Option Pricing with Regime Switching Lévy Processes using Fourier Space Time Stepping

Vladimir Surkov
University of Toronto

Financial Engineering and Applications
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Joint work with Ken Jackson and Sebastian Jaimungal, University of Toronto

The Option Pricing Problem

- Option payoff is given by $\varphi(S)$
- Stock price follows an exponential Lévy model:

$$S(t) = S(0)e^{\mu t + X(t)}, \qquad X(t) \text{ is a Lévy process}$$

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Generalizing PIDE for Lévy processes

$$\begin{cases} \partial_t v + \mathcal{L}v &= 0 \\ v(T,x) &= \varphi(S(0)\,e^x) \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator of the Lévy process:

$$\mathcal{L}f \;=\; \gamma\,\partial_x f + \frac{\sigma^2}{2}\partial_{xx} f + \int_{\mathbb{R}/\{0\}} \left[f(x+y) - f(x) - y\mathbb{1}_{\{|y|<1\}}\partial_x f(x)\right]\nu(dy)$$

Finite Difference Methods for Option Pricing

- Alternating Direction Implicit-FFT - Andersen and Andreasen (2000)
- Implicit-Explicit (IMEX) - Cont and Tankov (2004)
- IMEX Runge-Kutta - Briani, Natalini, and Russo (2004)
- Fixed Point Iteration - d'Halluin, Forsyth, and Vetzal (2005)

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Common Features:

- Treat the integral term explicitly to avoid solving a dense system of linear equations.
- Use the Fast Fourier Transform (FFT) to speed up the computation of the integral term (which can be regarded as a convolution)

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Drawbacks:

- Diffusive and integral terms are treated asymmetrically
- Large jump are truncated and small jumps approximated by diffusion
- Difficult to extend to higher dimensions

Infinitesimal Generator and Characteristic Exponent

The characteristic exponent of a Lévy-Khinchin representation can be factored from a Fourier transform of the operator (Sato 1999)

$$\begin{aligned}\mathcal{F}[\mathcal{L}v](\tau,\omega) &= \left\{i\gamma\omega - \frac{\sigma^2\omega^2}{2} + \int [e^{i\omega x} - 1 - i\omega y 1_{\{|\omega|<1\}}]\nu(dy)\right\}\mathcal{F}[v](\tau,\omega) \\ &= \psi(\omega)\mathcal{F}[v](\tau,\omega)\end{aligned}$$

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Model	**Characteristic Exponent** $\psi(\omega)$
Black-Scholes-Merton	$i\mu\omega - \frac{\sigma^2\omega^2}{2}$
Merton Jump-Diffusion	$i\mu\omega - \frac{\sigma^2\omega^2}{2} + \lambda(e^{i\tilde{\mu}\omega - \tilde{\sigma}^2\omega^2/2} - 1)$
Variance Gamma	$-\frac{1}{\kappa}\log(1 - i\mu\kappa\omega + \frac{\sigma^2\kappa\omega^2}{2})$
CGMY	$C\Gamma(-Y)\left[(M - i\omega)^Y - M^Y + (G + i\omega)^Y - G^Y\right]$

Numerical Method Derivation

- Apply the Fourier transform to the pricing PIDE

$$\begin{cases} \partial_t \mathcal{F}[v](t,\omega) + \Psi(\omega)\mathcal{F}[v](t,\omega) & = 0\ , \\ \mathcal{F}[v](T,\omega) & = \mathcal{F}[\varphi](\omega) \end{cases}$$

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$$\mathcal{F}[v](t_1,\omega) = \mathcal{F}[v](t_2,\omega) \cdot e^{(t_2 - t_1)\Psi(\omega)}$$

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- Apply the inverse Fourier transform

$$v(t_1,\mathbf{x}) = \mathcal{F}^{-1}\left\{\mathcal{F}[v](t_2,\boldsymbol{\omega}) \cdot e^{(t_2-t_1)\Psi(\boldsymbol{\omega})}\right\}(\mathbf{x})$$

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Fourier Space Time-stepping (FST) method

$$v^{n-1} = \mathrm{FFT}^{-1}[\mathrm{FFT}[v^n] \cdot e^{\Psi\,\Delta t}]$$

European Call Option Results

N	Value	Change	$\log_2$Ratio	CPU-Time
2048	0.04261423			0.002
4096	0.04263998	0.000026		0.005
8192	0.04264641	0.000006	2.0018	0.010
16384	0.04264801	0.000002	2.0010	0.019
32768	0.04264841	0.000000	2.0011	0.038

