Pricing and Hedging of Commodity Derivatives using the Fast Fourier Transform

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1 Generalized Model for Commodity Spot Prices

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3 Computing Option Greeks

4 Dynamic and Static Hedging
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WTI Crude Oil and Henry Hub Gas

- West Texas Intermediate Crude Oil
- Henry Hub Natural Gas

Spot Price ($/barrel) vs. Spot Price ($/mmbtu)

Year:
- 2004
- 2005
- 2006
- 2007
- 2008
- 2009
Great Britain System Sell Prices
Generalized Model for Commodity Spot Prices

Commodity prices

- Exhibit high volatilities and spikes and prices
- Tend to revert to long run equilibrium prices
Generalized Model for Commodity Spot Prices

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- Tend to revert to long run equilibrium prices

The Model

\[ dX(t) = (\Theta(t) - \kappa X(t)) \, dt + dW(t) + dJ(t) \]
\[ S(t) = S(0) \exp\{B X(t)\} \]
Generalized Model for Commodity Spot Prices

Commodity prices
- Exhibit high volatilities and spikes and prices
- Tend to revert to long run equilibrium prices

The Model

\[ dX(t) = (\Theta(t) - \kappa X(t)) \, dt + dW(t) + dJ(t) \]
\[ S(t) = S(0) \exp\{BX(t)\} \]

- Multi-factor, mean-reverting model with jumps
- Generalizes a number of well-known models, such as Gibson and Schwartz (1990), Clewlow and Strickland (2000), Hikspoors and Jaimungal (2007)
- Mean-reversion rate is time dependent - allows to incorporate seasonality
One-Factor Mean-Reverting Process with Jumps
Two-Factor MR Process with Different Decay Rates
One-Factor MR Processes with Co-dependent Jumps
Numerical Methods for Option Pricing

- Monte Carlo methods
- Tree methods
- Finite difference methods
  - Implicit-Explicit (IMEX) - Cont and Tankov (2004)
  - Fixed Point Iteration - d’Halluin, Forsyth, and Vetzal (2005)
- Quadrature methods
  - Reiner (2001)
  - Q-FFT - O’Sullivan (2005)
- Transform-based methods
  - Carr and Madan (1999)
  - Raible (2000)
  - Lewis (2001)
  - Lord, Fang, Bervoets, and Oosterlee (2008)
1 Generalized Model for Commodity Spot Prices

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Fourier Space Time-stepping Framework Overview

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces
Fourier Space Time-stepping Framework Overview

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

A framework for numerical pricing of financial derivatives

- Fast and precise pricing of a wide range of European and path-dependent, single- and multi-asset, vanilla and exotic derivatives
- Efficient handling of path-independent and discretely monitored derivatives
- Generic handling of different spot-price models and option payoffs
Option price at time $t$ is the discounted expected future payoff

$$V(t, S(t)) = e^{-r(T-t)}E[\varphi(S(T))]$$
Pricing Framework in Real Space

- Option price at time $t$ is the discounted expected future payoff
  $$V(t, S(t)) = e^{-r(T-t)}E[\varphi(S(T))]$$

- The discount-adjusted and log-transformed price process $\nu(t, X(t)) \triangleq e^{r(T-t)}V(t, S(0)e^{X(t)})$ satisfies a PIDE
  $$\begin{cases}
  (\partial_t + \mathcal{L}(t, x) - \kappa x' \partial_x) \nu(t, x) = 0, \\
  \nu(T, x) = \varphi(S(0) e^{Bx})
  \end{cases}$$

  where $\mathcal{L}$ acts on twice-differentiable functions $g(x)$ as follows:
  $$\mathcal{L}(t, x)g(x) = (\Theta(t)' \partial_x + \frac{1}{2} \partial_x' \Sigma \partial_x) g(x) + \int_{\mathbb{R}^n} (g(x+y) - g(x)) \nu(dy)$$
A function in the space domain $g(x)$ can be transformed to a function in the frequency domain $\hat{g}(\omega)$, where $\omega$ is given in radians per second, and vice-versa using the continuous Fourier transform

$$
\mathcal{F} [g](\omega) \triangleq \int_{-\infty}^{\infty} g(x) e^{-i\omega' x} dx \\
\mathcal{F}^{-1} [\hat{g}](x) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) e^{i\omega' x} d\omega
$$
Fourier Transform

