Simultaneous Search and Adverse Selection
—Online Appendix—

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Appendix B Additional Results and Proofs

B.1 Excessive Entry for Equilibria With a Single Pooling Market

In Appendix A.3, we introduce a threshold v_H^0, which guarantees that the price ranges of the separate markets to which high and low types apply cannot intersect. Using this threshold, we can show the following:

Proposition 8. *Assume $c_H < v_L - k$ and $v_H > v_H^0$. Then, as $\mathbb{N} \to \infty$, considering any sequence of equilibria featuring a single pooling market, the workers' market utility satisfies* (12).

Proof. Consider an arbitrary equilibrium with a single pooling market; that is, with a single wage level at which both H- and L-type workers send some applications. Let $\bar{p}$ denote the wage and $\bar{\mu}$ denote the effective queue length in that market. Towards a contradiction, suppose the equilibrium allocation satisfies

$$\lim_{N\to\infty} \sigma u_{N,L} + (1-\sigma)u_{N,H} = \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k. \tag{B.1}$$

Under this condition workers extract all the surplus. This means that in the limit there is no welfare loss: all workers are thus hired with probability one and all firms hire with probability one. By an analogous argument to the one used in the proof of Proposition 2 above, we can exclude the possibility that high types are hired at strictly higher wages than

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low types with a probability that is positive in the limit. It thus follows that, as $N \to \infty$ the probability of trades taking place outside the pooling market tends to zero. In order for firms to hire with probability one, the effective queue length in the pooling market $\bar{\mu}$ must then tend to ∞ as $N \to \infty$.

Let $\tilde{n}+1$ indicate the first application which L-types send to the pooling market. We allow $\tilde{n}$ to be equal to 0, in which case the first application of L-types is sent to the pooling market. When $\tilde{n} \geq 1$ the terms of trade in the separate markets where only L-types send applications, indexed by $n \leq \tilde{n}$, are determined as in the equilibrium where types are observable (see the argument in Appendix A.3). In equilibrium L-types must then prefer to send their $(\tilde{n}+1)$-th application to the pooling market rather than to the L-type market where they send their $(\tilde{n}+1)$-th application in the equilibrium with observable types (if this condition is violated, posting wage $p^*_{\tilde{n}+1,L}$ constitutes a profitable deviation for firms). Hence, $\tilde{n}$ must be such that[1]

$$\psi(\bar{\mu})(\bar{p} - c_L - u^*_{\tilde{n},L}) + u^*_{\tilde{n},L} \geq \psi(\mu^*_{\tilde{n}+1,L})(p^*_{\tilde{n}+1,L} - c_L - u^*_{\tilde{n},L}) + u^*_{\tilde{n},L}.$$

As argued above, for (B.1) to hold, $\bar{\mu}$ must tend to ∞ as $N \to \infty$. Notice further that $\bar{p}$ is bounded above by the size of the gains from trade, i.e. $\sigma v_L + (1-\sigma)v_H - k$. It then follows that the left-hand side of the above inequality converges to $u^*_{\tilde{n},L}$ as $N \to \infty$. In order for the inequality to hold, given $p^*_{\tilde{n},L} - c_L - u^*_{\tilde{n},L} > 0$, the effective queue length $\mu^*_{\tilde{n},L}$ must then also diverge to ∞ as $N \to \infty$. Hence, the index $\tilde{n}$ must tend to ∞ as $N \to \infty$: L-types send an infinite number of applications to L-type markets, where only such types apply.

Next, we can show that in any equilibrium with a single pooling market H-types send their first application to the pooling market. By the assumption $v_H > v^0_H$, there is a unique intersection between the upper envelope of the L-types' indifference curves associated with the first $\tilde{n}$ applications and Π_H. This intersection is with indifference curve $I_L(u_{\tilde{n}-1,L}, u_{\tilde{n}})$. Since the wage at this intersection is strictly greater than $\bar{p}$ (Π_H lies to the right of $\Pi_{\bar{\gamma}}$ in the (p, μ) space), there cannot be a market with a wage $p < \bar{p}$ to which only H-types apply and firms make non-negative profits. High types must therefore send their first application to the pooling market, as claimed.

For the allocation to be incentive compatible and ensure H-types do not want to deviate and apply to any L-type market, since $\bar{\mu} > \mu^*_{\tilde{n},L}$,[2] the H-types' outside option associated with their first application must be greater than the L-types' outside option associated with their $\tilde{n}$-th application, that is: $c_H \geq c_L + u^*_{\tilde{n}-1,L}$. Since $\tilde{n}$ tends to ∞ as $N \to \infty$, the market utility $u^*_{\tilde{n}-1,L}$ tends to $v_L - c_L - k$. Having assumed $c_H < v_L - k$, the term $c_L + u^*_{\tilde{n}-1,L}$ thus tends to a limit strictly greater than c_H as $N \to \infty$. The above inequality is violated in the

[1] When $\tilde{n} = 0$, $u^*_{0,L} = 0$.

[2] The effective queue length is increasing in the index of the low types' applications.

limit, which yields the contradiction.

□

B.2 Proof of Lemma 8

We show first that for wages $p < \bar{p}$ the market utility condition and (6) imply that firms' beliefs are $\gamma(p) = 1$. We begin by establishing the property for all $p \in (p^*_{l,L}, \bar{p})$. This is achieved by showing that, under the assumptions made, the following condition holds, for all $n = 1, .., N - l$:

$$\psi(\mu)(p - c_L) + (1 - \psi(\mu))\, u^*_{l-1,L} \;=\; u^*_{l,L}, \tag{B.2}$$

$$\psi(\mu)(p - c_H) + (1 - \psi(\mu))\, u_{n-1,H} \;\leq\; u_{n,H}. \tag{B.3}$$

Solving the first equation for $\psi(\mu)$ and substituting into the second inequality yields

$$\frac{u^*_{l,L} - u^*_{l-1,L}}{p - c_L - u^*_{l-1,L}} \leq \frac{u_{n,H} - u_{n-1,H}}{p - c_H - u_{n-1,H}}. \tag{B.4}$$

Recalling that, in the candidate equilibrium under consideration, $u^*_{l,L} = \psi(\bar{\mu})(\bar{p} - c_L) + (1 - \psi(\bar{\mu}))u^*_{l-1,L}$ and $u_{n,H} = \psi(\bar{\mu})(\bar{p} - c_H) + (1 - \psi(\bar{\mu}))u_{n-1,H}$, we have:

$$\frac{u^*_{l,L} - u^*_{l-1,L}}{\bar{p} - c_L - u^*_{l-1,L}} = \frac{u_{n,H} - u_{n-1,H}}{\bar{p} - c_H - u_{n-1,H}}. \tag{B.5}$$

Using this condition to substitute for $(u_{n,H} - u_{n-1,H}) \,/\, \left(u^*_{l,L} - u^*_{l-1,L}\right)$ in the above inequality and simplifying terms, we obtain:

$$p\left(c_H + u_{n-1,H} - c_L - u^*_{l-1,L}\right) \leq \bar{p}\left(c_H + u_{n-1,H} - c_L - u^*_{l-1,L}\right). \tag{B.6}$$

Finally, notice that for all $n \leq l - 1$, we have $u^*_{n,L} + c_L \leq c_H$ by definition of l and hence[3]

$$u^*_{l-1,L} + c_L \leq u_{n-1,H} + c_H, \text{ for all } n = 1, ..N. \tag{B.7}$$

Hence inequality (B.6) is satisfied whenever $p < \bar{p}$, which establishes the claim. A similar argument applies to wages weakly below $p^*_{l,L}$—for this case, it is in fact the same as for the separating equilibrium.

