin{cases} \left(\partial_t + \hat{\psi}(t,\boldsymbol{\omega}) + \mathrm{Tr}\boldsymbol{\kappa} + \boldsymbol{\omega}'\boldsymbol{\kappa}\partial_{\boldsymbol{\omega}}\right)\hat{U}(t,\boldsymbol{\omega}) & = \mathbf{0}\,, \\ \hat{U}(T,\boldsymbol{\omega}) & = \hat{\Phi}(\boldsymbol{\omega};T)\,, \end{cases}$$

where

$$\hat{\psi}(t,\boldsymbol{\omega}) = -i\boldsymbol{\omega}'(\boldsymbol{\mu} + \boldsymbol{\Sigma}\,\mathbf{B}(t;T)) - \tfrac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\,\boldsymbol{\omega} + \int_{\mathbb{R}^d} e^{-\mathbf{B}(t;T)\mathbf{z}}\left(e^{i\boldsymbol{\omega}'\mathbf{z}} - 1\right)\nu(d\mathbf{z})$$

The PIDE in Frequency Space

- Applying the Fourier transform to the pricing PIDE, obtain a PDE in frequency space

$$\begin{cases} \left(\partial_t + \hat{\psi}(t,\boldsymbol{\omega}) + \mathrm{Tr}\boldsymbol{\kappa} + \boldsymbol{\omega}'\boldsymbol{\kappa}\partial_{\boldsymbol{\omega}}\right)\hat{U}(t,\boldsymbol{\omega}) & = \mathbf{0}\,, \\ \hat{U}(T,\boldsymbol{\omega}) & = \hat{\Phi}(\boldsymbol{\omega};T)\,, \end{cases}$$

where

$$\hat{\psi}(t,\boldsymbol{\omega}) = -i\boldsymbol{\omega}'(\boldsymbol{\mu} + \boldsymbol{\Sigma}\,\mathbf{B}(t;T)) - \tfrac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\,\boldsymbol{\omega} + \int_{\mathbb{R}^d} e^{-\mathbf{B}(t;T)\mathbf{z}}\left(e^{i\boldsymbol{\omega}'\mathbf{z}} - 1\right)\nu(d\mathbf{z})$$

- Introduce a new coordinate system via frequency scaling

$$\tilde{g}(t,\boldsymbol{\omega}) = \hat{g}(t, e^{\boldsymbol{\kappa}'(t-t_\star)}\,\boldsymbol{\omega})$$

The PIDE in Frequency Space

- Applying the Fourier transform to the pricing PIDE, obtain a PDE in frequency space

$$\begin{cases} \left(\partial_t + \hat{\psi}(t,\boldsymbol{\omega}) + \mathrm{Tr}\boldsymbol{\kappa} + \boldsymbol{\omega}'\boldsymbol{\kappa}\partial_{\boldsymbol{\omega}}\right)\hat{U}(t,\boldsymbol{\omega}) & = \mathbf{0}\,, \\ \hat{U}(T,\boldsymbol{\omega}) & = \hat{\Phi}(\boldsymbol{\omega};T)\,, \end{cases}$$

  where

$$\hat{\psi}(t,\boldsymbol{\omega}) = -i\boldsymbol{\omega}'(\boldsymbol{\mu} + \boldsymbol{\Sigma}\,\mathbf{B}(t;T)) - \tfrac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\,\boldsymbol{\omega} + \int_{\mathbb{R}^d} e^{-\mathbf{B}(t;T)\mathbf{z}}\left(e^{i\boldsymbol{\omega}'\mathbf{z}} - 1\right)\nu(d\mathbf{z})$$

- Introduce a new coordinate system via frequency scaling

$$\tilde{g}(t,\boldsymbol{\omega}) = \hat{g}(t, e^{\boldsymbol{\kappa}'(t-t_\star)}\,\boldsymbol{\omega})$$