- A function in the space domain $g(x)$ can be transformed to a function in the frequency domain $\hat{g}(\omega)$, where $\omega$ is given in radians per second, and vice-versa using the continuous Fourier transform

\[
\mathcal{F}[g](\omega) \triangleq \int_{-\infty}^{\infty} g(x) e^{-i\omega'x} dx
\]

\[
\mathcal{F}^{-1}[\hat{g}](x) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega) e^{i\omega'x} d\omega
\]

- Continuous Fourier transform is a linear operator that maps spatial derivatives $\partial_x$ into multiplications in the frequency domain

\[
\mathcal{F}[\partial_x^n g](\omega) = i\omega \mathcal{F}[\partial_x^{n-1} g](\omega) = \cdots = (i\omega)^n \mathcal{F}[g](\omega)
\]
Pricing Framework in Fourier Space

- Applying the Fourier transform to the pricing PDE we obtain a PDE in frequency space

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} + \hat{L}(t, \omega) + \kappa + \kappa \omega \partial \omega \big) \hat{v}(t, \omega) = 0, \\
\hat{v}(T, \omega) = \hat{\Phi}(T, \omega)
\end{array} \right.
\]
Pricing Framework in Fourier Space

- Applying the Fourier transform to the pricing PDE we obtain a PDE in frequency space

\[
\begin{align*}
\left\{ \left( \partial_t + \hat{L}(t, \omega) + \kappa + \kappa \omega \partial_\omega \right) \hat{v}(t, \omega) & = 0, \\
\hat{v}(T, \omega) & = \hat{\Phi}(T, \omega)
\right. 
\end{align*}
\]

- The Fourier transform of the operator \( L(t, x) \) can be computed analytically

\[
\hat{L}(t, \omega) = i\omega \Theta(t) - \frac{1}{2} \omega' \Sigma \omega + \int \left( e^{i\omega'z} - 1 \right) \nu(dz)
\]
Pricing Framework in Fourier Space

- Applying the Fourier transform to the pricing PDE we obtain a PDE in frequency space

\[
\begin{cases}
\left( \partial_t + \hat{L}(t, \omega) + \kappa + \kappa \omega \partial_\omega \right) \hat{v}(t, \omega) = 0, \\
\hat{v}(T, \omega) = \hat{\Phi}(T, \omega)
\end{cases}
\]

- The Fourier transform of the operator \( \hat{L}(t, x) \) can be computed analytically

\[
\hat{L}(t, \omega) = i \omega \Theta(t) - \frac{1}{2} \omega' \Sigma \omega + \int \left( e^{i \omega' z} - 1 \right) \nu(dz)
\]

- Introduce a new coordinate system via frequency scaling

\[
\tilde{v}(t, \omega) = \hat{v}(t, e^{\kappa'(t-t_*)} \omega)
\]
Pricing Framework in Fourier Space

- Applying the Fourier transform to the pricing PDE we obtain a PDE in frequency space

\[
\begin{aligned}
& \left\{
\begin{array}{l}
\left( \partial_t + \hat{\mathcal{L}}(t, \omega) + \kappa + \kappa \omega \partial_\omega \right) \hat{v}(t, \omega) = 0, \\
\hat{v}(T, \omega) = \hat{\Phi}(T, \omega)
\end{array}
\right.
\end{aligned}
\]

- The Fourier transform of the operator \( \mathcal{L}(t, x) \) can be computed analytically

\[
\hat{\mathcal{L}}(t, \omega) = i\omega \Theta(t) - \frac{1}{2} \omega' \Sigma \omega + \int \left( e^{i\omega'z} - 1 \right) \nu(dz)
\]

- Introduce a new coordinate system via frequency scaling

\[
\tilde{v}(t, \omega) = \hat{v}(t, e^{\kappa'(t-t^*)} \omega)
\]