- *Option:* European call $S = 1.0, K = 1.0, T = 0.2$
- *Model:* Kou jump-diffusion
  $\sigma = 0.2, \lambda = 0.2, p = 0.5, \eta_- = 3, \eta_+ = 2, r = 0.0$
- *Quoted Price:* 0.0426761 Almendral and Oosterlee (2005)

American Put Option Results

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	512	9.22533163			0.181
8192	1024	9.22547180	0.0001402		0.958
16384	2048	9.22544621	0.0000256	2.4534	4.036
32768	4096	9.22543840	0.0000078	1.7117	21.303

- *Option:* American put $S = 90.0, K = 98.0, T = 0.25$
- *Model:* CGMY $C = 0.42, G = 4.37, M = 191.2, Y = 1.0102,$
  $r = 0.06$
- *Quoted Price:* 9.2254 Forsyth, Wan, and Wang (2006)

American Put Option Results with Infinite Activity

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	512	4.42077686			0.239
8192	1024	4.42077346	0.0000034		1.198
16384	2048	4.42077259	0.0000009	1.9616	4.614
32768	4096	4.42077245	0.0000001	2.6769	20.735

- *Option:* American put $S = 10.0$, $K = 10.0$, $T = 0.25$
- *Model:* CGMY $C = 1.0$, $G = 8.8$, $M = 9.2$, $Y = 1.8$, $r = 0.1$

Up-and-Out Barrier Call Option Results

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	512	0.25432521			0.149
8192	1024	0.25422752	0.0000977		0.669
16384	2048	0.25420350	0.0000240	2.0239	2.928
32768	4096	0.25419764	0.0000059	2.0335	15.691

- *Option:* Up-and-Out Barrier Call
  $S = 100.0$, $K = 100.0$, $B = 110$, $T = 1.0$
- *Model:* Black-Scholes-Merton $\sigma = 0.15$, $r = 0.05$, $q = 0.02$
- *Closed-Form Price:* 0.2541963 Hull (2005)

Multi-dimensional FST method

Pricing PIDE

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- Can be applied to pricing of European, American and other exotic, path dependent options

Spread Option Pricing Components

- Payoff depends on difference of two stock prices

$$\varphi(S_1(T), S_2(T)) = \max(\alpha S_2(T) - \beta S_1(T) - K, 0)$$

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- Stock price process is a 2D Merton jump-diffusion process

$$\frac{dS_i(t)}{S_i(t-)} = \mu_i dt + \sigma_i dW_i(t) + (J_i - 1)dN_i(t)$$

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- Characteristic exponent is given by:

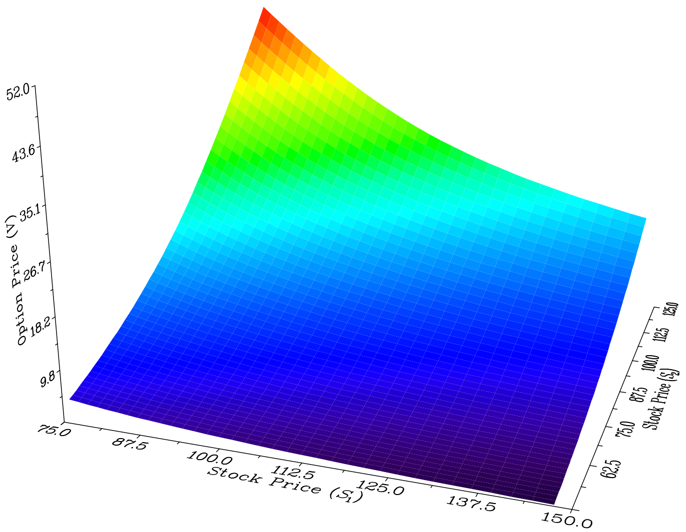
$$\begin{aligned}\Psi(\omega_1, \omega_2) &= i(\mu_1 - \frac{\sigma_1^2}{2})\omega_1 + i(\mu_2 - \frac{\sigma_2^2}{2})\omega_2 - \frac{\sigma_1^2\omega_1^2}{2} - \rho\sigma_1\sigma_2\omega_1\omega_2 - \frac{\sigma_2^2\omega_2^2}{2} \\ &\quad + \lambda_1\left(e^{i\tilde{\mu}_1\omega_1 - \tilde{\sigma}_1^2\omega_2^2/2} - 1\right) + \lambda_2\left(e^{i\tilde{\mu}_2\omega_2 - \tilde{\sigma}_2^2\omega_2^2/2} - 1\right)\end{aligned}$$