Next, we consider wages in the interval $(\bar{p}, p_H)$. We will show that for all $p \in (\bar{p}, p_H)$,

[3] Here we abstract from the non-generic case where $u_{l-1} + c_L = c_H$. In this knife-edge case, (B.3) will hold as equality for $n = 1$ and $p < \bar{p}$, which means that any $\gamma(p) \in [0, 1]$ will satisfy the market utility condition, including $\gamma(p) = 1$.

$\gamma(p) = 1$ again holds. By definition of l we have $u^*_{l,L} + c_L > c_H$. Hence for all $n = 0, 1, 2, ..., N - l - 1$, in the candidate equilibrium under consideration the following holds:

$$\begin{aligned} u_{l+n,L} + c_L &= \beta(n;\bar{\mu})\bar{p} + (1 - \beta(n;\bar{\mu}))(u^*_{l,L} + c_L) \\ &> \beta(n;\bar{\mu})\bar{p} + (1 - \beta(n;\bar{\mu}))c_H \\ &= u_{n,H} + c_H. \end{aligned} \tag{B.8}$$

This means that the reservation utility for the n-th application sent to the pooling market is greater for low than for high types, for all $n = 0, .., N - l - 1$. In particular, we have

$$u_{N-1,L} + c_L > u_{N-l-1,H} + c_H. \tag{B.9}$$

We also want to show that the reservation utility for the N-th application sent by low types to the pooling market is smaller than the one for the first application sent by high types to a H-type market, that is:

$$u_{N-1,L} + c_L < u_{N-l,H} + c_H. \tag{B.10}$$

Recalling that $u_{l+n,L} + c_L = \beta(n;\bar{\mu})\bar{p} + (1 - \beta(n;\bar{\mu}))(u^*_{l,L} + c_L)$, using the property $\beta(n;\cdot) = \beta(n-1;\cdot) + (1-\beta(n-1;\cdot))\beta(1;\cdot)$ and the fact that $u^*_{l,L} = \beta(1;\bar{\mu})(\bar{p} - c_L) + (1 - \beta(1;\bar{\mu}))u^*_{l-1,L}$, when $n = N - l - 1$, we obtain

$$\begin{aligned} u_{N-1,L} &= \beta(N - l - 1;\bar{\mu})(\bar{p} - c_L) + (1 - \beta(N - l - 1;\bar{\mu}))u^*_{l,L} \\ &= \beta(N - l;\bar{\mu})(\bar{p} - c_L) + (1 - \beta(N - l;\bar{\mu}))u^*_{l-1,L}. \end{aligned} \tag{B.11}$$

This implies

$$\begin{aligned} u_{N-1,L} + c_L &= \beta(N - l;\bar{\mu})\bar{p} + (1 - \beta(N - l;\bar{\mu}))(u^*_{l-1,L} + c_L) \\ &\leq \beta(N - l;\bar{\mu})\bar{p} + (1 - \beta(N - l;\bar{\mu}))c_H \\ &= u_{N-l,H} + c_H, \end{aligned} \tag{B.12}$$

where the inequality in the second line follows from $u^*_{l-1,L} + c_L \leq c_H$, which as we already pointed out, holds by definition of l. This then establishes (B.10).

Having shown (B.9) and (B.10), we want to prove that the following conditions hold for all $p \in (\bar{p}, p_H)$ and $n = 1, ..., N$:

$$\psi(\mu)(p - c_L) + (1 - \psi(\mu))\, u_{N-1,L} \;=\; u_{N,L} \tag{B.13}$$

$$\psi(\mu)(p - c_H) + (1 - \psi(\mu))\, u_{n-1,H} \;\leq\; u_{n,H} \tag{B.14}$$

Again solving for $\psi(\mu)$ the first equation and substituting into the second one yields

$$\frac{u_{N,L} - u_{N-1,L}}{(p - c_L - u_{N-1,L})} \leq \frac{u_{n,H} - u_{n-1,H}}{(p - c_H - u_{n-1,H})} \tag{B.15}$$

For $n \leq N - l$ the above inequality holds as an equality at $(\bar{\mu}, \bar{p})$. Following the same argument as above, we can use this equality to substitute for $(u_{n,H} - u_{n-1,H}) / (u_{N,L} - u_{N-1,L})$ and rewrite (B.15) as an inequality similar to (B.6):

$$p\left(c_H + u_{n-1,H} - c_L - u^*_{N-1,L}\right) \leq \bar{p}\left(c_H + u_{n-1,H} - c_L - u^*_{N-1,L}\right) \tag{B.16}$$

Due to condition (B.9), the terms in the brackets are negative for all $n \leq N - l$, so (B.15) holds. Hence, (B.13,B.14) is satisfied for $p \in (\bar{p}, p_H)$ and $n \leq N - l$.

Next, consider the applications that are sent by high types to the market with wage p_H: $n = N - l + 1, ..., N - l + \bar{n} - 1$. Using the property that for $n = N - l + 1, ..., N - l + \bar{n} - 1$ condition (B.15) holds as an equality at p_H (since in the candidate equilibrium we are considering, high types send those applications to p_H), we can again rewrite (B.15) as follows:

$$p\,(c_H + u_{n-1,H} - c_L - u_{N-1,L}) \leq p_H\,(c_H + u_{n-1,H} - c_L - u_{N-1,L})\,. \tag{B.17}$$

Under condition (B.10), we have $u_{N-1,L} + c_L < u_{n-1,H} + c_H$, so the inequality holds for all $p \in (\bar{p}, p_H)$.

For applications $n \geq N - l + \bar{n}$, the L-type incentive constraint is slack (by definition of $\bar{n}$) and the terms of trades for these applications are given by the unconstrained solution, described in (A.6). This implies that to attract applications from high types for which their reservation utility is given by $u_{N-l+\bar{n}-1}$, firms cannot make positive profits. Since for all $p \in (\bar{p}, p_H)$ and μ satisfying (B.13) firms would make positive profits if they could attract applications only from high types, i.e. $(1-e^{-\mu})(v_H - p) > k$, it follows that (B.14) is satisfied for all $n \geq N - l + \bar{n}$.

