# Pricing and Hedging of Commodity Derivatives using the Fast Fourier Transform

Vladimir Surkov
vladimir.surkov@utoronto.ca

Department of Statistical and Actuarial Sciences, University of Western Ontario
The Fields Institute, University of Toronto

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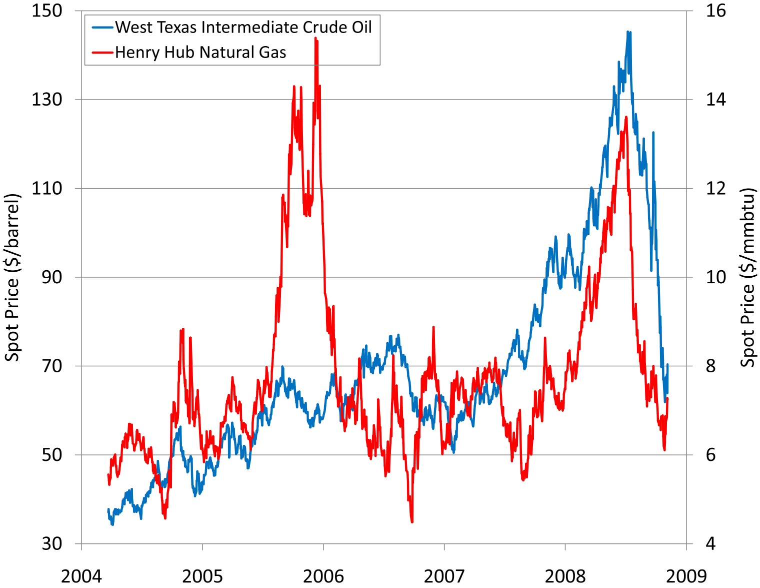
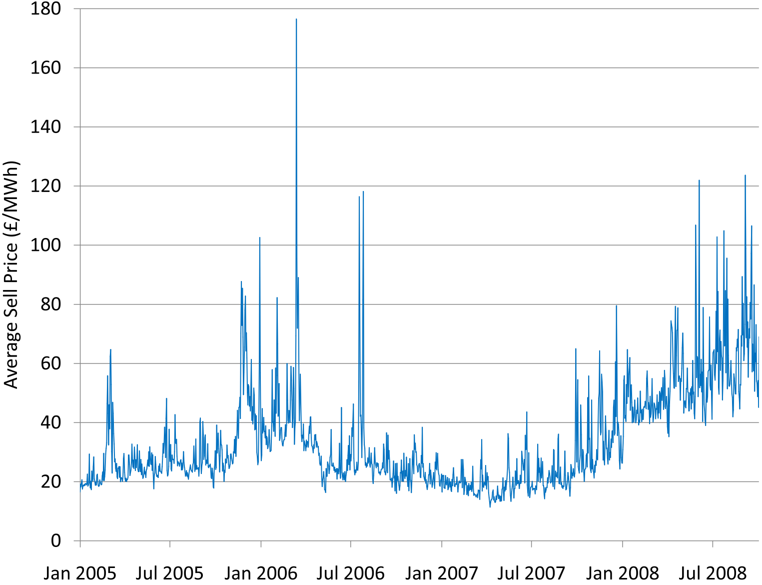
# WTI Crude Oil and Henry Hub Gas



# Great Britain System Sell Prices



# Generalized Model for Commodity Spot Prices

Commodity prices

- Exhibit high volatilities and spikes and prices
- Tend to revert to long run equilibrium prices

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## The Model

$$d\mathbf{X}(t) = (\mathbf{\Theta}(t) - \boldsymbol{\kappa}\mathbf{X}(t))\,dt + d\mathbf{W}(t) + d\mathbf{J}(t)$$
$$\mathbf{S}(t) = \mathbf{S}(0)\,\exp\{\mathbf{B}\mathbf{X}(t)\}$$