Spread Option Results

N	Value	Change	$\log_2$Ratio	CPU-Time
512	15.03639950			0.880245
1024	15.02776432	0.008635		2.821585
2048	15.02919574	0.001431	2.5928	11.919293
4096	15.02924971	0.000054	4.7293	48.371978
8192	15.02924214	0.000008	2.8345	209.806361

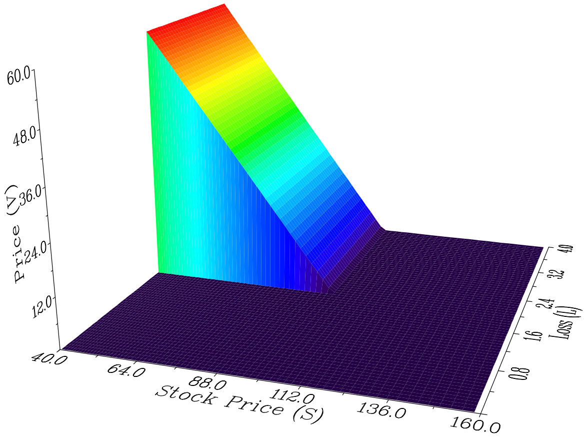
- *Option:* Spread call $S_1 = 96.0, S_2 = 100.0, K = 2.0, T = 1.0$
- *Model:* Merton jump-diffusion
  $\sigma_1 = 0.1, q_1 = 0.05, \lambda_1 = 0.25, \tilde{\mu_1} = -0.13, \tilde{\sigma}_1 = 0.37, \sigma_2 = 0.2, q_2 = 0.05, \lambda_2 = 0.5, \tilde{\mu_2} = 0.11, \tilde{\sigma}_2 = 0.41, \rho = 0.5, r = 0.1$
- *Kirk's Formula Price:* 15.03001533

Spread Option Price Surface



Catastrophe Equity Put (CatEPut) Payoff

- In the event of large (catastrophic) losses U, the insurer receives a put option on its own stock $\varphi(S(T), L(T)) = \mathbb{1}_{L(T)>U}(K - S(T))_+$



CatEPut Stock and Loss Processes

- Presence of losses drives the share value down, not an independent jump process

$$
\begin{aligned}
S(t) &= S(0)\exp\{-\alpha\, L(t) + \gamma\, t + \sigma\, W(t)\} \\
L(t) &= \sum_{n=1}^{N(t)} l_i
\end{aligned}
$$

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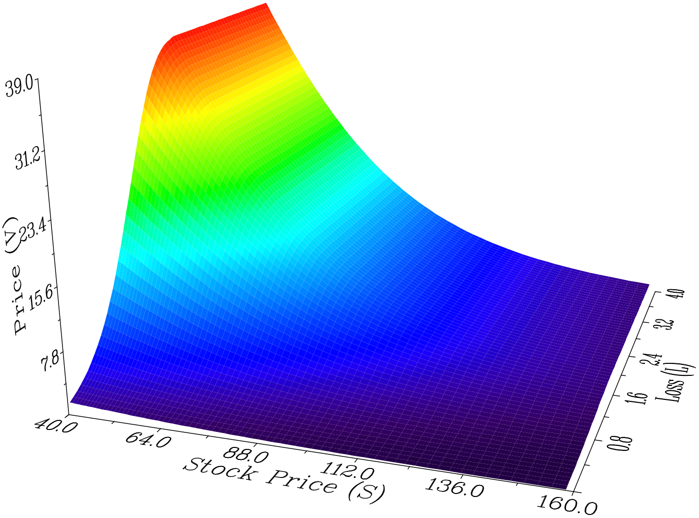
- When losses are modeled as a Gamma r.v., characteristic exponent is

$$\Psi(\omega_1, \omega_2) = i\,\gamma\,\omega_1 - \tfrac{1}{2}\sigma^2\,\omega_1^2 + \lambda\left[\left(1 - i(-\alpha\omega_1 + \omega_2)\frac{v}{m})\right)^{-\frac{m^2}{v}} - 1\right]$$

