Fourier Space Time-stepping Method for Option Pricing with Lévy Processes

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Joint work with Ken Jackson and Sebastian Jaimungal, University of Toronto

1. Fourier Space Time-stepping method
   - Infinitesimal generator and characteristic exponent
   - Method derivation
   - Numerical results
2. Extensions
   - Multi-asset options
   - Regime switching
3. Indifference pricing
   - Optimal investment problems
   - Application of FST to solution of HJB equations

The Option Pricing Problem

- Option payoff is given by $\varphi(S)$
- Stock price follows an exponential Lévy model:

$$S(t) = S(0)e^{\mu t + X(t)}, \qquad X(t) \text{ is a Lévy process}$$

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Generalizing PIDE for Lévy processes

$$\begin{cases} \partial_t v + \mathcal{L} v &= 0 \\ v(T, x) &= \varphi(S(0)\, e^x) \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator of the Lévy process:

$$\mathcal{L}f \;=\; \gamma\,\partial_x f + \frac{\sigma^2}{2}\partial_{xx} f + \int_{\mathbb{R}/\{0\}} \left[f(x+y) - f(x) - y\mathbb{1}_{\{|y|<1\}}\partial_x f(x)\right] \nu(dy)$$

Finite Difference Methods for Option Pricing

- Alternating Direction Implicit-FFT - Andersen and Andreasen (2000)
- Implicit-Explicit (IMEX) - Cont and Tankov (2004)
- IMEX Runge-Kutta - Briani, Natalini, and Russo (2004)
- Fixed Point Iteration - d'Halluin, Forsyth, and Vetzal (2005)

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Common Features:

- Treat the integral term explicitly to avoid solving a dense system of linear equations.
- Use the Fast Fourier Transform (FFT) to speed up the computation of the integral term (which can be regarded as a convolution)

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Common Features:

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Drawbacks:

- Diffusive and integral terms are treated asymmetrically
- Large jump are truncated and small jumps approximated by diffusion
- Difficult to extend to higher dimensions

Infinitesimal Generator and Characteristic Exponent

The characteristic exponent of a Lévy-Khinchin representation can be factored from a Fourier transform of the operator (Sato 1999)

$$\begin{aligned}\mathcal{F}[\mathcal{L}v](\tau,\omega) &= \left\{i\gamma\omega - \frac{\sigma^2\omega^2}{2} + \int [e^{i\omega x} - 1 - i\omega y 1_{\{|\omega|<1\}}]\nu(dy)\right\}\mathcal{F}[v](\tau,\omega) \\ &= \psi(\omega)\mathcal{F}[v](\tau,\omega)\end{aligned}$$

Model	Characteristic Exponent $\psi(\omega)$
Black-Scholes-Merton	$i\mu\omega - \frac{\sigma^2\omega^2}{2}$
Merton Jump-Diffusion	$i\mu\omega - \frac{\sigma^2\omega^2}{2} + \lambda(e^{i\tilde{\mu}\omega - \tilde{\sigma}^2\omega^2/2} - 1)$
Variance Gamma	$-\frac{1}{\kappa}\log(1 - i\mu\kappa\omega + \frac{\sigma^2\kappa\omega^2}{2})$
CGMY	$C\Gamma(-Y)\left[(M-i\omega)^Y - M^Y + (G+i\omega)^Y - G^Y\right]$

Numerical Method Derivation

- Apply the Fourier transform to the pricing PIDE

$$\begin{cases} \partial_t \mathcal{F}[v](t,\omega) + \Psi(\omega)\mathcal{F}[v](t,\omega) & = 0\ , \\ \mathcal{F}[v](T,\omega) & = \mathcal{F}[\varphi](\omega) \end{cases}$$

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- Resulting ODE has explicit solution

$$\mathcal{F}[v](t_1,\omega) = \mathcal{F}[v](t_2,\omega) \cdot e^{(t_2 - t_1)\Psi(\omega)}$$

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- Apply the inverse Fourier transform

$$v(t_1,\mathbf{x}) = \mathcal{F}^{-1}\left\{\mathcal{F}[v](t_2,\omega) \cdot e^{(t_2-t_1)\Psi(\omega)}\right\}(\mathbf{x})$$