□

B.3 Proof of Proposition 5

B.3.1 The case $c_H < v_L - k$.

Candidate equilibrium. The logic of the argument is very close to that used to prove Proposition 3. Consider the candidate equilibrium we constructed in the proof A.3 with $m = m' = N - l$; that is, with the last $N - l$ applications and the first $N - l$ applications sent respectively by low and high types to the pooling market. Let us reassign the same fraction of these applications both for low and high types to a second pooling market, with a higher wage and effective queue length.

Let $\hat{n} > l$ indicate the application after which the low type switches from the first pooling market to the second one. The low types' application strategy consists thus in sending the first l applications to L-type markets, where only low types are present, the next $\hat{n} - l$ applications to pooling market 1 and the last $N - \hat{n}$ applications to pooling market 2. The high types' application strategy consists in sending the first $\hat{n} - l$ to pooling market 1, the next $N - \hat{n}$ applications to pooling market 2, and the last l applications to H-type markets. We show next that the effective composition in the two pooling markets, resulting from this reassignment, is the same. Let us denote it by $\bar{\gamma}$, while $(\bar{\mu}_1, \bar{p}_1)$ denote the terms of trade in the first pooling market and $(\bar{\mu}_2, \bar{p}_2)$ those in the second pooling market.

Proceeding similarly to the proof of Proposition 3, we also indicate with $\tau_{2,H}$ the probability that a high type receives no wage offer strictly above $\bar{p}_2$. In pooling market 2 low types send $N - \hat{n}$ effective applications (since all offers received are accepted) , while high types only send $\tau_{2,H}(N - \hat{n})$ effective applications. The effective composition in this market is thus given by the following expression, analogous to (A.8):

$$\frac{\sigma(N - \hat{n})}{\sigma(N - \hat{n}) + (1 - \sigma)\tau_{2,H}(N - \hat{n})} = \frac{\sigma}{\sigma + (1 - \sigma)\tau_{2,H}}$$

Let $\beta(N - \hat{n}; \bar{\mu}_2)$ denote again the probability for any of the two types of receiving an offer in pooling market 2, with effective queue length $\bar{\mu}_2$, when sending $n \geq 1$ applications to that market. It thus follows that the effective composition in pooling market 1 is:

$$\frac{\sigma\,(\hat{n} - l)\,(1 - \beta(N - \hat{n}; \bar{\mu}_2))}{\sigma(\hat{n} - l)(1 - \beta(N - \hat{n}; \bar{\mu}_2)) + (1 - \sigma)(\hat{n} - l)\tau_{2,H}(1 - \beta(N - \hat{n}; \bar{\mu}_2))} = \frac{\sigma}{\sigma + (1 - \sigma)\tau_{2,H}},$$

the same as the effective compositions in pooling market 1.

The terms of trade in pooling market 1 are determined by the same condition (A.3) pinning down the terms of trade in the single pooling market in Appendix A.3. In pooling market 2 they are then determined as the unique solution satisfying $\bar{p}_2 > \bar{p}_1$ of the analogous condition:

$$(\bar{\mu}_2, \bar{p}_2) \in (\Pi_{\bar{\gamma}} \cap I_L(u_{\hat{n}-1,L}, u_{\hat{n},L}))\,. \tag{B.18}$$

with $u_{\hat{n},L}$ obtained analogously to $u_{N,L}$ in Appendix A.3. It is easy to see[4] that such a solution exists whenever $\hat{n}$ is sufficiently large. The terms of trade in the high quality markets are determined by the same procedure as in Appendix A.3,[5] starting from the utility attained by high types from their applications to pooling markets 1 and 2

$$u_{N-l,H} = \beta(N-\hat{n},\bar{\mu}_2)(\bar{p}_2 - c_H) + (1-\beta(N-\hat{n},\bar{\mu}_2))\underbrace{\beta(\hat{n}-l;\bar{\mu}_1)(\bar{p}_1 - c_L)}_{=u_{\hat{n}-l,H}},$$

and the wage $p_{2,H}$ lying at the intersection of the low types' indifference curve associated with their last application to pooling market 2 and the H-isoprofit curve.

Having found the effective queue lengths in the H-type markets, the high types' probability of being hired in one of these markets $\tau_{2,H}$ can be determined as a function of $\bar{\gamma}$ in the same way as in (A.7). Proceeding as in Appendix A.3 allows us then to prove that a fixed point for $\bar{\gamma}$ exists. This fixed point depends on the switching point $\hat{n}$, as do the other equilibrium variables (except for the terms of trade in the low quality markets). In what follows we make this dependence explicit by writing the variables as functions of $\hat{n}$.

It will be useful to establish some limit properties of these variables. First, since $u_{\hat{n},L}$ is strictly increasing in $\hat{n}$ and bounded above by the gains from trade $\sigma v_H + (1-\sigma)v_L - k$, the difference $u_{\hat{n},L} - u_{\hat{n}-1,L}$ converges to zero as $\hat{n} \to \infty$. Given this property and $\bar{p}_2(\hat{n}) > \bar{p}_1(\hat{n}) > c_L + u_{\hat{n}-1,L}$, condition (B.18) implies $\lim_{\hat{n}\to\infty} \bar{\mu}_2(\hat{n}) = \infty$. The fact that the effective queue length in the second pooling market tends ∞ implies that also the effective queue lengths in the high-type markets tend to ∞.[6] Noticing that the number of applications that high types send to these markets is l and thus independent $\hat{n}$, it follows that the probability with which high types receive an offer in one of the high-type markets tends to zero as $\hat{n} \to \infty$. Hence, $\lim_{\hat{n}\to\infty} \tau_{2,H}(\hat{n}) = 1$. Due to this property, the effective composition $\bar{\gamma}(\hat{n})$, as determined by (A.8) with $m' = m = \hat{n}$, tends to σ as $\hat{n} \to \infty$.

No profitable deviations. Next, we need to show that there are no profitable deviations. For wages $p < \bar{p}_1(\hat{n})$ and $p > p_{2,H}(\hat{n})$ the proof in Appendix A.3 directly applies. Considering wages $p \in (\bar{p}_1(\hat{n}), p_{2,H}(\hat{n}))$, we want to show that for any p in this interval, $\gamma(p) = 1$ holds

[4] A solution of (B.18) is always given by $\bar{\mu}_1, \bar{p}_1$. Note that the isoprofit curve of pooling market 1 is convex while the indifference curve of the $\hat{n}$-th application of the low types (sent to pooling market 1) is concave. Hence if the latter is steeper than the first one at $\bar{\mu}_1, \bar{p}_1$, a property satisfied for $\hat{n}$ sufficiently high, a second solution exists and features $\bar{p}_2 > \bar{p}_1$.