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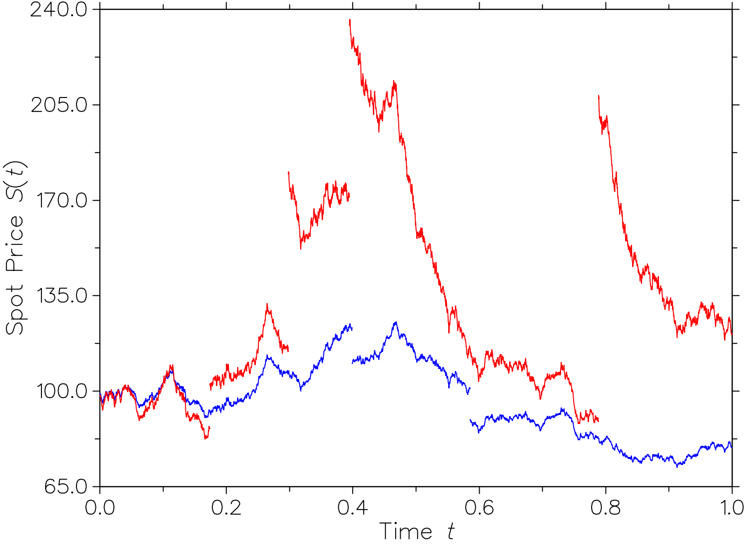
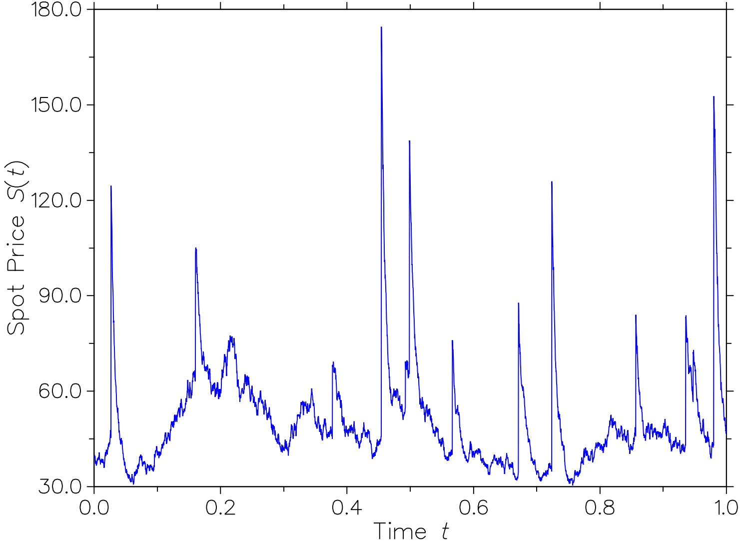
- Multi-factor, mean-reverting model with jumps
- Generalizes a number of well-known models, such as Gibson and Schwartz (1990), Clewlow and Strickland (2000), Hikspoors and Jaimungal (2007)
- Mean-reversion rate is time dependent - allows to incorporate seasonality

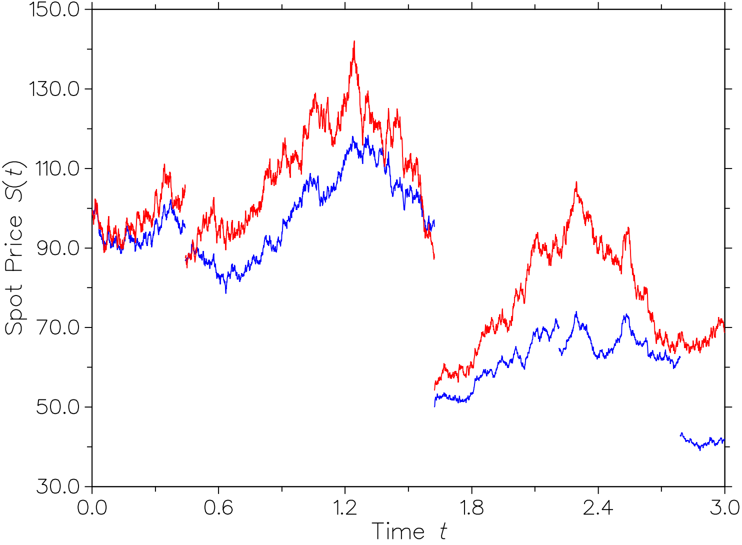
# One-Factor Mean-Reverting Process with Jumps