- The PDE reduces to an ODE in time parameterized by \( \omega \)

\[
\begin{aligned}
& \left\{
\begin{array}{l}
\left( \partial_t + \tilde{\mathcal{L}}(t, \omega) + \kappa \right) \tilde{v}(t, \omega) = 0, \\
\tilde{v}(T, \omega) = \tilde{\Phi}(T, \omega)
\end{array}
\right.
\end{aligned}
\]
Given the value of $\tilde{v}(t, \omega)$ at time $t_2 \leq T$, the constant coefficient ODE is easily solved to find the value at time $t_1 < t_2$:

$$\tilde{v}(t_1, \omega) = \tilde{v}(t_2, \omega) \cdot e^{\tilde{\Psi}_\kappa(t_1, \omega; t_2)},$$

where the frequency space propagator is

$$\tilde{\Psi}_\kappa(t_1, \omega; t_2) = \int_{t_1}^{t_2} \tilde{\mathcal{L}}(s, \omega) \, ds + \text{Tr} \, \kappa (t_2 - t_1)$$
Given the value of $\tilde{v}(t, \omega)$ at time $t_2 \leq T$, the constant coefficient ODE is easily solved to find the value at time $t_1 < t_2$:

$$\tilde{v}(t_1, \omega) = \tilde{v}(t_2, \omega) \cdot e^{\tilde{\Psi}_\kappa(t_1, \omega; t_2)},$$

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$$\tilde{\Psi}_\kappa(t_1, \omega; t_2) = \int_{t_1}^{t_2} \tilde{L}(s, \omega) \, ds + \text{Tr} \kappa (t_2 - t_1)$$

The solution in terms of original coordinates (with $t_* = t_1$) is given by

$$\hat{v}(t_1, \omega) = \hat{v}(t_2, e^{\kappa^*(t_2 - t_1)} \omega) \cdot e^{\hat{\Psi}_\kappa(t_1, \omega; t_2)}$$

and the frequency space propagator is

$$\hat{\Psi}_\kappa(t_1, \omega; t_2) = \int_{t_1}^{t_2} \hat{L}(s, e^{\kappa^*(s-t_1)} \omega) \, ds + \text{Tr} \kappa (t_2 - t_1)$$
The scaled option prices in frequency space can be obtained from the scaled option prices in real space:

\[
\mathcal{F}[g](t, e^{\kappa'(t_2-t_1)} \omega) = \mathcal{F}[\tilde{g}](t, \omega) \cdot e^{-\text{Tr} \kappa(t_2-t_1)},
\]

where \( \tilde{g}(t, x) \triangleq g(t, x e^{-\kappa'(t_2-t_1)}) \)
The scaled option prices in frequency space can be obtained from the scaled option prices in real space

\[ \mathcal{F}[g](t, e^{\kappa'(t_2-t_1)} \omega) = \mathcal{F}[\tilde{g}](t, \omega) \cdot e^{-\text{Tr} \kappa(t_2-t_1)}, \]

where \( \tilde{g}(t, x) \triangleq g(t, x e^{-\kappa'(t_2-t_1)}) \)

The final solution becomes

\[ \nu(t_1, x) = \mathcal{F}^{-1} \left[ \mathcal{F}[\check{\nu}](t_2, \omega) \cdot e^{\check{\psi}(t_1, \omega; t_2)} \right](x) \]
The scaled option prices in frequency space can be obtained from the scaled option prices in real space

$$\mathcal{F}[g](t, e^{\kappa'(t_2-t_1)} \omega) = \mathcal{F}[\tilde{g}](t, \omega) \cdot e^{-\text{Tr} \kappa (t_2-t_1)},$$

where $$\tilde{g}(t, x) \triangleq g(t, x e^{-\kappa'(t_2-t_1)})$$

The final solution becomes

$$\nu(t_1, x) = \mathcal{F}^{-1} \left[ \mathcal{F}[\tilde{\nu}](t_2, \omega) \cdot e^{\hat{\psi}(t_1, \omega; t_2)} \right](x)$$

**FST Method for Propagating Option Prices**

$$\nu_{m-1} = \text{FFT}^{-1} \left[ \text{FFT}[\tilde{\nu}_m] \cdot e^{\hat{\psi}(t_{m-1}, \omega; t_m)} \right]$$
European options

\[ \mathbf{v}_0 = \text{FFT}^{-1} \left[ \text{FFT} \left[ \tilde{\mathbf{v}}_1 \right] \cdot e^{\hat{\Psi}(t, \omega; T)} \right] \]
Fourier Space Time-stepping Numerical Method

- European options

\[ v_0 = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{v}_1] \cdot e^{\hat{\psi}(t, \omega; T)} \right] \]