[5] In particular, see equations (A.6), (A.5) and the text immediately below them.

[6] Recall that the effective queue length increases in the index of the application—in this case the application of high types.

except at $p = \bar{p}_2(\hat{n})$. For wages in the interval $(\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}))$ we can again apply the proof in Appendix A.3, conditions (B.9) and (B.10), simply replacing N with $\hat{n}$. Thereby, we obtain $u_{\hat{n}-1,L}(\hat{n})+c_L \geq u_{\hat{n}-l-1,H}(\hat{n})+c_H$ and $u_{\hat{n}-1,L}(\hat{n})+c_L < u_{\hat{n}-l,H}(\hat{n})+c_H$, thus proving $\gamma(p) = 1$ for all $p \in (\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}))$.

Next, consider the interval $(\bar{p}_2(\hat{n}), p_{2,H}(\hat{n}))$. To show $\gamma(p) = 1$ for wages in this interval, we must prove that analogous inequalities hold: $u_{N-1,L}(\hat{n}) + c_L \geq u_{N-l-1,H}(\hat{n}) + c_H$ and $u_{N-1,L}(\hat{n})+c_L < u_{N-l,H}(\hat{n})+c_H$. We have argued above that $u_{\hat{n},L}(\hat{n})+c_L \geq u_{\hat{n}-l,H}(\hat{n})+c_H$ is satisfied. Using this property, we obtain:

$$
\begin{aligned}
u_{N-1,L}(\hat{n}) + c_L &= \beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n}))\bar{p}_2(\hat{n}) + (1-\beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n})))(u_{\hat{n},L}(\hat{n}) + c_L) \\
&\geq \beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n}))\bar{p}_2(\hat{n}) + (1-\beta(N-1-\hat{n}; \bar{\mu}_2(\hat{n})))(u_{\hat{n}-l,H}(\hat{n}) + c_H) \\
&= u_{N-l-1,H}(\hat{n}) + c_H,
\end{aligned}
$$

which establishes the first inequality. To prove the second inequality, $u_{N-1,L}(\hat{n}) + c_L < u_{N-l,H}(\hat{n}) + c_H$, it is sufficient to notice that $u_{\hat{n},L}(\hat{n}) = \beta(1, \bar{\mu}_1(\hat{n}))(\bar{p}_1(\hat{n}) - c_L - u_{\hat{n}-1,L}(\hat{n})) + u_{\hat{n}-1,L}(\hat{n})$ holds (low types are indifferent between sending their $\hat{n}$-th application to the first or second pooling market). With $\beta(n; \cdot) = \beta(n-1; \cdot)+(1-\beta(n-1; \cdot))\beta(1; \cdot)$, we can follow the same steps as in (B.11-B.12), Appendix A.3, to establish that $u_{N-1,L}(\hat{n})+c_L < u_{N-l,H}(\hat{n})+c_H$ holds. We thus have $\gamma(p) = 1$ for all $p \in (\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}))$.

Given $\gamma(p) = 1$ for $p \in (\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n})) \cup (\bar{p}_2(\hat{n}), p_H(\hat{n}))$, the associated profits for firms are weakly below k as long as $\bar{p}_1(\hat{n}), \bar{p}_2(\hat{n}) \geq v_L - k$ is satisfied (see the argument in Appendix A.3 following (A.9)). Given the assumption $v_H > \hat{v}_H$, we can choose $\hat{n}$ sufficiently large, and hence $\bar{\gamma}(\hat{n})$ sufficiently close to σ, such that $\bar{p}_1(\hat{n}) \geq v_L - k$ holds. By construction, we have $\bar{p}_2(\hat{n}) > \bar{p}_1(\hat{n})$, hence $\bar{p}_2(\hat{n}) \geq v_L - k$ holds as well.

Taken together, we conclude that for N sufficiently large, we can find a threshold $\hat{n}_0$ sufficiently high such that there is an equilibrium with two pooling markets for each switching point $\hat{n} \in \{\hat{n}_0, N-1\}$.

Expected payoffs. We are now ready to prove the statement in the proposition. Fix ε arbitrarily close to zero and let δ_1, δ_2 be a pair of positive numbers such that

$$\delta_1(v_H - v_L) + \frac{\delta_2}{1-\delta_2}k \leq \varepsilon.$$

Since, as shown earlier, $\lim_{\hat{n}\to\infty} \bar{\mu}_2(\hat{n}) = \infty$ and $\lim_{\hat{n}\to\infty} \bar{\gamma}(\hat{n}) = \sigma$, we can find a value for $\hat{n}$ such that $\bar{\gamma}(\hat{n}) < \sigma + \delta_1$ and $1 - e^{-\bar{\mu}_2(\hat{n})} > 1 - \delta_2$. In what follows we fix then the number of applications sent to pooling market 1 to be equal to a value of $\hat{n}$ such that these inequalities are satisfied. As $N \to \infty$, the number of applications sent to the first pooling market is then fixed to $\hat{n}-l$, while the number of applications sent to the second pooling market tends to infinity. We want to show that we can find N large enough so that (13) holds.

Using the inequalities $\bar{\gamma}(\hat{n})<\sigma+\delta_1$ and $1-e^{-\bar{\mu}_2(\hat{n})}>1-\delta_2$ together with the free-entry condition imposed by (B.18) yields:

$$\bar{p}_2(\hat{n})>(\sigma+\delta_1)v_L+(1-(\sigma+\delta_1))v_H-\frac{k}{1-\delta_2}.$$

The level of total surplus attained by workers in equilibrium satisfies the following:

$$\begin{aligned}
& \sigma u_{N,L}(\hat{n})+(1-\sigma)u_{N,H}(\hat{n}) \\
> \; & \sigma u_{N,L}(\hat{n})+(1-\sigma)u_{N-l,H}(\hat{n}) \\
= \; & \sigma\left[\beta(N-\hat{n};\bar{\mu}_2(\hat{n}))(\bar{p}_2(\hat{n})-c_L)+(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))u_{\hat{n},L}(\hat{n})\right] \\
& +(1-\sigma)\left[\beta(N-\hat{n};\bar{\mu}_2(\hat{n}))(\bar{p}_2(\hat{n})-c_H)+(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))u_{\hat{n}-l,H}(\hat{n})\right] \\
= \; & \beta(N-\hat{n};\bar{\mu}_2(\hat{n}))(\bar{p}_2(\hat{n})-\sigma c_L-(1-\sigma)c_H) \\
& +(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))(\sigma u_{\hat{n},L}(\hat{n})+(1-\sigma)u_{\hat{n}-l,H}(\hat{n})) \\
\geq \; & \beta(N-\hat{n};\bar{\mu}_2(\hat{n}))\left(\sigma(v_L-c_L)+(1-\sigma)(v_H-c_H)-k-\delta_1(v_H-v_L)-\frac{\delta_2}{1-\delta_2}k\right) \\
& +(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))(\sigma u_{\hat{n},L}(\hat{n})+(1-\sigma)u_{\hat{n}-l,H}(\hat{n})) \\
\geq \; & \beta(N-\hat{n};\bar{\mu}_2(\hat{n}))\left(\sigma(v_L-c_L)+(1-\sigma)(v_H-c_H)-k-\varepsilon\right) \\
& +(1-\beta(N-\hat{n};\bar{\mu}_2(\hat{n})))(\sigma u_{\hat{n},L}(\hat{n})+(1-\sigma)u_{\hat{n}-l,H}(\hat{n}))
\end{aligned}$$

Since $\hat{n}$ is fixed, $\bar{\mu}_2(\hat{n})$ is bounded and $\beta(N-\hat{n};\bar{\mu}_2(\hat{n}))$ tends to 1 as $N\to\infty$ (workers send infinitely many applications to a market with a finite effective queue length). We thus have

$$\lim_{N\to\infty}(\sigma u_{N,L}(\hat{n})+(1-\sigma)u_{N,H}(\hat{n}))\geq\sigma(v_L-c_L)+(1-\sigma)(v_H-c_H)-k-\varepsilon.$$

B.3.2 The case $c_H\in[v_L-k,\sigma v_L+(1-\sigma)v_H-k)$.

Candidate equilibrium. We consider the construction of a candidate equilibrium with a single pooling market in the proof of Proposition 3, with m as the number of applications low types send to the pooling market and m' as the number of applications high types send to the pooling market. For any $m,m'\geq 1$ there exists a value of the wage $\bar{p}(m,m')$, queue length $\bar{\mu}(m,m')$ and effective fraction of low types $\bar{\gamma}(m,m')$ in the pooling market satisfying (A.3) and (A.8).

Next, we impose the following condition on m,m' : for any m, let m' be determined as follows

$$m' = \arg\max\{\tilde{m} \geq 0 : \beta(m-\tilde{m}; \bar{\mu}(m,\tilde{m}))\bar{p}(m,\tilde{m}) + (1-\beta(m-\tilde{m}, \bar{\mu}(m,\tilde{m})))(u^*_{N-m,L}+c_L) \geq c_H\}. \tag{B.19}$$

A solution to (B.19) always exists provided N, m are sufficiently large so that $\bar{p}(m,m') > c_H$. To see this, note that, for any sequence of values $m, m' \to \infty$, with $m - m'$ bounded (converging to some number greater or equal than 1), we have $\bar{\gamma}(m,m') \to \sigma$. If in addition the switching point $N - m \to \infty$ we have $\bar{\mu}(m,m') \to \infty$ and then also $\bar{p}(m,m') \to \sigma v_L + (1-\sigma)v_H - k$. Hence for $m, m', N - m$ sufficiently large and $\frac{m'}{m}$ sufficiently close to 1 we have

$$\bar{\gamma}(m,m')v_L + (1-\bar{\gamma}(m,m'))v_H - k > c_H \tag{B.20}$$

and also, since $\sigma v_L + (1-\sigma)v_H - k > c_H$, $\bar{p}(m,m') > c_H$.

No profitable deviations. Next, we verify that firms have no incentives to deviate. For wages p in the interval $(p^*_{N-m,L}, \bar{p})$, we can follow steps (B.2-B.6), replacing l with $N - m$, to establish that $\gamma(p) = 1$ and hence no deviation to wages in this range is profitable. The analogous condition to (B.7) is $u^*_{N-m-1,L} + c_L < c_H + u_{n-1,H}$ for all $n = 1, ..., N$, which follows from

$$u^*_{N-m-1,L} + c_L < v_L - k \leq c_H,$$

and holds then for all $N - m$. Hence the switching point to the pooling market $N - m$ can now take an arbitrarily large value.

Consider next wages $p \in (\bar{p}, p_H)$, Since m' satisfies (B.19), we have

$$u_{N-m',L} + c_L = \beta(m-m'; \bar{\mu})\bar{p} + (1-\beta(m-m', \bar{\mu}))(u^*_{N-m,L} + c_L) \geq c_H.$$

Hence, proceding similarly as in (B.8), we obtain:

$$\begin{aligned} u_{N-1,L} + c_L &= \beta(m'-1; \bar{\mu})\bar{p} + (1-\beta(m'-1; \bar{\mu}))(u_{N-m',L} + c_L) \\ &\geq \beta(m'-1; \bar{\mu})\bar{p} + (1-\beta(m'-1; \bar{\mu}))c_H \\ &= u_{m'-1,H} + c_H, \end{aligned}$$

the analogue of condition (B.9) in our candidate equilibrium, saying that the reservation utility for the last application sent to the pooling market is greater for the low than for the high types.

The analogue of (B.10) in our candidate equilibrium is $u_{N-1,L} + c_L < u_{m',H} + c_H$, requiring that the reservation utility for the last application sent by low types to the pooling market is smaller than the one for the first application sent by high types to a high quality market. Since m' is the largest value of $\tilde{m}$ satisfying the inequality in (B.19), we have

$$u_{N-m'-1,L} + c_L = \beta(m-(m'+1);\bar{\mu})\bar{p} + (1-\beta(m-(m'+1),\bar{\mu}))(u^*_{N-m,L}+c_L) < c_H. \quad \text{(B.21)}$$

We proceed then similarly as in (B.12) to obtain:

$$\begin{aligned} u_{N-1,L} + c_L &= \beta(m'-1;\bar{\mu})\bar{p} + (1-\beta(m'-1;\bar{\mu}))(u_{N-m',L}+c_L) \\ &= \beta(m';\bar{\mu})\bar{p} + (1-\beta(m';\bar{\mu}))(u_{N-m'-1,L}+c_L) \\ &< \beta(m';\bar{\mu})\bar{p} + (1-\beta(m';\bar{\mu}))c_H \\ &= u_{m',H} + c_H, \end{aligned}$$

where the inequality sign follows from (B.21). This establishes the analogue of (B.10) we intended to show.

Having shown these properties, we can follow the steps of the proof of Proposition 3, conditions (B.13-B.17), to show that $\gamma(p) = 1$ for all $p \in (\bar{p}, p_H)$. To show that no deviation to a wage in this interval is profitable it remains then to show that $\eta(\mu(p))(v_L - p) \leq k$ holds for $\mu(p)$ satisfying $(\mu(p), p) \in I_L(u_{N-1,L}, u_{N,L})$. This is true since $\bar{p} \geq v_L - k$, always holds here, as $\bar{p} > c_H$ and $c_H \geq v_L - k$.