- The PDE reduces to an ODE in time parameterized by $\boldsymbol{\omega}$

$$\begin{cases} \left(\partial_t + \tilde{\psi}(t,\boldsymbol{\omega}) + \mathrm{Tr}\boldsymbol{\kappa}\right)\tilde{U}(t,\boldsymbol{\omega}) & = \mathbf{0}\,, \\ \tilde{U}(T,\boldsymbol{\omega}) & = \tilde{\Phi}(\boldsymbol{\omega};T) \end{cases}$$

The PIDE in Frequency Space

- Since $\psi(t,\boldsymbol{\omega})$ has no partial derivatives with respect to $\boldsymbol{\omega}$, the constant coefficient ODE is easily solved

$$\tilde{U}(t,\boldsymbol{\omega}) = \tilde{U}(s,\boldsymbol{\omega}) \cdot e^{\tilde{\Psi}(t,\boldsymbol{\omega};s)+\mathrm{Tr}\,\boldsymbol{\kappa}\,(s-t)}$$

where

$$\tilde{\Psi}(t,\boldsymbol{\omega};s) = \int_t^s \tilde{\psi}(u,\boldsymbol{\omega})\,du$$

The PIDE in Frequency Space

- Since $\psi(t, \boldsymbol{\omega})$ has no partial derivatives with respect to $\boldsymbol{\omega}$, the constant coefficient ODE is easily solved

$$\tilde{U}(t, \boldsymbol{\omega}) = \tilde{U}(s, \boldsymbol{\omega}) \cdot e^{\tilde{\Psi}(t,\boldsymbol{\omega};s)+\mathrm{Tr}\,\boldsymbol{\kappa}\,(s-t)}$$

  where

$$\tilde{\Psi}(t, \boldsymbol{\omega}; s) = \int_t^s \tilde{\psi}(u, \boldsymbol{\omega})\, du$$

- The ODE solution can be expressed in terms of original coordinates

$$\hat{U}(t, \boldsymbol{\omega}) = \hat{U}(s, e^{\boldsymbol{\kappa}'(s-t)}\,\boldsymbol{\omega}) \cdot e^{\hat{\Psi}(t,\,\boldsymbol{\omega};\,s)+\mathrm{Tr}\,\boldsymbol{\kappa}\,(s-t)}\,,$$

  where

$$\hat{\Psi}(t, \boldsymbol{\omega}; s) = \int_t^s \hat{\psi}(u, e^{\boldsymbol{\kappa}'(u-t)}\,\boldsymbol{\omega})\, du$$

The PIDE in Frequency Space

- Since $\psi(t,\boldsymbol{\omega})$ has no partial derivatives with respect to $\boldsymbol{\omega}$, the constant coefficient ODE is easily solved

$$\tilde{U}(t,\boldsymbol{\omega}) = \tilde{U}(s,\boldsymbol{\omega}) \cdot e^{\tilde{\Psi}(t,\boldsymbol{\omega};s)+\mathrm{Tr}\,\boldsymbol{\kappa}\,(s-t)}$$

  where

$$\tilde{\Psi}(t,\boldsymbol{\omega};s) = \int_t^s \tilde{\psi}(u,\boldsymbol{\omega})\,du$$

- The ODE solution can be expressed in terms of original coordinates

$$\hat{U}(t,\boldsymbol{\omega}) = \hat{U}(s, e^{\boldsymbol{\kappa}'(s-t)}\,\boldsymbol{\omega}) \cdot e^{\hat{\Psi}(t,\,\boldsymbol{\omega};\,s)+\mathrm{Tr}\,\boldsymbol{\kappa}\,(s-t)}\,,$$

  where

$$\hat{\Psi}(t,\boldsymbol{\omega};s) = \int_t^s \hat{\psi}(u, e^{\boldsymbol{\kappa}'(u-t)}\,\boldsymbol{\omega})\,du$$

- The propagator $\hat{\Psi}(t,\boldsymbol{\omega};s)$ can be computed analytically for Hull-White, Vasicek-EJ++ and G2++ models or approximated numerically for Vasicek-GJ++ model.