European CatEPut Price Surface



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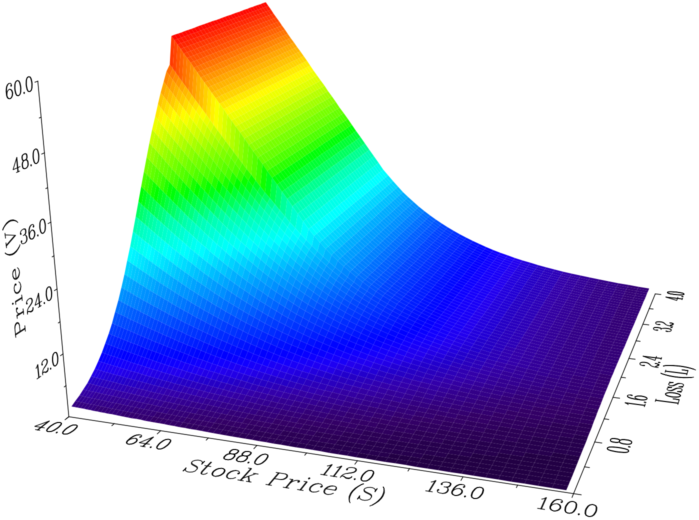

Why Regime Switching ?

- Capture long term implied volatility surfaces
- Model defaultable securities by associating one state as an absorbing, default state
- Model catastrophic losses which have moderate and extreme periods

Regime Switching Modeling Framework

- Let $\mathbb{K} := \{1, \ldots, K\}$ denote the possible hidden states of the world, driven by a continuous time Markov chain Z_t.

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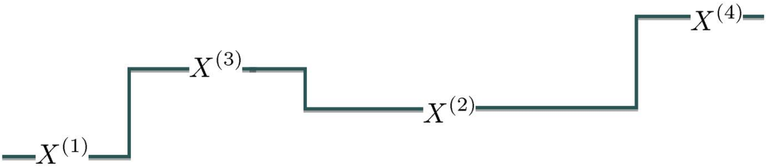

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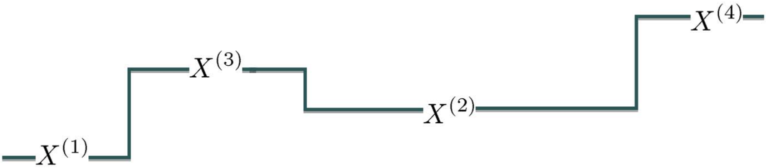

- The transition probability from state k at time t_1 to state l at time t_2 is given by

$$P_{kl}^{t_1 t_2} = \mathbb{Q}(Z_{t_2} = l | Z_{t_1} = k) = (\exp\{(t_2 - t_1)\mathbf{A}\})_{kl}$$

where $\mathbf{A}$ is the Markov chain generator

FST Method with Regime Switching

Pricing PIDE

$$\begin{cases} \left[\partial_t + \left(A_{kk} + \mathcal{L}^{(k)}\right)\right] v(\mathbf{x}, k, t) + \sum_{j \neq k} A_{jk}\, v(\mathbf{x}, j, t) & = 0 \\ v(\mathbf{x}, k, T) & = \varphi(\mathbf{S}(0) e^{\mathbf{x}}) \end{cases}$$

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FST Method Extension

$$v^{n-1} = \mathrm{FFT}^{-1}[\mathrm{FFT}[v^n] \cdot e^{\widetilde{\Psi}\,\Delta t}]$$

where

$$\widetilde{\Psi}(\omega)_{kl} = \begin{cases} A_{kk} + \Psi^{(k)}(\omega), & k = l \\ A_{kl}, & k \neq l \end{cases}$$

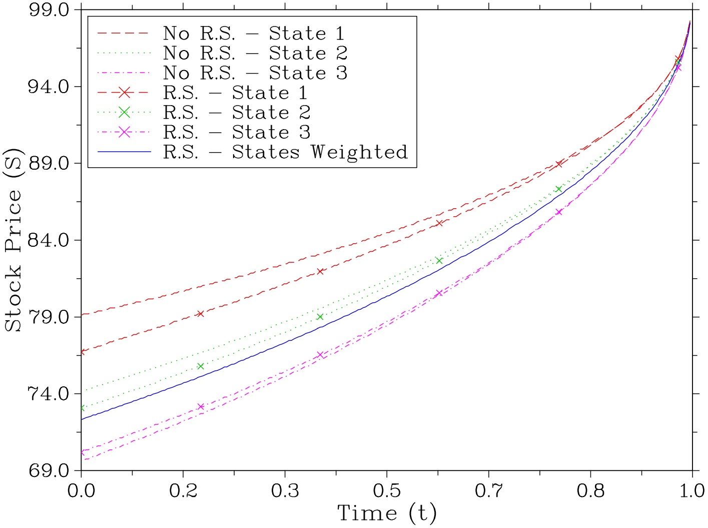
- No time-stepping required for European options