The non profitability of deviations to wages $p < p^*_{N-m,L}$ and $p > p_H$ follows then directly by the same argument as in the proof of Proposition 3.

Expected payoffs. In the next and final step, we use a similar argument as for the case $c_H < v_L - k$, taking the switching point for low types to the pooling market large enough. Fix ε arbitrarily close to zero and let δ be a positive number such that

$$\frac{\delta}{1-\delta} k < \varepsilon. \quad \text{(B.22)}$$

Recalling that $\mu^*_{n-1,L} \to \infty$ as $n \to \infty$, let the low types' switching point to the pooling market $N - m$ be the smallest number κ satisfying $1 - e^{-\mu^*_{\kappa,L}} \geq 1 - \delta$. For δ small, this condition implies $\bar{p} \geq c_H$ as long as N is sufficiently large. Having set $N - m = \kappa$, we can write all equilibrium variables as a function of N. For any N, the number of applications low types send to the pooling market is $m = N - \kappa$ and the number of applications high types send to the pooling market, $m'(N - \kappa)$, is determined by (B.19).

We consider then $N \to \infty$. Since $(\bar{\mu}(N-\kappa), \bar{p}(N-\kappa))$ lies on the indifference curve associated with the κ-th application of the low types, as $N \to \infty$ both $\bar{\mu}(N-\kappa)$ and $\bar{p}(N-\kappa)$ tend to a finite limit. This implies that also $m - m' = N - \kappa - m'(N-\kappa)$ has a finite limit as $N \to \infty$.[7] Hence $\lim_{N\to\infty} m'(N-\kappa) = \infty$ and $\lim_{N\to\infty} \frac{m'(N-\kappa)}{N-\kappa} = 1$. Also $\lim_{N\to\infty} \bar{\gamma}(N-\kappa) = \sigma$.

Using the above properties, we want to show that we can find N large enough so that (13) holds. Since L-type workers send their $\kappa+1$-th application to the pooling market which features a higher effective queue length than their κ-th application, sent to a low market, we have

$$\eta(\bar{\mu}(m)) > \eta(\mu_{\kappa,L}) \geq 1-\delta.$$

Together with the free-entry condition $\eta(\bar{\mu}(m))(\bar{\gamma}(m)v_L + (1-\bar{\gamma}(m))v_H - \bar{p}(m)) = k$, this implies:

$$\lim_{N\to\infty} \bar{p}(N-\kappa) = \sigma v_L + (1-\sigma)v_H - \lim_{N\to\infty} \frac{k}{\eta(\bar{\mu}(N-\kappa))} \geq \sigma v_L + (1-\sigma)v_H - \frac{k}{1-\delta}.$$

Taking then the limit of the expression of total surplus in equilibrium, as $N \to \infty$, we obtain:

$$\begin{aligned}
& \lim_{N\to\infty} \left(\sigma u_{N,L}(N) + (1-\sigma)u_{N,H}(N)\right) \\
\geq & \lim_{N\to\infty} \left(\sigma u_{N,L}(N) + (1-\sigma)u_{N-\kappa,H}(N-\kappa)\right) \\
= & \lim_{N\to\infty} \left(\begin{array}{c} \sigma \left[\beta(N-\kappa);\bar{\mu}(N-\kappa))(\bar{p}(N-\kappa)-c_L) + (1-\beta(N-\kappa);\bar{\mu}(N-\kappa)))u^*_{\kappa,L}(N-\kappa)\right] \\ +(1-\sigma)\beta(N-\kappa;\bar{\mu}(N-\kappa))(\bar{p}(N-\kappa)-c_H) \end{array} \right) \\
= & \lim_{N\to\infty} \bar{p}(N-\kappa) - \sigma c_L - (1-\sigma)c_H \\
\geq & \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \frac{\delta}{1-\delta}k \\
> & \sigma(v_L - c_L) + (1-\sigma)(v_H - c_H) - k - \varepsilon
\end{aligned}$$

where we used $\lim_{N\to\infty} \beta(N-\kappa);\bar{\mu}(N-\kappa)) = 1$ and, in the last inequality, condition (B.22). This proves that (13) is satisfied.

□

B.4 Proof of Proposition 7

Because both the market and the planner need to respect the free entry condition, firms' payoffs are zero. Hence, ex-ante welfare equals $W = \sigma u_{2,L} + (1-\sigma)\,u_{2,H}$, where low- and high-type workers' payoffs are equal to $u_{2,L}$ and $u_{2,H}$, respectively. We focus on the limit $c_H \to c_L$, but the results extend to the case where c_H and c_L are different but close enough by continuity. In the limit, both types of workers have identical preferences and must therefore obtain identical payoffs. That is, $u_{2,H} \to u_{2,L}$ and thus $W \to u_{2,L}$.

[7] Take m, N large enough so that a solution to (B.19) exists. Let then $N \to \infty$ so that also $m = N-\kappa \to \infty$. The solution for m' obtained from (B.19) is such that $m - m'$ is either unchanged or decreases.

Equilibrium. For c_H sufficiently close to c_L, we are in the case of mild averse selection with $l = 1$. Hence, the partial-pooling equilibrium features 3 markets, (μ_L, p_L), $(\bar{\mu}, \bar{p})$ and (μ_H, p_H), with low types applying to the first two and high types applying to the last two. The market (μ_L, p_L) coincides with the first best and maximizes $\psi(\mu)(p - c_L)$ subject to the free entry condition $\eta(\mu)(v_L - p) - k = 0$. Substitution of the free entry condition into the objective yields

$$u_{1,L}^{\text{eqm}} = u_{1,L}^{*} \equiv \max_{\mu} \psi(\mu)(v_L - c_L) - \frac{k}{\mu}. \tag{B.23}$$

Let $\bar{\gamma} > \sigma$ be the equilibrium composition of the pooling market $(\bar{\mu}, \bar{p})$. This market then lies on the iso-profit curve determined by

$$\eta(\bar{\mu})(\bar{\gamma} v_L + (1 - \bar{\gamma}) v_H - \bar{p}) - k = 0. \tag{B.24}$$

In the limit $c_H \to c_L$, welfare is equal to the payoff of a low-type worker sending two applications, which is

$$\begin{aligned} u_{2,L}^{\text{eqm}} &\equiv \psi(\bar{\mu})\left(\bar{p} - c_L - u_{1,L}^{*}\right) + u_{1,L}^{\text{eqm}} \\ &= \psi(\overline{\mu})\left(\overline{\gamma} v_L + (1 - \overline{\gamma}) v_H - c_L - u_{1,L}^{\text{eqm}}\right) + u_{1,L}^{\text{eqm}} - \frac{k}{\overline{\mu}}, \end{aligned}$$

where the second line follows from substituting (B.24)

Planner with One Market. Consider a planner who implements full pooling by opening a single market (μ, p) to which both types of workers send both their applications. With a single market, both types of workers get job offers with identical probability, which means that the composition of the market must equal the population composition. The free entry condition is therefore given by

$$\eta(\mu)(\mathbb{E}[v] - p) - k = 0, \tag{B.25}$$

where $\mathbb{E}[v] \equiv \sigma v_L + (1 - \sigma) v_H$.