# Two-Factor MR Process with Different Decay Rates



# One-Factor MR Processes with Co-dependent Jumps



# Numerical Methods for Option Pricing

- Monte Carlo methods
- Tree methods
- Finite difference methods
  - Alternating Direction Implicit-FFT - Andersen and Andreasen (2000)
  - Implicit-Explicit (IMEX) - Cont and Tankov (2004)
  - IMEX Runge-Kutta - Briani, Natalini, and Russo (2004)
  - Fixed Point Iteration - d'Halluin, Forsyth, and Vetzal (2005)
- Quadrature methods
  - Reiner (2001)
  - QUAD - Andricopoulos, Widdicks, Duck, and Newton (2003)
  - Q-FFT - O'Sullivan (2005)
- Transform-based methods
  - Carr and Madan (1999)
  - Raible (2000)
  - Lewis (2001)
  - Lord, Fang, Bervoets, and Oosterlee (2008)

# Fourier Space Time-stepping Framework Overview

## The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

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A framework for numerical pricing of financial derivatives

- Fast and precise pricing of a wide range of European and path-dependent, single- and multi-asset, vanilla and exotic derivatives
- Efficient handling of path-independent and discretely monitored derivatives
- Generic handling of different spot-price models and option payoffs

# Pricing Framework in Real Space

- Option price at time $t$ is the discounted expected future payoff

$$V(t, \mathbf{S}(t)) = e^{-r(T-t)}\mathbb{E}\left[\varphi(\mathbf{S}(T))\right]$$

# Pricing Framework in Real Space

- Option price at time $t$ is the discounted expected future payoff

$$V(t,\mathbf{S}(t)) = e^{-r(T-t)}\mathbb{E}\left[\varphi(\mathbf{S}(T))\right]$$

- The discount-adjusted and log-transformed price process $v(t,\mathbf{X}(t)) \triangleq e^{r(T-t)}V(t,\mathbf{S}(0)e^{\mathbf{X}(t)})$ satisfies a PIDE

$$\begin{cases} (\partial_t + \mathcal{L}(t,\mathbf{x}) - \boldsymbol{\kappa}\mathbf{x}'\partial_{\mathbf{x}})\, v(t,\mathbf{x}) & = 0\,, \\ v(T,\mathbf{x}) & = \varphi(\mathbf{S}(0)\, e^{\mathbf{B}\mathbf{x}}) \end{cases}$$

where $\mathcal{L}$ acts on twice-differentiable functions $g(\mathbf{x})$ as follows:

$$\mathcal{L}(t,\mathbf{x})g(\mathbf{x}) = \left(\boldsymbol{\Theta}(t)'\partial_{\mathbf{x}} + \tfrac{1}{2}\,\partial_{\mathbf{x}}'\boldsymbol{\Sigma}\partial_{\mathbf{x}}\right) g(\mathbf{x}) + \int_{\mathbb{R}^n} \left(g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x})\right) \nu(d\mathbf{y})$$

# Fourier Transform

- A function in the space domain $g(\mathbf{x})$ can be transformed to a function in the frequency domain $\hat{g}(\boldsymbol{\omega})$, where $\boldsymbol{\omega}$ is given in radians per second, and vice-versa using the continuous Fourier transform

$$\mathcal{F}[g](\boldsymbol{\omega}) \triangleq \int_{-\infty}^{\infty} g(\mathbf{x}) e^{-i\boldsymbol{\omega}'\mathbf{x}} d\mathbf{x}$$

$$\mathcal{F}^{-1}[\hat{g}](\mathbf{x}) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega}'\mathbf{x}} d\boldsymbol{\omega}$$