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Fourier Space Time-stepping (FST) method

$$v^{n-1} = \text{FFT}^{-1}[\text{FFT}[v^n] \cdot e^{\Psi\,\Delta t}]$$

European Call Option

N	Value	Change	$\log_2$Ratio	CPU-Time
2048	0.04261423			0.002
4096	0.04263998	0.000026		0.005
8192	0.04264641	0.000006	2.0018	0.010
16384	0.04264801	0.000002	2.0010	0.019
32768	0.04264841	0.000000	2.0011	0.038

- *Option:* European call $S = 1.0, K = 1.0, T = 0.2$
- *Model:* Kou jump-diffusion
  $\sigma = 0.2, \lambda = 0.2, p = 0.5, \eta_- = 3, \eta_+ = 2, r = 0.0$
- *Quoted Price:* 0.0426761 Almendral and Oosterlee (2005)

American Put Option

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	512	9.22533163			0.181
8192	1024	9.22547180	0.0001402		0.958
16384	2048	9.22544621	0.0000256	2.4534	4.036
32768	4096	9.22543840	0.0000078	1.7117	21.303

- *Option:* American put $S = 90.0$, $K = 98.0$, $T = 0.25$
- *Model:* CGMY $C = 0.42$, $G = 4.37$, $M = 191.2$, $Y = 1.0102$, $r = 0.06$
- *Quoted Price:* 9.2254 Forsyth, Wan, and Wang (2006)

American Put Option

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	512	4.42077686			0.239
8192	1024	4.42077346	0.0000034		1.198
16384	2048	4.42077259	0.0000009	1.9616	4.614
32768	4096	4.42077245	0.0000001	2.6769	20.735

- *Option:* American put $S = 10.0$, $K = 10.0$, $T = 0.25$
- *Model:* CGMY $C = 1.0$, $G = 8.8$, $M = 9.2$, $Y = 1.8$, $r = 0.1$

Up-and-Out Barrier Call Option

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	512	0.25432521			0.149
8192	1024	0.25422752	0.0000977		0.669
16384	2048	0.25420350	0.0000240	2.0239	2.928
32768	4096	0.25419764	0.0000059	2.0335	15.691

- *Option:* Up-and-Out Barrier Call
  $S = 100.0$, $K = 100.0$, $B = 110$, $T = 1.0$
- *Model:* Black-Scholes-Merton $\sigma = 0.15$, $r = 0.05$, $q = 0.02$
- *Closed-Form Price:* 0.2541963 Hull (2005)

Multi-asset options

Pricing PIDE

$$\begin{cases} (\partial_t + \mathcal{L})\, v &= 0\,, \\ v(T, \mathbf{x}) &= \varphi(\mathbf{S}(0)\, e^{\mathbf{x}}) \end{cases}$$

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Multi-dimensional FST method

$$\mathbf{v}^{n-1} = \mathrm{FFT}^{-1}[\mathrm{FFT}[\mathbf{v}^n] \cdot e^{\mathbf{\Psi}\,\Delta t}]$$

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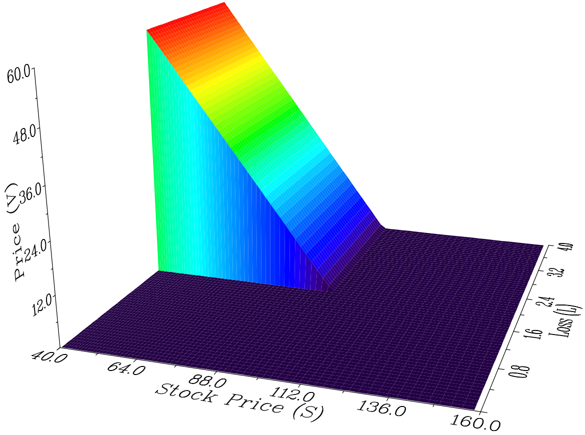
- Jackson, Jaimungal and Surkov (2007) discuss application of FST to pricing of spread, American spread and catastrophe equity put options

Catastrophe Equity Put (CatEPut)

- In the event of large (catastrophic) losses U, the insurer receives a put option on its own stock $\varphi(S(T), L_T) = \mathbb{1}_{L_T > U}(K - S(T))_+$



Catastrophe Equity Put (CatEPut)

- Presence of losses drives the share value down, not an independent jump process

$$
\begin{aligned}
S(t) &= S(0)\exp\{-\alpha\, L(t) + \gamma\, t + \sigma\, W_t\} \\
L(t) &= \sum_{n=1}^{N(t)} l_i
\end{aligned}
$$