The planner chooses (μ, p) to maximize welfare, subject to this free entry condition. In the limit $c_H \to c_L$, welfare equals the payoff of the low-type worker, which after substitution of (B.25) is equal to

$$u_{2,L}^{\text{pl},1} = \max_{\mu} \psi(\mu)(\mathbb{E}[v] - c_L) - \frac{k}{\mu} + (1 - \psi(\mu))\left[\psi(\mathbb{E}[v] - c_L) - \frac{k}{\mu}\right].$$

Planner with Two Markets. Consider now a planner who creates two markets, (μ_1, p_1) and (μ_2, p_2), in such a way that both types of workers send application i to (μ_i, p_i). Both markets must lie on the same iso-profit curve, which again is defined by (B.25). In the limit $c_H \to c_L$, welfare again equals the payoff of the low-type worker, which after substitution of (B.25) is equal to

$$u_{2,L}^{\text{pl},2} \equiv \max_{\mu_1,\mu_2} \psi(\mu_2)(\mathbb{E}[v] - c_L) - \frac{k}{\mu_2} + (1 - \psi(\mu_2)) \left[\psi(\mu_1)(\mathbb{E}[v] - c_L) - \frac{k}{\mu_1} \right].$$

It is straightforward to see that the planner's choice of μ_1 differs from μ_2, which means that the planner who pools types but spreads applications across two markets creates more welfare than the planner who implements full pooling of all types and applications.[8]

For the comparison with equilibrium welfare, we proceed in two steps. First, we compare the low type's payoff from their first application. Since $\mathbb{E}[v] > v_L$, we have

$$\begin{aligned} u_{1,L}^{\text{pl},2} &\equiv \max_{\mu_1} \psi(\mu_1)(\mathbb{E}[v] - c_L) - \frac{k}{\mu_1} \\ &> \max_{\mu} \psi(\mu)(v_L - c_L) - \frac{k}{\mu} \equiv u_{1,L}^{\text{eqm}}. \end{aligned}$$

That is, the low-type worker obtains a higher payoff with their first application under the planner with two markets than in equilibrium.

We then proceed by comparing the payoff from the portfolio with two applications. Note that

$$\begin{aligned} u_{2,L}^{\text{pl},2} &\equiv \max_{\mu_2} \psi(\mu_2)\left(\sigma v_L + (1-\sigma) v_H - c_L - u_{1,L}^{\text{pl},2}\right) - \frac{k}{\mu_2} + u_{1,L}^{\text{pl},2} \\ &> \max_{\mu_2} \psi(\mu_2)\left(\overline{\gamma} v_L + (1-\overline{\gamma}) v_H - c_L - u_{1,L}^{\text{pl},2}\right) - \frac{k}{\mu_2} + u_{1,L}^{\text{pl},2} \\ &> \max_{\mu_2} \psi(\mu_2)\left(\overline{\gamma} v_L + (1-\overline{\gamma}) v_H - c_L - u_{1,L}^{\text{eqm}}\right) - \frac{k}{\mu_2} + u_{1,L}^{\text{eqm}} \\ &\geq \psi(\overline{\mu})\left(\overline{\gamma} v_L + (1-\overline{\gamma}) v_H - c_L - u_{1,L}^{\text{eqm}}\right) - \frac{k}{\overline{\mu}} + u_{1,L}^{\text{eqm}} \\ &\equiv u_{2,L}^{\text{eqm}}, \end{aligned}$$

where the inequalities follow because $\overline{\gamma} > \sigma$, $u_{1,L}^{\text{pl},2} > u_{1,L}^{\text{eqm}}$, and the maximization. Hence, the planner with two markets creates more welfare than the market equilibrium.

[8] Note that this optimization problem coincides with the one in Kircher (2009) if we assign the workers in his model a productivity $\mathbb{E}[v]$ and outside option c_L.

Appendix C Equilibrium Definition

Let $\mathbf{p} = (p_1, \ldots p_N)$ be an application portfolio and let $\mathbf{p}_{-n} = (p_1, \ldots p_{n-1}, p_{n+1}, \ldots p_N)$ be the portfolio excluding application p_n. We then define $G_i(\mathbf{p})$ as the application distribution of a worker of type i, with corresponding marginals $G_{n,i}(p)$. Similarly, $\overline{G}_{-n,i}(\mathbf{p}_{-n}; p_n)$ denotes the distribution of $\mathbf{p}_{-n}$, conditional on the worker sending application n to p_n. We can then define equilibrium as follows.

Definition 2. *An equilibrium is a measure of vacancies ϕ, a distribution of wages F, application distributions (G_L, G_H), effective queue lengths $\mu(p)$, and effective queue compositions $\gamma(p)$ such that*

1. *For any $n \in \{1, \ldots, N\}$ and $p \in \mathcal{F}$, $\lambda_{n,L}(p)$ satisfies*

$$\phi \int_0^p \lambda_{n,L}(p')\, dF(p') = \sigma G_{n,L}(p)$$

   and $\lambda_{n,H}(p)$ satisfies

$$\phi \int_0^p \lambda_{n,H}(p')\, dF(p') = (1-\sigma) G_{n,H}(p).$$

2. *For any $i \in \{L, H\}$, $n \in \{1, \ldots, N\}$, and $p \in \mathcal{F}$, $\mu_{n,i}(p)$ satisfies*

$$\mu_{n,i}(p) = \lambda_{n,i}(p) \int_{\mathcal{F}^{n-1}} \prod_{j=n+1}^{N} \left(1 - \frac{1-e^{-\mu(p_j)}}{\mu(p_j)}\right) d\overline{G}_{-n,i}(\mathbf{p}_{-n}; p).$$

3. *For any $p \in \mathcal{F}$, $\mu(p)$ satisfies*

$$\mu(p) = \sum_{n=1}^{N} \sum_{i=L,H} \mu_{n,i}(p).$$

4. *For any $p \in \mathcal{F}$, $\gamma(p)$ must satisfy*

$$\gamma(p) = \frac{\sum_{n=1}^{N} \mu_{n,L}(p)}{\mu(p)}.$$

5. *For any* $i \in \{L, H\}$ *and* $n \in \{1, \ldots, N\}$*, every* $p \in$ *supp* $G_{n,i}$ *solves*

$$u_{n,i} = \frac{1 - e^{-\mu(p)}}{\mu\left(p\right)} \left(p - c_i - u_{n-1,i}\right) + u_{n-1,i}.$$