- Bermudan/American options

\[ v_{m-1}^* = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{v}_m] \cdot e^{\hat{\psi}(t_{m-1}, \omega; t_m)} \right], \]

\[ v_{m-1} = \max \{ v_{m-1}^*, v_M \}, \]

where \( v_{m-1}^* \) represents the holding value of the option.
Fourier Space Time-stepping Numerical Method

- **European options**

  \[ v_0 = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{v}_1] \cdot e^{\hat{\psi}(t,\omega;T)} \right] \]

- **Bermudan/American options**

  \[ v_{m-1}^* = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{v}_m] \cdot e^{\hat{\psi}(t_{m-1},\omega;t_m)} \right], \]

  \[ v_{m-1} = \max \{v_{m-1}^*, v_M\}, \]

  where \( v_{m-1}^* \) represents the holding value of the option

- **Barrier options**

  \[ v_{m-1} = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{v}_m] \cdot e^{\hat{\psi}(t_{m-1},\omega;t_m)} \right] \cdot \mathbb{1}_{\{x < B\}} + R \cdot \mathbb{1}_{\{x \geq B\}} \]
Fourier Space Time-stepping Numerical Method

- European options

\[ \mathbf{v}_0 = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{\mathbf{v}}_1] \cdot e^{\hat{\psi}(t,\omega;T)} \right] \]

- Bermudan/American options

\[ \mathbf{v}_{m-1}^* = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{\mathbf{v}}_m] \cdot e^{\hat{\psi}(t_{m-1},\omega;t_m)} \right] , \]

\[ \mathbf{v}_{m-1} = \max \left\{ \mathbf{v}_{m-1}^*, \mathbf{v}_M \right\} , \]

where \( \mathbf{v}_{m-1}^* \) represents the holding value of the option

- Barrier options

\[ \mathbf{v}_{m-1} = \text{FFT}^{-1} \left[ \text{FFT} [\tilde{\mathbf{v}}_m] \cdot e^{\hat{\psi}(t_{m-1},\omega;t_m)} \right] \cdot \mathbb{1}_{\{x<B\}} + R \cdot \mathbb{1}_{\{x\geq B\}} \]

- Exotic options, such as swings, can also be handled
Discrete Barrier Option Results

<table>
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<tr>
<th>N</th>
<th>M</th>
<th>Value</th>
<th>Change</th>
<th>Convergence</th>
<th>Time (sec.)</th>
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<td>0.974</td>
</tr>
</tbody>
</table>

- **Option:** Down-and-out barrier put $S = 100$, $K = 105$, $T = 1$, $B = 90$, $R = 3$ with daily monitoring
- **Model:** Merton jump-diffusion with mean reversion
  $\sigma = 0.2$, $\lambda = 1.0$, $\tilde{\mu} = -0.1$, $\tilde{\sigma} = 0.25$, $\theta = 90.0$, $\kappa = 0.75$, $r = 0.05$
- **Monte Carlo:** $2.77533300$ – 95% CI width of $0.00323116$ @ 114 sec.
1 Generalized Model for Commodity Spot Prices

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4 Dynamic and Static Hedging
Computation of Greeks - State Variables

Delta – $\partial S_k \nu(t, x) = \partial_{x_k} \nu(t, x)/S_k(t)$
Computation of Greeks - State Variables

Delta – $\partial_{S_k} \nu(t, x) = \partial_{x_k} \nu(t, x)/S_k(t)$

- Differentiation in real space computed via scaling in Fourier space

\[ \partial_{x_k} \nu(t, x) = \mathcal{F}^{-1} [i\omega_k \cdot \hat{\nu}(t, \omega)](x). \]
Computation of Greeks - State Variables

Delta – $\partial_{S_k} v(t, x) = \partial_{x_k} v(t, x)/S_k(t)$

- Differentiation in real space computed via scaling in Fourier space
  \[
  \partial_{x_k} v(t, x) = \mathcal{F}^{-1} \left[ i\omega_k \cdot \hat{v}(t, \omega) \right](x).
  \]

- The discrete method for computing Deltas is then given by
  \[
  \Delta_{k, m-1} = \text{FFT}^{-1} \left[ i\omega_k \cdot \hat{v}_{m-1} \right]
  \]
Delta – $\partial_S v(t, x) = \frac{\partial_x v(t, x)}{S_k(t)}$

- Differentiation in real space computed via scaling in Fourier space
  $\partial_x v(t, x) = \mathcal{F}^{-1} [i\omega_k \cdot \hat{v}(t, \omega)](x)$.