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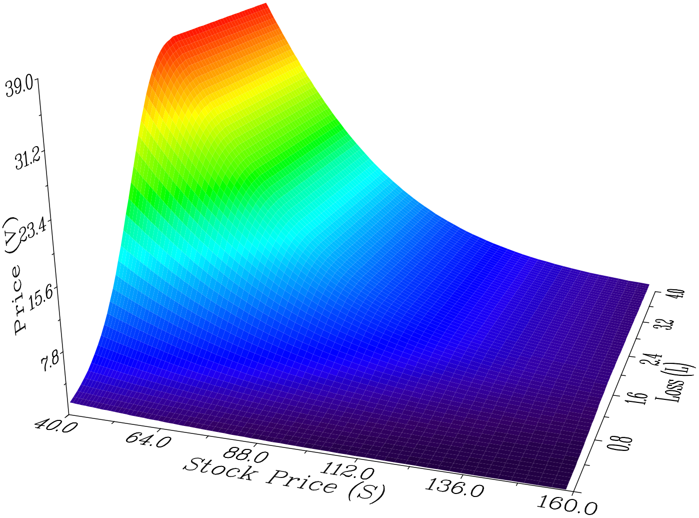
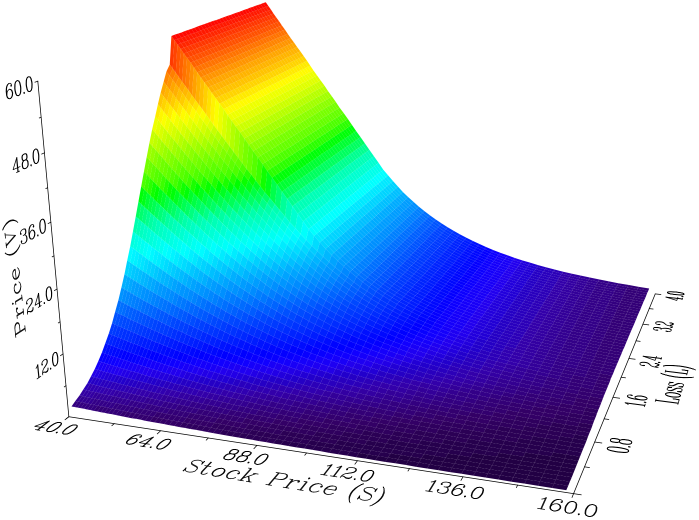
- When losses are modeled as a Gamma r.v., characteristic exponent is

$$\Psi(\omega_1, \omega_2) = i\,\gamma\,\omega_1 - \tfrac{1}{2}\sigma^2\,\omega_1^2 + \lambda\left[\left(1 - i(-\alpha\omega_1 + \omega_2)\frac{v}{m})\right)^{-\frac{m^2}{v}} - 1\right]$$

European CatEPut



American CatEPut



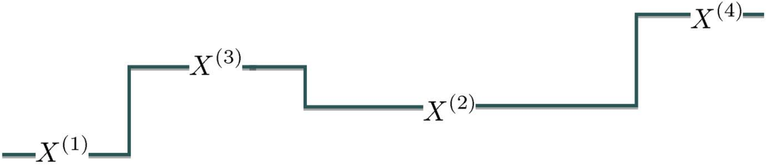
Regime Switching

- Let $\mathbb{K} := \{1, \ldots, K\}$ denote the possible hidden states of the world, driven by a continuous time Markov chain Z_t.
- The transition probability from state k at time t_1 to state l at time t_2 is given by

$$P_{kl}^{t_1 t_2} = \mathbb{Q}(Z_{t_2} = l | Z_{t_1} = k) = (\exp\{(t_2 - t_1)\mathbf{A}\})_{kl}$$

where $\mathbf{A}$ is the Markov chain generator

- Within state k, log-stock follows Lévy model k



Regime Switching

Pricing PIDE

$$
\begin{cases}
\left[\partial_t + \left(A_{kk} + \mathcal{L}^{(k)}\right)\right] v(\mathbf{x}, k, t) + \sum_{j \neq k} A_{jk}\, v(\mathbf{x}, j, t) & = 0 \\
v(\mathbf{x}, k, T) & = \varphi(\mathbf{S}(0) e^{\mathbf{x}})
\end{cases}
$$