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- Continuous Fourier transform is a linear operator that maps spatial derivatives $\partial_x$ into multiplications in the frequency domain

$$\mathcal{F}[\partial_{\mathbf{x}}^n g](\boldsymbol{\omega}) = i\boldsymbol{\omega}\, \mathcal{F}\left[\partial_{\mathbf{x}}^{n-1} g\right](\boldsymbol{\omega}) = \cdots = (i\boldsymbol{\omega})^n \mathcal{F}[g](\boldsymbol{\omega})$$

# Pricing Framework in Fourier Space

- Applying the Fourier transform to the pricing PDE we obtain a PDE in frequency space

$$\begin{cases} \left(\partial_t + \hat{\mathcal{L}}(t,\boldsymbol{\omega}) + \boldsymbol{\kappa} + \boldsymbol{\kappa}\boldsymbol{\omega}\partial_{\boldsymbol{\omega}}\right) \hat{v}(t,\boldsymbol{\omega}) & = 0\,, \\ \hat{v}(T,\boldsymbol{\omega}) & = \hat{\Phi}(T,\boldsymbol{\omega}) \end{cases}$$

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- The Fourier transform of the operator $\mathcal{L}(t,\mathbf{x})$ can be computed analytically

$$\hat{\mathcal{L}}(t,\boldsymbol{\omega}) = i\boldsymbol{\omega}\boldsymbol{\Theta}(t) - \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} + \int \left(e^{i\boldsymbol{\omega}'\mathbf{z}} - 1\right)\nu(d\mathbf{z})$$

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- Introduce a new coordinate system via frequency scaling

$$\tilde{v}(t,\boldsymbol{\omega}) = \hat{v}(t, e^{\boldsymbol{\kappa}'(t-t_\star)}\boldsymbol{\omega})$$

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$$\tilde{v}(t,\boldsymbol{\omega}) = \hat{v}(t, e^{\boldsymbol{\kappa}'(t-t_\star)}\,\boldsymbol{\omega})$$

- The PDE reduces to an ODE in time parameterized by $\boldsymbol{\omega}$

$$\begin{cases} \left(\partial_t + \tilde{\mathcal{L}}(t,\boldsymbol{\omega}) + \boldsymbol{\kappa}\right)\ \tilde{v}(t,\boldsymbol{\omega}) & = 0\,, \\ \tilde{v}(T,\boldsymbol{\omega}) & = \tilde{\Phi}(T,\boldsymbol{\omega}) \end{cases}$$

# Pricing Framework in Fourier Space (cont.)

- Given the value of $\tilde{v}(t,\boldsymbol{\omega})$ at time $t_2 \leq T$, the constant coefficient ODE is easily solved to find the value at time $t_1 < t_2$:

$$\tilde{v}(t_1,\boldsymbol{\omega}) = \tilde{v}(t_2,\boldsymbol{\omega}) \cdot e^{\tilde{\Psi}_{\boldsymbol{\kappa}}(t_1,\boldsymbol{\omega};t_2)},$$

where the frequency space propagator is

$$\tilde{\Psi}_{\boldsymbol{\kappa}}(t_1,\boldsymbol{\omega};t_2) = \int_{t_1}^{t_2} \tilde{\mathcal{L}}(s,\boldsymbol{\omega})\,ds + \mathrm{Tr}\,\boldsymbol{\kappa}\,(t_2 - t_1)$$

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- The solution in terms of original coordinates (with $t_\star = t_1$) is given by

$$\hat{v}(t_1, \boldsymbol{\omega}) = \hat{v}(t_2, e^{\boldsymbol{\kappa}'(t_2 - t_1)}\, \boldsymbol{\omega}) \cdot e^{\hat{\Psi}_{\boldsymbol{\kappa}}(t_1, \boldsymbol{\omega}; t_2)}$$

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# Pricing Framework in Fourier Space (cont.)