- The discrete method for computing Deltas is then given by
  $\Delta_{k,m-1} = \text{FFT}^{-1} [i\omega_k \cdot \hat{v}_{m-1}]$

- Higher order derivatives computed in similar manner
Computation of Greeks - State Variables

Delta – $\partial_{S_k} \nu(t, x) = \partial_{x_k} \nu(t, x) / S_k(t)$

- Differentiation in real space computed via scaling in Fourier space
  
  \[
  \partial_{x_k} \nu(t, x) = F^{-1} [i\omega_k \cdot \hat{\nu}(t, \omega)](x).
  \]

- The discrete method for computing Deltas is then given by
  
  $\Delta_{k, m-1} = FFT^{-1} [i\omega_k \cdot \hat{\nu}_{m-1}]$

- Higher order derivatives computed in similar manner

Theta – $\partial_t \nu(t, x)$
Computation of Greeks - State Variables

Delta – $\partial_{S_k} \nu(t,x) = \partial_{x_k} \nu(t,x)/S_k(t)$

- Differentiation in real space computed via scaling in Fourier space
  \[ \partial_{x_k} \nu(t,x) = \mathcal{F}^{-1} [i\omega_k \cdot \hat{\nu}(t,\omega)](x). \]

- The discrete method for computing Deltas is then given by
  \[ \Delta_{k,m-1} = \text{FFT}^{-1} [i\omega_k \cdot \hat{\nu}_{m-1}] \]

- Higher order derivatives computed in similar manner

Theta – $\partial_t \nu(t,x)$

- Obtained directly from the pricing ODE
  \[ \partial_t \tilde{\nu}(t,\omega) = -(\tilde{\mathcal{L}}(t,\omega) + \kappa) \tilde{\nu}(t,\omega) \]
Computation of Greeks - State Variables

Delta – $\partial_S k \nu(t, x) = \partial_x k \nu(t, x)/S_k(t)$

- Differentiation in real space computed via scaling in Fourier space
  $$\partial_x k \nu(t, x) = F^{-1} [i \omega_k \cdot \hat{\nu}(t, \omega)](x).$$

- The discrete method for computing Deltas is then given by
  $$\Delta_{k,m-1} = FFT^{-1} [i \omega_k \cdot \hat{\nu}_{m-1}]$$

- Higher order derivatives computed in similar manner

Theta – $\partial_t \nu(t, x)$

- Obtained directly from the pricing ODE
  $$\partial_t \tilde{\nu}(t, \omega) = - (\tilde{\mathcal{L}}(t, \omega) + \kappa) \tilde{\nu}(t, \omega)$$

- The discrete method for computing Theta is then given by
  $$\Theta_{m-1} = FFT^{-1} \left[ - (\hat{\mathcal{L}}(t, \omega) + \kappa) \cdot \hat{\nu}_{m-1} \right]$$
In Fourier space, the sensitivity satisfies an ODE with source term

\[ \partial_x \left\{ ( \partial_t + \tilde{\mathcal{L}}_\kappa ) \tilde{\nu}(t, \omega) \right\} = ( \partial_t + \tilde{\mathcal{L}}_\kappa ) \partial_x \tilde{\nu}(t, \omega) + \partial_x \tilde{\mathcal{L}}_\kappa \cdot \tilde{\nu}(t, \omega) = 0 \]
Computation of Greeks - Model Parameters

- In Fourier space, the sensitivity satisfies an ODE with source term

\[ \partial_x \left\{ (\partial_t + \tilde{L}_\kappa) \tilde{v}(t, \omega) \right\} = (\partial_t + \tilde{L}_\kappa) \partial_x \tilde{v}(t, \omega) + \partial_x \tilde{L}_\kappa \cdot \tilde{v}(t, \omega) = 0 \]

- The ODE can be solved explicitly

\[ \partial_x v(t, x) = \mathcal{F}^{-1} \left[ \partial_x \hat{\Psi}_\kappa(t, e^{\kappa'(T-t)} \omega; T) \cdot \hat{v}(t, \omega) \right](x) \]
Computation of Greeks - Model Parameters