Regime Switching

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$$\begin{cases} \left[\partial_t + \left(A_{kk} + \mathcal{L}^{(k)}\right)\right] v(\mathbf{x},k,t) + \sum_{j\neq k} A_{jk}\, v(\mathbf{x},j,t) &= 0 \\ v(\mathbf{x},k,T) &= \varphi(\mathbf{S}(0)e^{\mathbf{x}}) \end{cases}$$

FST Method

$$v^{n-1} = \mathrm{FFT}^{-1}[\mathrm{FFT}[v^n] \cdot e^{\widetilde{\Psi}\,\Delta t}]$$

where

$$\widetilde{\Psi}(\omega)_{kl} = \begin{cases} A_{kk} + \Psi^{(k)}(\omega), & k = l \\ A_{kl}, & k \neq l \end{cases}$$

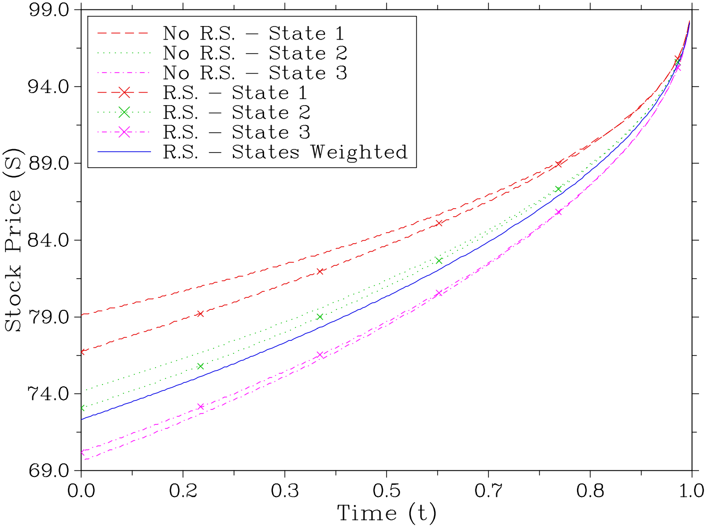
- No time-stepping required for European options

American put option with 3 regimes

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	256	14.25029309			0.634
8192	512	14.25025450	0.0000386		4.694
16384	1024	14.25024472	0.0000098	1.9802	14.385
32768	2048	14.25024245	0.0000023	2.1119	63.443

- *Option:* American put $S = 100.0, K = 100.0,\ T = 1.0$
- *Model:* Merton jump-diffusion $\sigma = 0.15, \tilde{\mu} = -0.5, \tilde{\sigma} = 0.45,$
  $r = 0.05, q = 0.02,\ \lambda \in [0.3, 0.5, 0.7],\ p = [0.2, 0.3, 0.5],$
  $A = [-0.8, 0.6, 0.2; 0.2, -1, 0.8; 0.1, 0.3, -0.4]$

Exercise boundary for American put option with 3 regimes



FST Method Strengths

- Stable and robust, even for options with discontinuous payoffs
- Easily extendable to various stochastic processes and no loss of performance for infinite activity processes
- Can be applied to multi-dimensional and regime-switching problems in a natural manner
- Computationally efficient
  - Computational cost is $O(MNlogN)$ while the error is $O(\Delta x^2 + \Delta t^2)$
  - European options priced in a single time-step
  - Bermudan style options do not require time-stepping between monitoring dates

Indifference Pricing

- In incomplete markets (non-traded assets, transaction costs, portfolio constraints, etc.) perfect replication is impossible
- Investor can still maximize the expected utility of wealth through dynamic trading
- The price of a claim is the initial wealth forgone so that the investor is no worse off in expected utility terms at maturity
- The framework incorporates wealth dependence, non-linear pricing and risk-aversion

The Modelling Framework

- Utility function is a strictly increasing and concave function ranking investor's preferences of wealth
  Popular choices include CRRA model $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ or CARA model $U(x) = 1 - \frac{1}{\gamma}e^{-\gamma x}$

The Modelling Framework

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  Popular choices include CRRA model $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ or CARA model $U(x) = 1 - \frac{1}{\gamma}e^{-\gamma x}$
- An economic agent over a fixed-time horizon attempts to optimally allocate his investment between risky (S_t) and risk-free (B_t) assets