American Put Option with 3 Regimes Results

N	M	Value	Change	$\log_2$Ratio	CPU-Time
4096	256	14.25029309			0.634
8192	512	14.25025450	0.0000386		4.694
16384	1024	14.25024472	0.0000098	1.9802	14.385
32768	2048	14.25024245	0.0000023	2.1119	63.443

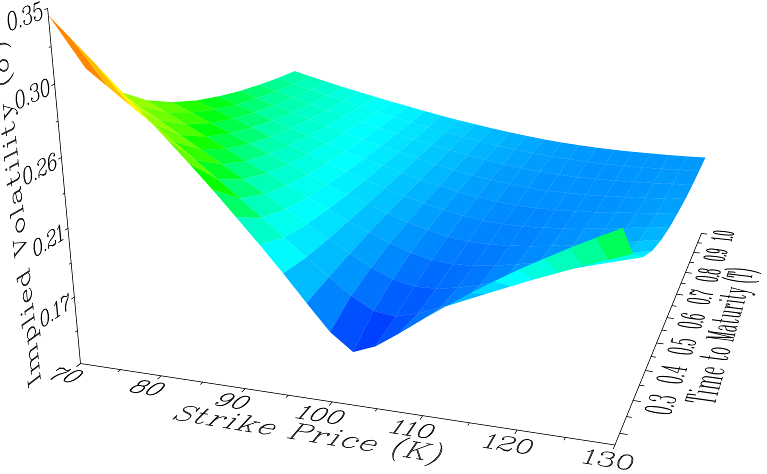
- *Option:* American put $S = 100.0, K = 100.0, T = 1.0$
- *Model:* Merton jump-diffusion $\sigma = 0.15, \tilde{\mu} = -0.5, \tilde{\sigma} = 0.45,$
  $r = 0.05, q = 0.02,\ \lambda \in [0.3, 0.5, 0.7],\ p = [0.2, 0.3, 0.5],$
  $A = [-0.8, 0.6, 0.2; 0.2, -1, 0.8; 0.1, 0.3, -0.4]$

American Put Option with 3 Regimes Exercise Boundary



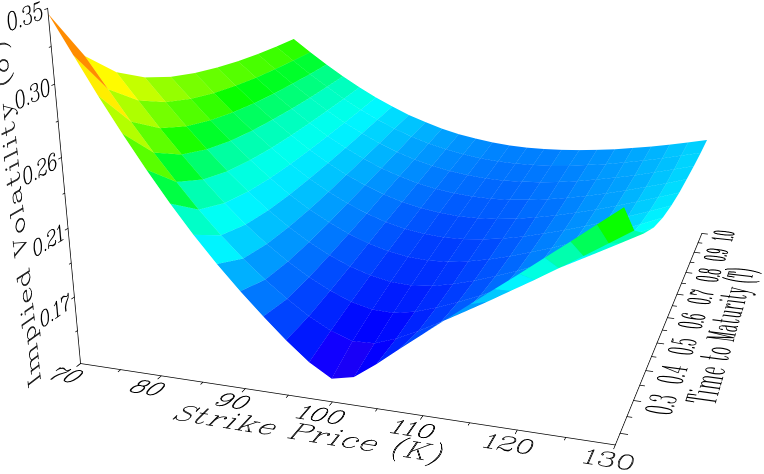
Implied Volatility with Lévy Processes

Lévy processes unable to capture the smile of long term options



Implied Vol. with Lévy Processes and Regime Switching

The feature can be captured using two state regime switching Lévy model



Fourier Space Time-stepping Method Summary

- Stable and robust, even for options with discontinuous payoffs
- Easily extendable to various stochastic processes and no loss of performance for infinite activity processes
- Can be applied to multi-dimensional and regime-switching problems in a natural manner
- Computationally efficient
  - Computational cost is $O(MNlogN)$ while the error is $O(\Delta x^2 + \Delta t^2)$
  - European options priced in a single time-step
  - Bermudan style options do not require time-stepping between monitoring dates

Future Work

- Exotic, multi-asset options
- Mean reverting processes in energy markets
- Stochastic volatility processes
- HJB equations arising from optimal control problems
  - Efficient policy iteration algorithm
  - Optimal control with jump-diffusions

Thank You!

http://128.100.73.155/fst/