- In Fourier space, the sensitivity satisfies an ODE with source term

$$\partial_{\ast} \left\{ \left( \partial_t + \tilde{\mathcal{L}}_\kappa \right) \tilde{v}(t, \omega) \right\} = \left( \partial_t + \tilde{\mathcal{L}}_\kappa \right) \partial_{\ast} \tilde{v}(t, \omega) + \partial_{\ast} \tilde{\mathcal{L}}_\kappa \cdot \tilde{v}(t, \omega) = 0$$

- The ODE can be solved explicitly

$$\partial_{\ast} v(t, x) = \mathcal{F}^{-1} \left[ \partial_{\ast} \hat{\Psi}_\kappa(t, e^{\kappa'(T-t)} \omega; T) \cdot \hat{v}(t, \omega) \right](x)$$

- The discrete method for computing the sensitivity is then given by

$$\nabla_{\ast, m-1} = \text{FFT}^{-1} \left[ \partial_{\ast} \hat{\Psi}_\kappa(t_{m-1}, e^{\kappa' \Delta t_m} \omega; t_m) \cdot \hat{v}_{m-1} \right]$$
Computation of Greeks - Model Parameters

- In Fourier space, the sensitivity satisfies an ODE with source term

\[
\partial_x \left\{ (\partial_t + \tilde{L}_\kappa) \tilde{v}(t, \omega) \right\} = (\partial_t + \tilde{L}_\kappa) \partial_x \tilde{v}(t, \omega) + \partial_x \tilde{L}_\kappa \cdot \tilde{v}(t, \omega) = 0
\]

- The ODE can be solved explicitly

\[
\partial_x v(t, x) = \mathcal{F}^{-1} \left[ \partial_x \hat{\Psi}_\kappa(t, e^{\kappa(T-t)} \omega; T) \cdot \hat{v}(t, \omega) \right](x)
\]

- The discrete method for computing the sensitivity is then given by

\[
\nabla_{*, m-1} = \text{FFT}^{-1} \left[ \partial_x \hat{\Psi}_\kappa(t_{m-1}, e^{\kappa' \Delta t_m} \omega; t_m) \cdot \hat{v}_{m-1} \right]
\]

- Higher order derivatives computed in similar manner
Greeks Computation Errors

- Price Error
- Delta Error
- Gamma Error
- Vega Error
- Theta Error
- Rho Error
1. Generalized Model for Commodity Spot Prices

2. Fourier Space Time-stepping framework

3. Computing Option Greeks

4. Dynamic and Static Hedging
Dynamic Hedging

Hedging portfolio for the option $V$ consists of $B$ units of cash, $e$ units of the underlying asset $S$ and $N$ hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, S(t)) + eS(t) + B - V(t, S(t))$$
Dynamic Hedging

- Hedging portfolio for the option $V$ consists of $B$ units of cash, $e$ units of the underlying asset $S$ and $N$ hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, S(t)) + eS(t) + B - V(t, S(t))$$

- The portfolio’s value remains unchanged under small movements in price

$$\partial_S \Pi = \vec{\phi} \cdot \partial_S \vec{I}(t, S(t)) + e - \partial_S V(t, S(t)) = 0$$
Dynamic Hedging

- Hedging portfolio for the option $V$ consists of $B$ units of cash, $e$ units of the underlying asset $S$ and $N$ hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, S(t)) + eS(t) + B - V(t, S(t))$$

- The portfolio’s value remains unchanged under small movements in price

$$\partial_S \Pi = \vec{\phi} \cdot \partial_S \vec{I}(t, S(t)) + e - \partial_S V(t, S(t)) = 0$$

- Can also hedge against small movements in interest-rates, volatility, etc.

$$\partial_\star \Pi = \vec{\phi} \cdot \partial_\star \vec{I}(t, S(t)) - \partial_\star V(t, S(t)) = 0$$
Dynamic Hedging

- Hedging portfolio for the option \( V \) consists of \( B \) units of cash, \( e \) units of the underlying asset \( S \) and \( N \) hedging instruments \( \vec{I} \) with weights \( \vec{\phi} \)

\[
\Pi = \vec{\phi} \cdot \vec{I}(t, S(t)) + eS(t) + B - V(t, S(t))
\]

- The portfolio’s value remains unchanged under small movements in price

\[
\partial_S \Pi = \vec{\phi} \cdot \partial_S \vec{I}(t, S(t)) + e - \partial_S V(t, S(t)) = 0
\]

- Can also hedge against small movements in interest-rates, volatility, etc.