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t \\
dX_t &= \pi_t X_t \frac{dS_t}{S_t} + (1-\pi_t) X_t \frac{dB_t}{B_t} \\
&= (\pi_t(\mu - r) + r) X_t dt + \pi_t \sigma X_t dW_t
\end{aligned}
$$

where π_t is the share of wealth allocated in stocks

Two Optimal Control Problems

- Investor maximizes the expected utility of wealth at time horizon T, given initial wealth x at t:

$$V^0(t,x) = \sup_{\pi_t} E\left[U(X_T^\pi | X_t = x)\right]$$

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- In addition to initial endowment x, investor receives k derivative contracts with payoff $C(S_T)$:

$$V^k(t,x,s) = \sup_{\pi_t} E\left[U(X_T^\pi + k \cdot C(S_T) | X_t = x, S_t = s)\right]$$

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Indifference Pricing Principle

Indifference buy price $p_{buy}^k(s)$ and sell price $p_{sell}^k(s)$ satisfy

$$V^0(t,x) = V^k(t, x - p_{buy}^k(s), s) \qquad V^0(t,x) = V^{-k}(t, x + p_{sell}^k(s), s)$$

Optimal Investment Problem

Optimal value HJB equation

$$\begin{cases} \partial_t V^0(t,x) + \sup_{\pi_t}\{\mathcal{A}^\pi V^0(t,x)\} &= 0 \\ V^0(T,x) &= U(x) \end{cases}$$

where

$$\mathcal{A}^\pi = (\pi_t(\mu - r) + r)x\partial_x + \frac{1}{2}\pi_t^2\sigma^2x^2\partial_{xx}$$

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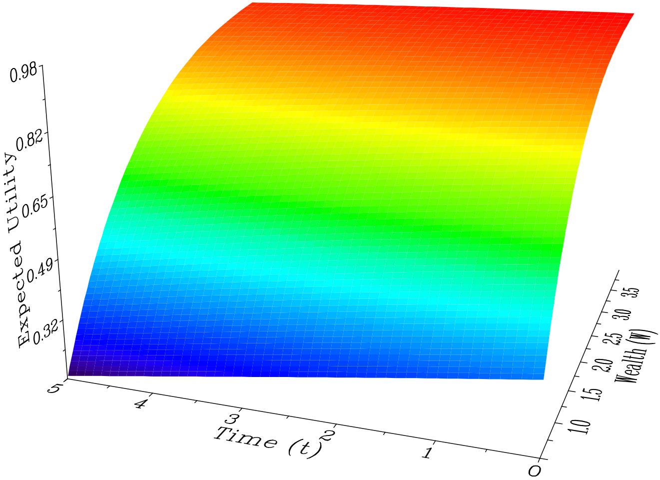
Example:
CARA utility $\gamma = 0.8$; GBM $\mu = 0.1$, $\sigma = 0.2$, $r = 0.04$

Optimal Expected Utility



Optimal Investment Problem with Option

Optimal value 2D HJB equation

$$\begin{cases} \partial_t V^k(t,x,s) + \sup_{\pi_t}\{\mathcal{A}^\pi V^k(t,x,s)\} & = 0 \\ V^k(T,x,s) & = U(x + k\cdot C(s)) \end{cases}$$

where

$$\mathcal{A}^\pi = (\pi_t(\mu - r) + r)x\partial_x + \frac{1}{2}\pi_t^2\sigma^2x^2\partial_{xx} + \mu s\partial_s + \frac{1}{2}\sigma^2 ss\partial_{ss} + \pi\sigma^2 xs\partial_{xs}$$