Computing Solution in Real Space

- The scaled frequencies of option prices $\hat{U}(s, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega})$ pose challenges numerically due to need of extrapolation in frequency space

Computing Solution in Real Space

- The scaled frequencies of option prices $\hat{U}(s, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega})$ pose challenges numerically due to need of extrapolation in frequency space

- Using the scaling property of Fourier transforms, the option prices in frequency space can be obtained from the option prices in real space:

$$\mathcal{F}[g](t, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega}) = \mathcal{F}[\breve{g}](t, \boldsymbol{\omega}) \cdot e^{-\mathrm{Tr}\,\boldsymbol{\kappa}\,(s-t)},$$

where

$$\breve{g}(t, \mathbf{x}) \triangleq g(t, \mathbf{x}\, e^{-\boldsymbol{\kappa}'(s-t)})$$

Computing Solution in Real Space

- The scaled frequencies of option prices $\hat{U}(s, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega})$ pose challenges numerically due to need of extrapolation in frequency space

- Using the scaling property of Fourier transforms, the option prices in frequency space can be obtained from the option prices in real space:

$$\mathcal{F}[g](t, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega}) = \mathcal{F}[\breve{g}](t, \boldsymbol{\omega}) \cdot e^{-\operatorname{Tr}\boldsymbol{\kappa}(s-t)},$$

where

$$\breve{g}(t, \mathbf{x}) \triangleq g(t, \mathbf{x}\, e^{-\boldsymbol{\kappa}'(s-t)})$$

- A convenient representation for European option prices at time t, given the value of U at time $s > t$

$$U(t, \mathbf{x}) = \mathcal{F}^{-1}\left[\mathcal{F}\left[\breve{U}\right](s, \boldsymbol{\omega}) \cdot e^{\hat{\Psi}(t, \boldsymbol{\omega}; s)}\right](\mathbf{x}).$$

Numerical Method

irFST Method

$$\mathbf{V}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{V}}_m\right] \cdot e^{\hat{\mathbf{\Psi}}_{m-1,m}}\right] \cdot \mathbf{P}_{m-1,m}$$

Numerical Method

irFST Method

$$\mathbf{V}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{V}}_m\right] \cdot e^{\hat{\mathbf{\Psi}}_{m-1,m}}\right] \cdot \mathbf{P}_{m-1,m}$$

- European options

$$\mathbf{V}_0 = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{V}}_1\right] \cdot e^{\hat{\mathbf{\Psi}}_{0,1}}\right] \cdot \mathbf{P}_{0,1}$$

Numerical Method

irFST Method

$$\mathbf{V}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{V}}_m\right] \cdot e^{\hat{\mathbf{\Psi}}_{m-1,m}}\right] \cdot \mathbf{P}_{m-1,m}$$

- European options

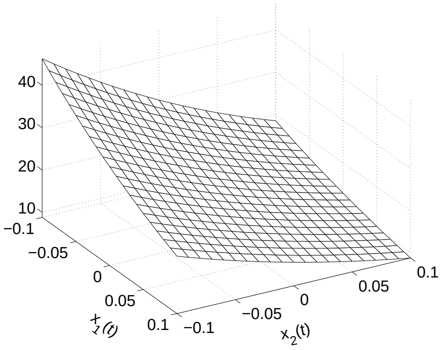
$$\mathbf{V}_0 = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{V}}_1\right] \cdot e^{\hat{\mathbf{\Psi}}_{0,1}}\right] \cdot \mathbf{P}_{0,1}$$

- Bermudan and callable options

$$\mathbf{V}_{m-1} = \max\left\{\mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\check{\mathbf{V}}_m\right] \cdot e^{\hat{\mathbf{\Psi}}_{m-1,m}}\right] \cdot \mathbf{P}_{m-1,m}\, , \mathbf{\Phi}_{m-1}\right\}\ ,$$