- The scaled option prices in frequency space can be obtained from the scaled option prices in real space

$$\mathcal{F}[g](t, e^{\boldsymbol{\kappa}'(t_2-t_1)}\boldsymbol{\omega}) = \mathcal{F}[\breve{g}](t,\boldsymbol{\omega}) \cdot e^{-\mathrm{Tr}\,\boldsymbol{\kappa}\,(t_2-t_1)},$$

where $\breve{g}(t,\mathbf{x}) \triangleq g(t, \mathbf{x}\, e^{-\boldsymbol{\kappa}'(t_2-t_1)})$

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where $\breve{g}(t, \mathbf{x}) \triangleq g(t, \mathbf{x}\, e^{-\boldsymbol{\kappa}'(t_2-t_1)})$

- The final solution becomes

$$v(t_1, \mathbf{x}) = \mathcal{F}^{-1}\left[\mathcal{F}\,[\breve{v}](t_2, \boldsymbol{\omega}) \,\cdot\, e^{\hat{\Psi}(t_1, \boldsymbol{\omega}; t_2)}\right](\mathbf{x})$$

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## FST Method for Propagating Option Prices

$$\mathbf{v}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}[\breve{\mathbf{v}}_m] \cdot e^{\hat{\Psi}(t_{m-1},\boldsymbol{\omega};t_m)}\right]$$

# Fourier Space Time-stepping Numerical Method

- European options

$$\mathbf{v}_0 = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\breve{\mathbf{v}}_1\right] \cdot e^{\hat{\Psi}(t,\boldsymbol{\omega};T)}\right]$$

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- Bermudan/American options

$$\mathbf{v}^{\star}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\breve{\mathbf{v}}_m\right] \cdot e^{\hat{\Psi}(t_{m-1},\boldsymbol{\omega};t_m)}\right] ,$$
$$\mathbf{v}_{m-1} = \max\left\{\mathbf{v}^{\star}_{m-1}, \mathbf{v}_M\right\} ,$$

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- Barrier options

$$\mathbf{v}_{m-1} = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\breve{\mathbf{v}}_m\right] \cdot e^{\hat{\Psi}(t_{m-1},\boldsymbol{\omega};t_m)}\right] \cdot \mathbb{1}_{\{\mathbf{x}<\mathbf{B}\}} + R \cdot \mathbb{1}_{\{\mathbf{x}\geq\mathbf{B}\}}$$

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- Exotic options, such as swings, can also be handled

# Discrete Barrier Option Results

| N | M | Value | Change | Convergence | Time (sec.) |
|---|---|---|---|---|---|
| 2048 | 252 | 2.75818698 | | | 0.048 |
| 4096 | 252 | 2.77495289 | 0.0167659 | | 0.099 |
| 8192 | 252 | 2.77164607 | 0.0033068 | 2.3420 | 0.210 |
| 16384 | 252 | 2.77315499 | 0.0015089 | 1.1319 | 0.523 |
| 32768 | 252 | 2.77395701 | 0.0008020 | 0.9118 | 0.974 |

- *Option:* Down-and-out barrier put $S = 100, K = 105, T = 1,\ B = 90, R = 3$ with daily monitoring
- *Model:* Merton jump-diffusion with mean reversion $\sigma = 0.2, \lambda = 1.0, \widetilde{\mu} = -0.1, \widetilde{\sigma} = 0.25, \theta = 90.0, \kappa = 0.75, r = 0.05$
- *Monte Carlo:* 2.77533300 – 95% CI width of 0.00323116 @ 114 sec.