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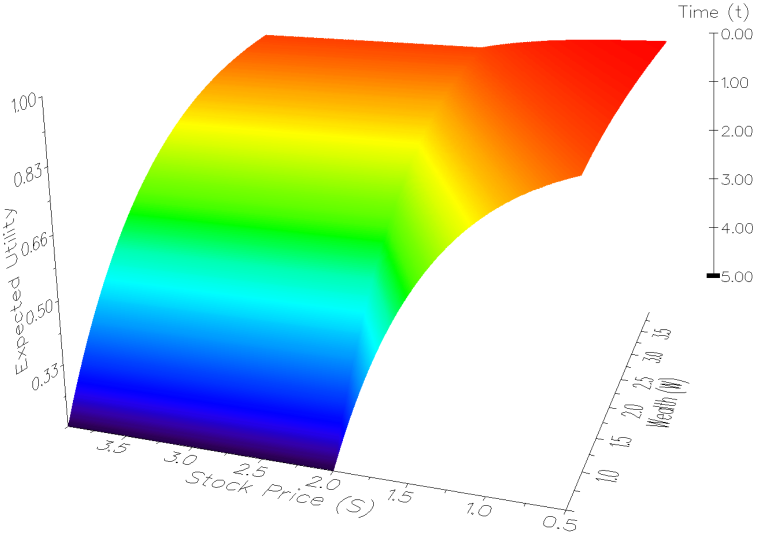
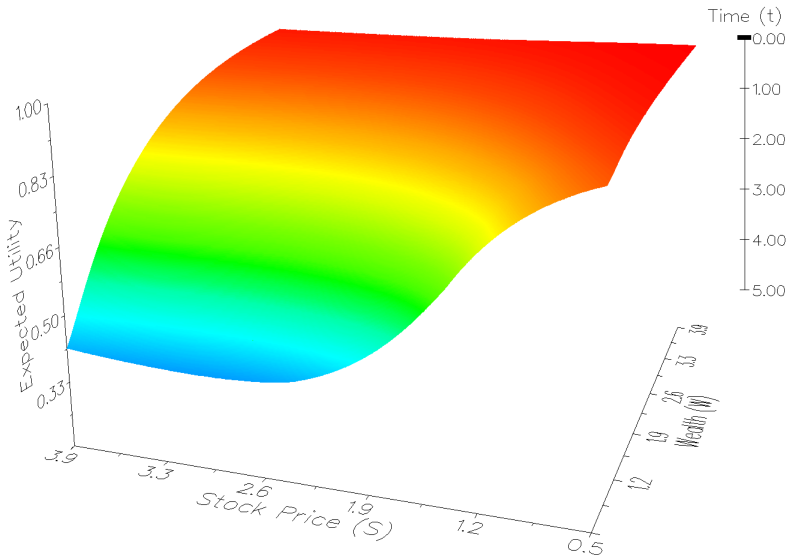
Example continued:
European put $K = 2$, $T = 5$; $k = 2$

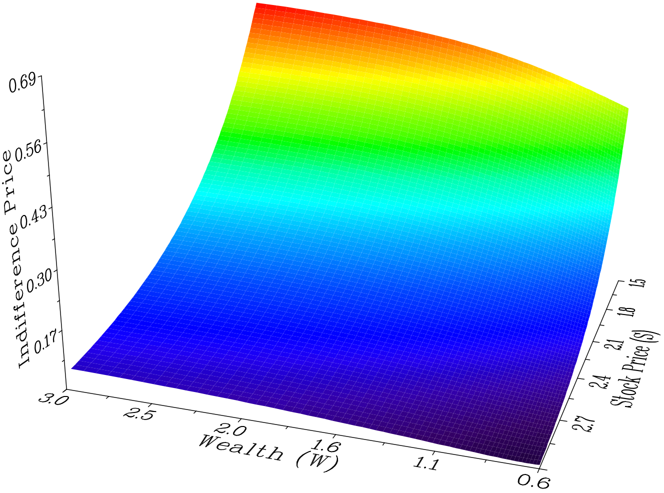
Optimal Expected Utility (t=5)



Optimal Expected Utility (t=0)



Indifference Buy Price



Application of FST to solution of HJB equations

General Approach:

- Fix $\pi_t(x)$ over the time-step $[t_1, t_2]$ and solve the resulting PIDE
- Iterate to converge to the optimal policy

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Optimization Algorithm 1: Fix $\pi_t(x) = \pi_t$ over the entire space

- $\mathcal{F}[\mathcal{A}^\pi]$ has an analytic form and the resulting ODE can be solved explicitly (just like in option pricing)
- Inefficient if optimal policy is uniformly distributed in space

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Optimization Algorithm 2: Let $\pi_t(x)$ vary in space

- $\mathcal{F}[\mathcal{A}^\pi]$ involves convolutions of $\mathcal{F}[\mathcal{A}^\pi]$ and $\mathcal{F}[\pi(x)]$, $\mathcal{F}[\pi^2(x)]$
- Can apply a policy iteration approach of Wang and Forsyth (2006)
- Working in Fourier space we can solve an Integral HJB

Future Work

- Exotic, multi-asset options
- Mean reverting processes in energy markets
- HJB equations arising from optimal control problems
  - Efficient policy iteration algorithm
  - Optimal control with jump-diffusions

Thank You!

http://128.100.73.155/fst/