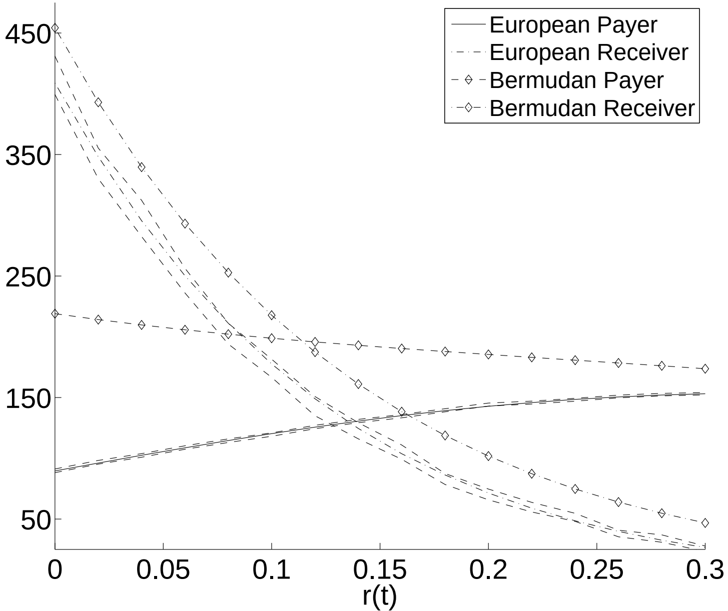
where $\mathbf{\Phi}_{m-1}$ is the exercise value of the option at time t_{m-1}.

Options Prices on a Zero-bond under a G2++ model

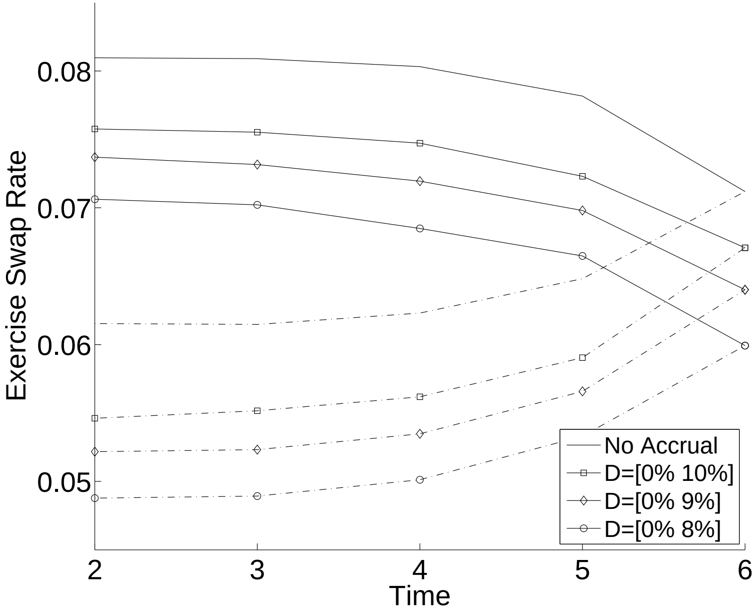


(a) European (b) Bermudan

Option Prices under Vasicek-EJ++ model



Bermudan Option Exercise Boundary



Option Greeks in Fourier Space

- The first-order sensitivity of option prices to changes in model parameters can be obtained by differentiating the ODE solution with respect to the model parameter $\star$:

$$\partial_\star \tilde{U}(t,\boldsymbol{\omega}) = \partial_\star \tilde{U}(s,\boldsymbol{\omega}) \cdot e^{\tilde{\Psi}_\kappa(t,\boldsymbol{\omega};s)} + \partial_\star \tilde{\Psi}_\kappa(t,\boldsymbol{\omega};s) \cdot \tilde{U}(t,\boldsymbol{\omega})$$

where $\tilde{\Psi}_\kappa(t,\omega;s) = \tilde{\Psi}(t,\omega;s) + \mathrm{Tr}\boldsymbol{\kappa}(s-t)$

Option Greeks in Fourier Space

- The first-order sensitivity of option prices to changes in model parameters can be obtained by differentiating the ODE solution with respect to the model parameter $\star$:

$$\partial_\star \tilde{U}(t,\boldsymbol{\omega}) = \partial_\star \tilde{U}(s,\boldsymbol{\omega}) \cdot e^{\tilde{\Psi}_\kappa(t,\boldsymbol{\omega};s)} + \partial_\star \tilde{\Psi}_\kappa(t,\boldsymbol{\omega};s) \cdot \tilde{U}(t,\boldsymbol{\omega})$$

where $\tilde{\Psi}_\kappa(t,\boldsymbol{\omega};s) = \tilde{\Psi}(t,\boldsymbol{\omega};s) + \mathrm{Tr}\boldsymbol{\kappa}(s-t)$