\[
\partial_\star \Pi = \vec{\phi} \cdot \partial_\star \vec{I}(t, S(t)) - \partial_\star V(t, S(t)) = 0
\]

- What about large movements?
Static Hedging - Minimize Portfolio Variance

- Minimize portfolio price variance under expected asset price movement, Kennedy, Forsyth, Vetzal (2009):

\[
\arg\min_{\vec{e}_n, \vec{\phi}_n} \xi \mathbb{E}_t \left[ \vec{\phi}_n \cdot \Delta \vec{I}_n + e_n \Delta S_n - \Delta V_n \right]^2 + (1 - \xi) \Upsilon_n.
\]

where \( \Upsilon_n \) is the transaction cost to rebalance the portfolio:

\[
\Upsilon_n = \sum_{k=1}^{N} \left[ \tilde{\alpha}_k (\vec{\phi}_{k,n} - \vec{\phi}_{k,n-1}) \vec{I}_{k,n} \right]^2 + \left[ \beta (e_n - e_{n-1}) S_n \right]^2,
\]
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\[
\arg \min_{\xi, \phi_n, e_n} \xi \mathbb{E}_t \left[ \phi_n \cdot \Delta \tilde{I}_n + e_n \Delta S_n - \Delta V_n \right]^2 + (1 - \xi) \gamma_n .
\]

where \( \gamma_n \) is the transaction cost to rebalance the portfolio:

\[
\gamma_n = \sum_{k=1}^{N} \left[ \alpha_k (\phi_{k,n} - \phi_{k,n-1}) I_{k,n} \right]^2 + \left[ \beta (e_n - e_{n-1}) S_n \right]^2 ,
\]

- Since the objective function is quadratic, the optimality requires

\[
\frac{\partial F}{\partial \phi_{k,n}} = \xi \mathbb{E}_t \left[ \phi \cdot \Delta \tilde{I} + e \Delta S - \Delta V \right] (2 \Delta I_k) + (1 - \xi) \partial_{\phi_{k,n}} \gamma_n = 0
\]

\[
\frac{\partial F}{\partial e_n} = \xi \mathbb{E}_t \left[ \phi \cdot \Delta \tilde{I} + e \Delta S - \Delta V \right] (2 \Delta S) + (1 - \xi) \partial_{e_n} \gamma_n = 0
\]
Static Hedging - Minimize Portfolio Variance
Minimize portfolio price and Greeks variance under expected asset price movement

\[
\arg \min_{\mathbf{e}_n, \mathbf{\phi}_n} \sum_{D} w_D \mathbb{E}_{t_n} \left[ \mathbf{\phi}_n \cdot \Delta (D \mathbf{I}_n) + e_n \Delta (D S_n) - \Delta (D V_n) \right]^2 + (1 - \xi) \Upsilon_n
\]
Static Hedging - Minimize Price and Greeks Variance

- Minimize portfolio price and Greeks variance under expected asset price movement

$$\arg\min_{\phi_n, e_n} \xi \sum_D w_D \mathbb{E}_{t_n} \left[ \phi_n \cdot \Delta(D I_n) + e_n \Delta(D S_n) - \Delta(D V_n) \right]^2 + (1 - \xi) \gamma_n$$

- Since the objective function is quadratic, the optimality requires

$$\frac{\partial F}{\partial \phi_{k,n}} = \xi \sum_D w_D \mathbb{E}_{t_n} \left[ (\phi \cdot \Delta(D I) + e \Delta(D S) - \Delta(D V)) (2\Delta(D I_k)) \right]$$

$$+ (1 - \xi) \partial \phi_{k,n} \gamma_n = 0$$

$$\frac{\partial F}{\partial e_n} = \xi \sum_D w_D \mathbb{E}_{t_n} \left[ (\phi \cdot \Delta(D I) + e \Delta(D S) - \Delta(D V)) (2\Delta(D S)) \right]$$

$$+ (1 - \xi) \partial e_n \gamma_n = 0$$
Static Hedging - Minimize Price and Greeks Variance
Loss Distribution and VaR - Constant Volatility
Loss Distribution and VaR - Dynamic Volatility
The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces
FST Framework Summary

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Independent-increment, mean-reverting and interest-rate Lévy models are handled generically
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- One extra FFT required to compute each Greek
- Second order convergence in space and second order convergence in time for American options with penalty method
Thank You!