# Computation of Greeks - State Variables

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$$\partial_{\mathbf{x}_k} v(t, \mathbf{x}) = \mathcal{F}^{-1} \left[ i\boldsymbol{\omega}_k \cdot \hat{v}(t, \boldsymbol{\omega}) \right](\mathbf{x}) \,.$$

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- The discrete method for computing Deltas is then given by

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- Obtained directly from the pricing ODE

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- The discrete method for computing Theta is then given by

$$\boldsymbol{\Theta}_{m-1} = \mathrm{FFT}^{-1} \left[ -\left( \hat{\mathcal{L}}(t, \boldsymbol{\omega}) + \boldsymbol{\kappa} \right) \cdot \hat{\mathbf{v}}_{m-1} \right]$$

# Computation of Greeks - Model Parameters

- In Fourier space, the sensitivity satisfies an ODE with source term

$$\partial_\star \Big\{ \big(\partial_t + \tilde{\mathcal{L}}_{\boldsymbol{\kappa}}\big) \tilde{v}(t, \boldsymbol{\omega}) \Big\} = \big(\partial_t + \tilde{\mathcal{L}}_{\boldsymbol{\kappa}}\big) \partial_\star \tilde{v}(t, \boldsymbol{\omega}) + \partial_\star \tilde{\mathcal{L}}_{\boldsymbol{\kappa}} \cdot \tilde{v}(t, \boldsymbol{\omega}) = \mathbf{0}$$

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- The ODE can be solved explicitly

$$\partial_\star v(t,\mathbf{x}) = \mathcal{F}^{-1}\left[\partial_\star\hat{\Psi}_{\boldsymbol{\kappa}}(t, e^{\boldsymbol{\kappa}'(T-t)}\boldsymbol{\omega}; T) \cdot \hat{v}(t,\boldsymbol{\omega})\right](\mathbf{x})$$

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- The discrete method for computing the sensitivity is then given by

$$\nabla_{\star,\,m-1} = \mathrm{FFT}^{-1}\left[\partial_\star\hat{\Psi}_{\boldsymbol{\kappa}}(t_{m-1}, e^{\boldsymbol{\kappa}'\Delta t_m}\boldsymbol{\omega}; t_m)\cdot\hat{\mathbf{v}}_{m-1}\right]$$

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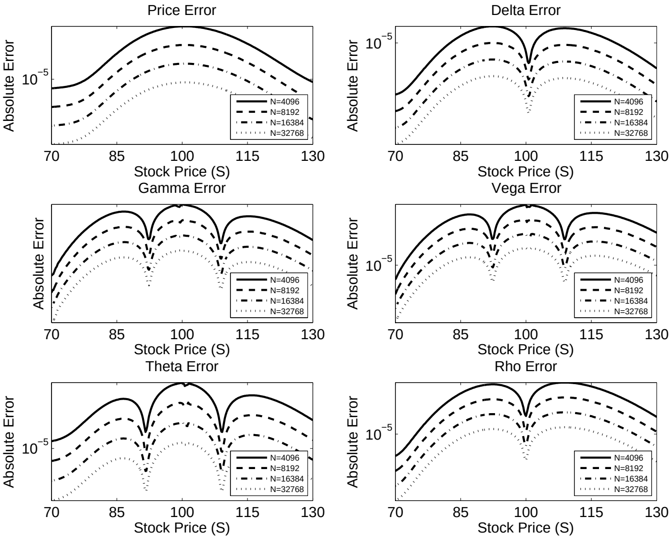
$$\nabla_{\star,\,m-1} = \mathrm{FFT}^{-1}\left[\partial_\star\hat{\Psi}_{\boldsymbol{\kappa}}(t_{m-1}, e^{\boldsymbol{\kappa}'\Delta t_m}\boldsymbol{\omega}; t_m) \cdot \hat{\mathbf{v}}_{m-1}\right]$$

- Higher order derivatives computed in similar manner

# Greeks Computation Errors



# Dynamic Hedging

- Hedging portfolio for the option $V$ consists of $B$ units of cash, $e$ units of the underlying asset $S$ and $N$ hedging instruments $\vec{I}$ with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, \mathbf{S}(t)) + e\mathbf{S}(t) + B - V(t, \mathbf{S}(t))$$

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- What about large movements?