- Converting to original coordinates and applying the inverse Fourier transform gives the solution in real space to the first-order Greek:

$$\partial_\star U(t,\mathbf{x}) = \mathcal{F}^{-1}\left[\partial_\star \hat{U}(s, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega}) \cdot e^{\Psi_\kappa(t,\boldsymbol{\omega};s)} + \partial_\star \Psi_\kappa(t,\boldsymbol{\omega};s) \cdot \hat{U}(t, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega})\right](\mathbf{x})$$

- Greeks under the $\mathbb{Q}$ measure are then given by

$$\partial_\star V(t,\mathbf{x}) = \partial_\star U(t,\mathbf{x}) \cdot P(t,\mathbf{x};T) + U(t,\mathbf{x}) \cdot \partial_\star P(t,\mathbf{x};T)$$

Option Greeks in Fourier Space

- The first-order sensitivity of option prices to changes in model parameters can be obtained by differentiating the ODE solution with respect to the model parameter $\star$:

$$\partial_\star \tilde{U}(t, \boldsymbol{\omega}) = \partial_\star \tilde{U}(s, \boldsymbol{\omega}) \cdot e^{\tilde{\Psi}_\kappa(t,\boldsymbol{\omega};s)} + \partial_\star \tilde{\Psi}_\kappa(t, \boldsymbol{\omega}; s) \cdot \tilde{U}(t, \boldsymbol{\omega})$$

  where $\tilde{\Psi}_\kappa(t, \boldsymbol{\omega}; s) = \tilde{\Psi}(t, \boldsymbol{\omega}; s) + \mathrm{Tr}\boldsymbol{\kappa}(s - t)$

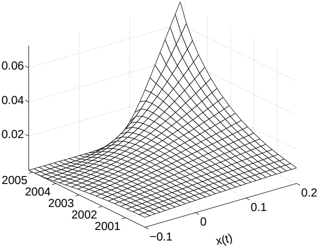
- Converting to original coordinates and applying the inverse Fourier transform gives the solution in real space to the first-order Greek:

$$\partial_\star U(t, \mathbf{x}) = \mathcal{F}^{-1}\left[\partial_\star \hat{U}(s, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega}) \cdot e^{\Psi_\kappa(t,\boldsymbol{\omega};s)} + \partial_\star \Psi_\kappa(t, \boldsymbol{\omega}; s) \cdot \hat{U}(t, e^{\boldsymbol{\kappa}'(s-t)}\boldsymbol{\omega})\right](\mathbf{x})$$

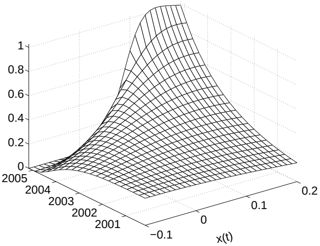
- Greeks under the $\mathbb{Q}$ measure are then given by

$$\partial_\star V(t, \mathbf{x}) = \partial_\star U(t, \mathbf{x}) \cdot P(t, \mathbf{x}; T) + U(t, \mathbf{x}) \cdot \partial_\star P(t, \mathbf{x}; T)$$

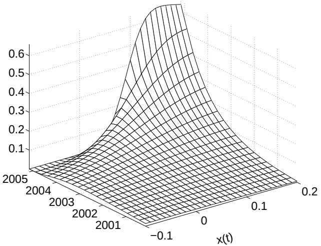
- The numerical methods readily obtained by replacing CFTs by FFTs



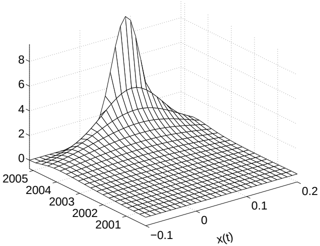
(a) Price



(b) Vega



(c) Delta



(d) Gamma

Dynamic Hedging

- Hedging portfolio for the option V consists of B units of cash, e units of the underlying asset S and N hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, \mathbf{S}(t)) + e\mathbf{S}(t) + B - V(t, \mathbf{S}(t))$$