6. *For any* $p \notin \mathcal{F}$*,* $\mu\left(p\right)$ *solves*

$$u_{n,i} \geq \frac{1 - e^{-\mu(p)}}{\mu\left(p\right)} \left(p - c_i - u_{n-1,i}\right) + u_{n-1,i} \tag{C.1}$$

with weak inequality for any (n, i)*, and with equality for at least one* (n, i) *if* $\mu(p) > 0$*.*

7. *For any* $p \notin \mathcal{F}$*,* $\gamma\left(p\right)$ *satisfies*

$$\begin{cases} \gamma\left(p\right)\mu\left(p\right) = 0 & \textit{if (C.1) holds with strict inequality for } i = L \textit{ and all } n \\ \left(1 - \gamma\left(p\right)\right)\mu\left(p\right) = 0 & \textit{if (C.1) holds with strict inequality for } i = H \textit{ and all } n \end{cases}$$

8. *Any* $p \in \mathcal{F}$ *solves*

$$\left(1 - e^{-\mu(p)}\right)\left[\gamma\left(p\right)v_L + \gamma\left(p\right)v_H - p\right] = \pi^* \equiv \max_{p'} \left(1 - e^{-\mu(p')}\right)\left[\gamma\left(p'\right)v_L + \gamma\left(p'\right)v_H - p'\right].$$

9. $\phi \geq 0$ *and* $\pi^* \leq k$*, with complementary slackness.*

Appendix D Endogenizing Applications

Suppose workers can choose how many applications to send facing a fixed, equal cost z per application. Let N_i denote the total number of applications a worker of type $i = L, H$ chooses to send in equilibrium. Given N_H, N_L, the definition of an equilibrium is analogous to the one in Definition 1. In addition, to assess the optimality of N_i, recall that for all $n \in \mathbb{N}$, the benefit for a worker of type i from sending one additional application to an optimally chosen market, after having sent $n-1$ of them, is equal to

$$u_{n,i} - u_{n-1,i} = \max_{p \in \mathcal{F}} \psi\left(\mu(p)\right)\left(p - c_i - u_{n-1,i}\right).$$

For N_i to be optimal, we need that for all $n \leq N_i$, the benefit $u_{n,i} - u_{n-1,i}$ exceeds the application cost z, while it is lower than z for all $n > N_i$. The fact that $u_{n-1,i}$ is increasing in n directly implies that the utility gain $u_{n,i} - u_{n-1,i}$ is decreasing in n. Hence, the total number of applications a worker of type i sends in equilibrium, N_i, is uniquely pinned down by the following condition:

$$N_i = \max\{n \in \mathbb{N} : u_{n,i} - u_{n-1,i} \geq z\}$$

To examine the consequences for the properties of equilibrium allocations, assume first that the lemons condition holds, $c_H \geq v_L - k$, and consider the separating equilibrium characterized in Proposition 1. As we saw, the markets for L-type workers coincide with the unconstrained solution described in Section 3.2. Hence, the total number of applications low types send is given by the largest number N_L that satisfies $u^*_{N_L,L} - u^*_{N_L-1,L} \geq z$. This condition ensures that L-type workers do not wish to send an additional application to a separate market (for which the utility gain is $u^*_{N_L+1,L} - u^*_{N_L,L} < z$). We also need that they have no incentives to send an additional application to the lowest wage to which high types apply.[9] Letting (μ_H, p_H) describe this market, we must have

$$\psi(\mu_H)(p_H - c_L - u_{N_L,L}) \leq z \tag{D.1}$$

for $u_{N_L,L} = u^*_{N_L,L}$. Since $c_H \geq v_L - k > u^*_{N_L,L} + c_L$, inequality (D.1) implies $\psi(\mu_H)(p_H - c_H) < z$. Hence, in equilibrium, incentive constraints limit the gains high types can achieve by trading in the market so much that they will prefer not to participate at all. Hence, with endogenous applications, a separating equilibrium exists under the conditions of Proposition

[9] This condition is different than the L-type incentive constraint relative to his last application, $u^*_{N_L,L} - u^*_{N_L-1,L} \geq \psi\left(\mu_H\right)\left(p_H - c_L - u^*_{N_L-1,L}\right)$, which only guarantees that the L-type has no incentives to divert his last application to wage p_H.

1 and features N_L application of low types and 0 applications of high types.

Turning then to the case $c_H < v_L - k$, consider the equilibrium with one pooling market described in Proposition 3. As we show in the proof of this proposition, in the equilibrium allocation we constructed, market utilities satisfy the condition

$$u_{\ell+n-1,L} + c_L < u_{n,H} + c_H \leq u_{\ell+n,L} + c_L \text{ for all } n \geq 0. \tag{D.2}$$

Letting $(\bar{\mu}, \bar{p})$ denote again the terms of trade in the pooling market, the total number N_L of applications that the low type sends must then be the largest number satisfying

$$\psi(\bar{\mu})(\bar{p} - c_L - u_{N_L-1,L}) \geq z. \tag{D.3}$$

Using (D.2), this implies $\psi(\bar{\mu})(\bar{p} - c_H - u_{N_L-\ell-1,H}) \geq z$, i.e., when low types are willing to send $N_L - \ell$ applications to the pooling market together with ℓ applications to separate markets, high types are also happy to send $N_L - \ell$ applications to the pooling market.

Suppose now that an H-type market exists with terms of trade (μ_H, p_H). For the considered allocation to be an equilibrium with endogenous applications, it must be that low types do not want to send any additional application to this market (inequality (D.1) is satisfied) nor to redirect any of their N_L applications to that market (ensured by the incentive constraints already imposed in the construction used in the proof of Proposition 3). By the second inequality in (D.2) we have $u_{N_L-\ell,H} + c_H \leq u_{N_L,L} + c_L$, so it is possible that $\psi(\mu_H)(p_H - u_{N_L-\ell,H} - c_H) \geq z$ and (D.1) are both satisfied. If that is the case, high types find it profitable to send one application to market (μ_H, p_H) while low types do not. However, due to the first inequality in (D.2), we also have $u_{N_L-\ell+1,H} + c_H > u_{N_L,L} + c_L$, so that sending a second application to market (μ_H, p_H) is never profitable. Hence, under the conditions stated in Proposition 3, there exists an equilibrium with a pooling market whenever z satisfies (D.3) for some $N_L > \ell$, and high types send at most one application to a separate market.