# Static Hedging - Minimize Portfolio Variance

- Minimize portfolio price variance under expected asset price movement, Kennedy, Forsyth, Vetzal (2009):

$$\underset{e_n,\vec{\phi}_n}{\arg\min}\ \xi\,\mathbb{E}_{t_n}\left[\vec{\phi}_n \cdot \Delta\vec{I}_n + e_n\Delta S_n - \Delta V_n\right]^2 + (1-\xi)\Upsilon_n\,.$$

where $\Upsilon_n$ is the transaction cost to rebalance the portfolio:

$$\Upsilon_n = \sum_{k=1}^{N}\left[\vec{\alpha}_k(\vec{\phi}_{k,n} - \vec{\phi}_{k,n-1})\vec{I}_{k,n}\right]^2 + \left[\beta(e_n - e_{n-1})S_n\right]^2,$$

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- Since the objective function is quadratic, the optimality requires

$$\frac{\partial F}{\partial \phi_{k,n}} = \xi\,\mathbb{E}_{t_n}\left[\left(\vec{\phi}\cdot\Delta\vec{I} + e\Delta S - \Delta V\right)\left(2\Delta I_k\right)\right] + (1-\xi)\,\partial_{\phi_{k,n}}\Upsilon_n = 0$$

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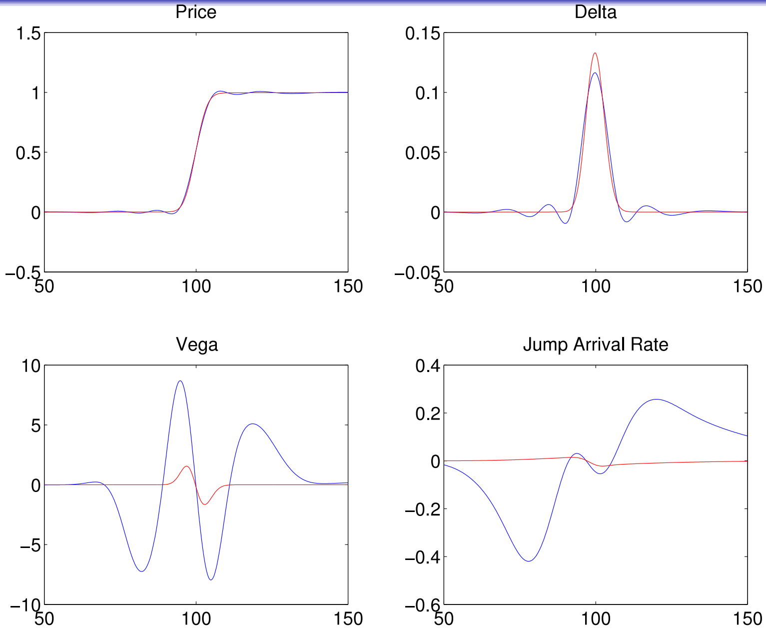

# Static Hedging - Minimize Price and Greeks Variance

- Minimize portfolio price and Greeks variance under expected asset price movement

$$\underset{e_n,\vec{\phi}_n}{\arg\min}\ \xi \sum_{\mathcal{D}} w_{\mathcal{D}}\, \mathbb{E}_{t_n}\left[\vec{\phi}_n \cdot \Delta(\mathcal{D}\vec{I}_n) + e_n\Delta(\mathcal{D}S_n) - \Delta(\mathcal{D}V_n)\right]^2 + (1-\xi)\Upsilon_n$$