Dynamic Hedging

- Hedging portfolio for the option V consists of B units of cash, e units of the underlying asset S and N hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, \mathbf{S}(t)) + e\mathbf{S}(t) + B - V(t, \mathbf{S}(t))$$

- The portfolio's value remains unchanged under small movements in price

$$\partial_S \Pi = \vec{\phi} \cdot \partial_S \vec{I}(t, \mathbf{S}(t)) + e - \partial_S V(t, \mathbf{S}(t)) = 0$$

Dynamic Hedging

- Hedging portfolio for the option V consists of B units of cash, e units of the underlying asset S and N hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, \mathbf{S}(t)) + e\mathbf{S}(t) + B - V(t, \mathbf{S}(t))$$

- The portfolio's value remains unchanged under small movements in price

$$\partial_S \Pi = \vec{\phi} \cdot \partial_S \vec{I}(t, \mathbf{S}(t)) + e - \partial_S V(t, \mathbf{S}(t)) = 0$$

- Can also hedge against small movements in interest-rates, volatility, etc.

$$\partial_\star \Pi = \vec{\phi} \cdot \partial_\star \vec{I}(t, \mathbf{S}(t)) - \partial_\star V(t, \mathbf{S}(t)) = 0$$

Dynamic Hedging

- Hedging portfolio for the option V consists of B units of cash, e units of the underlying asset S and N hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, \mathbf{S}(t)) + e\mathbf{S}(t) + B - V(t, \mathbf{S}(t))$$

- The portfolio's value remains unchanged under small movements in price

$$\partial_S \Pi = \vec{\phi} \cdot \partial_S \vec{I}(t, \mathbf{S}(t)) + e - \partial_S V(t, \mathbf{S}(t)) = 0$$

- Can also hedge against small movements in interest-rates, volatility, etc.

$$\partial_\star \Pi = \vec{\phi} \cdot \partial_\star \vec{I}(t, \mathbf{S}(t)) - \partial_\star V(t, \mathbf{S}(t)) = 0$$

- What about large movements?

Static Hedging - Minimize Portfolio Variance

- Minimize portfolio price variance under expected asset price movement, Kennedy, Forsyth, Vetzal (2009):

$$\arg\min_{e_n,\vec{\phi}_n} \xi\, \mathbb{E}_{t_n}\left[\vec{\phi}_n \cdot \Delta\vec{I}_n + e_n \Delta S_n - \Delta V_n\right]^2 + (1-\xi)\Upsilon_n\,.$$

where Υ_n is the transaction cost to rebalance the portfolio:

$$\Upsilon_n = \sum_{k=1}^{N}\left[\vec{\alpha}_k(\vec{\phi}_{k,n} - \vec{\phi}_{k,n-1})\vec{I}_{k,n}\right]^2 + \left[\beta(e_n - e_{n-1})S_n\right]^2,$$

Static Hedging - Minimize Portfolio Variance

- Minimize portfolio price variance under expected asset price movement, Kennedy, Forsyth, Vetzal (2009):

$$\underset{e_n,\vec{\phi}_n}{\arg\min}\ \xi\, \mathbb{E}_{t_n}\left[\vec{\phi}_n \cdot \Delta\vec{I}_n + e_n \Delta S_n - \Delta V_n\right]^2 + (1-\xi)\Upsilon_n\,.$$

where Υ_n is the transaction cost to rebalance the portfolio:

$$\Upsilon_n = \sum_{k=1}^{N}\left[\vec{\alpha}_k(\vec{\phi}_{k,n} - \vec{\phi}_{k,n-1})\vec{I}_{k,n}\right]^2 + \left[\beta(e_n - e_{n-1})S_n\right]^2,$$

- Since the objective function is quadratic, the optimality requires

$$\frac{\partial F}{\partial \phi_{k,n}} = \xi\, \mathbb{E}_{t_n}\left[\left(\vec{\phi}\cdot\Delta\vec{I} + e\Delta S - \Delta V\right)\left(2\Delta I_k\right)\right] + (1-\xi)\,\partial_{\phi_{k,n}}\Upsilon_n = 0$$

$$\frac{\partial F}{\partial e_n} = \xi\, \mathbb{E}_{t_n}\left[\left(\vec{\phi}\cdot\Delta\vec{I} + e\Delta S - \Delta V\right)\left(2\Delta S\right)\right] + (1-\xi)\,\partial_{e_n}\Upsilon_n = 0$$

Static Hedging - Minimize Price and Greeks Variance

- Minimize portfolio price and Greeks variance under expected asset price movement

$$\arg\min_{e_n, \vec{\phi}_n} \xi \sum_{\mathcal{D}} w_{\mathcal{D}} \mathbb{E}_{t_n} \left[\vec{\phi}_n \cdot \Delta(\mathcal{D}\vec{I}_n) + e_n \Delta(\mathcal{D}S_n) - \Delta(\mathcal{D}V_n) \right]^2 + (1-\xi)\Upsilon_n$$

Static Hedging - Minimize Price and Greeks Variance

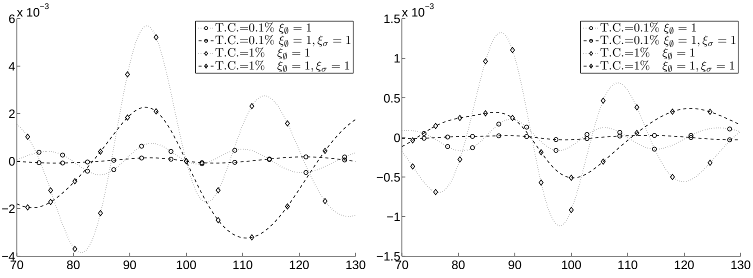
- Minimize portfolio price and Greeks variance under expected asset price movement

$$\underset{e_n,\vec{\phi}_n}{\arg\min}\ \xi \sum_{\mathcal{D}} w_{\mathcal{D}}\, \mathbb{E}_{t_n}\left[\vec{\phi}_n \cdot \Delta(\mathcal{D}\vec{I}_n) + e_n \Delta(\mathcal{D}S_n) - \Delta(\mathcal{D}V_n)\right]^2 + (1-\xi)\Upsilon_n$$

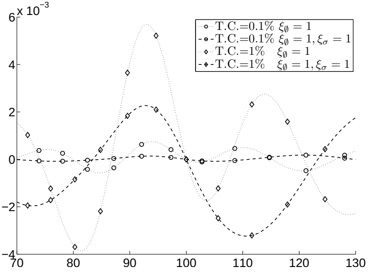
- Since the objective function is quadratic, the optimality requires

$$\frac{\partial F}{\partial \phi_{k,n}} = \xi \sum_{\mathcal{D}} w_{\mathcal{D}}\, \mathbb{E}_{t_n}\Big[\big(\vec{\phi} \cdot \Delta(\mathcal{D}\vec{I}) + e\Delta(\mathcal{D}S) - \Delta(\mathcal{D}V)\big)\big(2\Delta(\mathcal{D}I_k)\big)\Big] + (1-\xi)\,\partial_{\phi_{k,n}}\Upsilon_n = 0$$

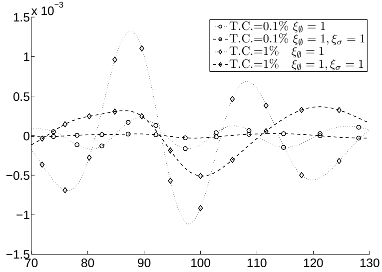
$$\frac{\partial F}{\partial e_n} = \xi \sum_{\mathcal{D}} w_{\mathcal{D}}\, \mathbb{E}_{t_n}\Big[\big(\vec{\phi} \cdot \Delta(\mathcal{D}\vec{I}) + e\Delta(\mathcal{D}S) - \Delta(\mathcal{D}V)\big)\big(2\Delta(\mathcal{D}S)\big)\Big] + (1-\xi)\,\partial_{e_n}\Upsilon_n = 0$$