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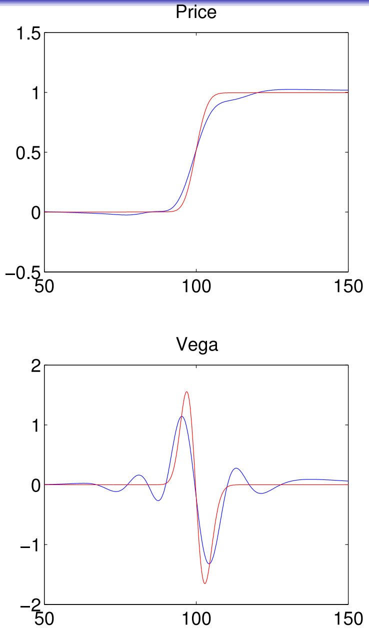
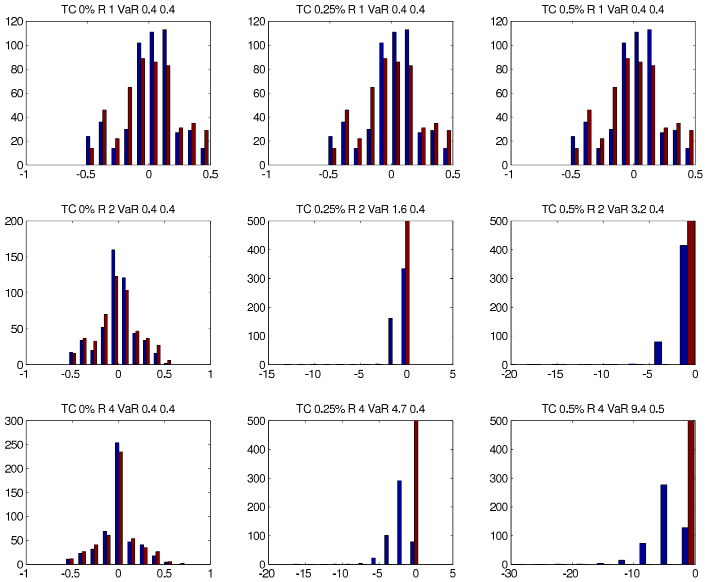
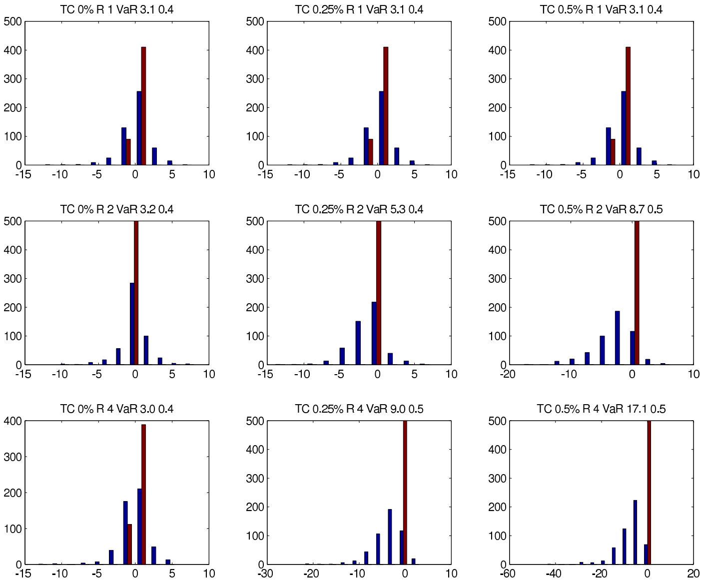


# Loss Distribution and VaR - Constant Volatility



# Loss Distribution and VaR - Dynamic Volatility



# FST Framework Summary

## The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
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- Second order convergence in space and second order convergence in time for American options with penalty method

# Thank You!

Jackson, K. R., S. Jaimungal, and V. Surkov (2008).
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Working paper.

More at http://ssrn.com/author=879101