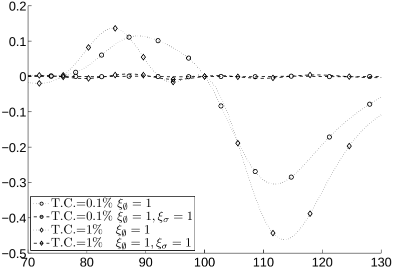
(a) Price (b) Delta



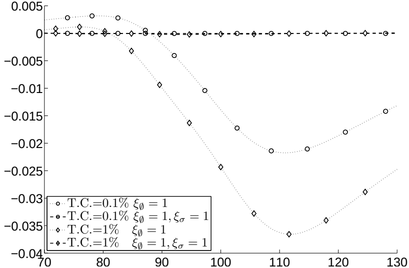
(a) Price



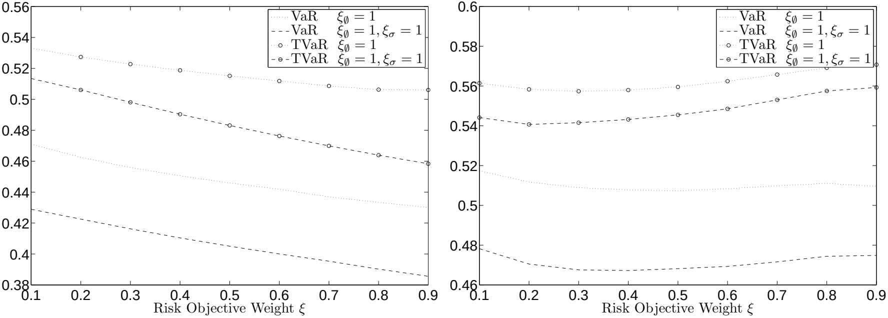
(b) Delta



(c) Vega



(d) Jump Arrival


VaR $\xi_\emptyset = 1$
VaR $\xi_\emptyset = 1, \xi_\sigma = 1$
TVaR $\xi_\emptyset = 1$
TVaR $\xi_\emptyset = 1, \xi_\sigma = 1$
0.56
0.54
0.52
0.5
0.48
0.46
0.44
0.42
0.4
0.38
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
Risk Objective Weight ξ
0.6
0.58
0.56
0.54
0.52
0.5
0.48
0.46
Risk Objective Weight ξ

(a) TC = 0.1%

(b) TC = 1%

FST Framework Summary

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

FST Framework Summary

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- Independent-increment, mean-reverting and interest-rate Lévy models are handled generically

FST Framework Summary

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- Independent-increment, mean-reverting and interest-rate Lévy models are handled generically
- Option values are obtained for a range of spot prices - readily price path-dependent options

FST Framework Summary

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- Independent-increment, mean-reverting and interest-rate Lévy models are handled generically
- Option values are obtained for a range of spot prices - readily price path-dependent options
- Two FFTs per time-step are required; no time-stepping for European options or between monitoring dates of discretely monitored options

FST Framework Summary

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

- Independent-increment, mean-reverting and interest-rate Lévy models are handled generically
- Option values are obtained for a range of spot prices - readily price path-dependent options
- Two FFTs per time-step are required; no time-stepping for European options or between monitoring dates of discretely monitored options
- Two extra FFTs required to compute each Greek

Thank You!

Davison, M. and V. Surkov (2010).
Efficient construction of robust hedging strategies under jump models.
Working paper, available at http://ssrn.com/abstract=1562685.

Jackson, K. R., S. Jaimungal, and V. Surkov (2008).
Fourier space time-stepping for option pricing with Lévy models.
Journal of Computational Finance 12(2), 1–28.

Jaimungal, S. and V. Surkov (2008).
Levy based cross-commodity models and derivative valuation.
Working paper, available at http://ssrn.com/abstract=1302887.

Jaimungal, S. and V. Surkov (2010).
Valuing early exercise interest rate options with multi-factor affine models.
Working paper, available at http://ssrn.com/abstract=1562638.

More at http://ssrn.com/author=879101