Meetings and Mechanisms

Online Appendix

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B Social Planner

B.1 Boundary.

Submarket with Only Low-Type Buyers. Consider a submarket that has a queue λ with only low-type buyers. Then, the marginal contribution of these buyers is $T_1(0, \lambda) = m'(\lambda)$, while sellers' marginal contribution is $R(0, \lambda) = m(\lambda) - \lambda m'(\lambda)$. For future reference, we can therefore define a function g which maps the marginal contribution to surplus of sellers to that of low-type buyers. That is, for any $\lambda > 0$, we have

$$T_1(0, \lambda) = g(R(0, \lambda)). \tag{B.1}$$

Alternatively, we can define g explicitly as

$$g(R) = \begin{cases} m'\left((m - \lambda m')^{-1}(R)\right) & \text{for } R \in [0, 1) \\ 0 & \text{for } R \geq 1, \end{cases}$$

where $(m - \lambda m')^{-1}$ is the inverse function of $m - \lambda m'$. Since $\frac{d}{d\lambda}R(0, \lambda) = -\lambda m''(\lambda)$ and $\frac{d}{d\lambda}T_1(0, \lambda) = -m''(\lambda)$, we have

$$g'(R) = -\frac{1}{\lambda} \quad \text{if} \quad R = m(\lambda) - \lambda m'(\lambda). \tag{B.2}$$

When $R \geq 1$, we have $g'(R) = 0$. For the stochastic capacity meeting technology with $m(\lambda) = \lambda/(1 + \lambda)$, one can verify that $g(R) = (1 - \sqrt{R})^2$ when $R \in [0, 1)$ and $g(R) = 0$ for $R \geq 1$.

Proposition 2 tells us that pooling is optimal if $\overline{S}'(B_1) \geq 0$, i.e. the marginal contribution of low-type buyers is greater in the segment with the high-type buyers than in a separate segment with an ε amount of low-type buyers and where sellers are optimally allocated. In the following,

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we are especially interested in the cutting-edge case where $\overline{S}'(B_1)$ in equation (14) is exactly zero. That is, $(B_2, B_1 + B_2)$ is a solution to the following equation

$$T_1(\mu, \lambda) = g(R(\mu, \lambda)). \tag{B.3}$$

In this case, the planner's solution is pooling, but $(B_2, B_1 + B_2)$ lies on the boundary of the pooling area.

Among the solutions to equation (B.3), we distinguish between two cases: i) $R(\mu, \lambda) < 1$, and ii) $R(\mu, \lambda) \geq 1$, which implies $T_1(\mu, \lambda) = g(R(\mu, \lambda)) = 0$. In the first case, by the definition of g, there exists a λ_0 such that $R(\mu, \lambda) = R(0, \lambda_0)$ and $T_1(\mu, \lambda) = T_1(0, \lambda_0)$, which, by equations (4) and (6), implies that

$$\frac{m(\lambda_0) - m(\lambda) - (\lambda_0 - \lambda)m'(\lambda_0)}{m'(\lambda) - m'(\lambda_0)} = \frac{\phi(\mu, \lambda) - \mu\phi_\mu(\mu, \lambda)}{-\phi_\lambda(\mu, \lambda)} \tag{B.4}$$

By taking the derivative with respect to λ_0, one can see that the right-hand side above is strictly increasing in λ with its infimum being 0 as $\lambda_0 \to \lambda$ and its supremum being $(1 - m(\lambda))/m'(\lambda)$, the Mills ratio of function $m(\lambda)$, as $\lambda_0 \to \infty$. For future use, we introduce a new function to denote the solution of λ_0 to the above equation. Specifically, we define

$$\Lambda(\mu, \lambda) = \begin{cases} \lambda_0 & \text{the solution in equation (B.4),} \quad \text{if} \quad \frac{\phi(\mu,\lambda) - \mu\phi_\mu(\mu,\lambda)}{-\phi_\lambda(\mu,\lambda)} \in \left(0, \frac{1-m(\lambda)}{m'(\lambda)}\right) \\ \infty, & \text{if} \quad \frac{\phi(\mu,\lambda) - \mu\phi_\mu(\mu,\lambda)}{-\phi_\lambda(\mu,\lambda)} \geq \frac{1-m(\lambda)}{m'(\lambda)} \end{cases} \tag{B.5}$$

Note that $\Lambda(\mu, \lambda)$ is not well-defined at points (μ, λ) where $\phi_\lambda(\mu, \lambda) > 0$, i.e. the meeting externalities are positive. To simplify exposition and to focus on the more realistic case where buyers crowd each other out, we will therefore sometimes impose the following assumption.

Assumption 4. $\phi_\lambda(\mu, \lambda) < 0$ *for* $0 < \mu \leq \lambda$.

Note that the inequality in this assumption is strict, which means that it is satisfied by the stochastic capacity technology as long as $\sigma < 1$. Note that for any meeting technology, $\phi(0, \lambda) = 0$, which implies that $\phi_\lambda(0, \lambda) = 0$ for any λ. Hence in the above assumption we require $\mu > 0$. □

B.2 Special Cases

B.2.1 Stochastic Capacity Technology

Before solving the planner's problem, we first calculate $\phi(\mu, \lambda)$, the result of which is given by the following lemma.

Lemma 8. *For the stochastic capacity technology, we have*

$$\phi(\mu, \lambda) = \frac{\mu}{1 + \sigma\mu + (1 - \sigma)\lambda}. \tag{B.6}$$

Proof. The seller's meeting capacity n_C follows a geometric distribution with support $\mathbb{N}_1$ and mean $(1 - \sigma)^{-1}$. That is, $\mathbb{P}(n_C = n) = (1 - \sigma)\sigma^{n-1}$ for $n = 1, 2, \ldots$. Meanwhile, the number of buyers who visit the seller, n_A, also follows a geometric distribution but with support $\mathbb{N}_0$ and mean λ, i.e. $\mathbb{P}(n_A = n) = \frac{1}{1+\lambda}(\frac{\lambda}{1+\lambda})^n$ for $n = 0, 1, 2, \ldots$. The actual number of buyers that the seller meets, n, is then $\min\{n_C, n_A\} \in \mathbb{N}_0$. Hence $P_n(\lambda) \equiv \mathbb{P}[\min\{n_C, n_A\} = n \,|\, \lambda]$. Since the capacity constraint n_C is at least one, $P_0(\lambda) = \frac{1}{1+\lambda}$. For $n \geq 1$, we have

$$P_n(\lambda) = (1-\sigma)\sigma^{n-1} \sum_{j=n}^{\infty} \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda}\right)^j + \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda}\right)^n \sum_{j=n+1}^{\infty} (1-\sigma)\sigma^{j-1}.$$

The first term on the right-hand side denotes the case where the number of buyers is (weakly) larger than n while the meeting capacity equals n. The second term denotes the case where the number of buyers equals n while the meeting capacity is strictly larger than n. Simplifying the summations yields

$$\text{(B.7)} \qquad P_n(\lambda) = \begin{cases} \frac{1}{1+\lambda} & \text{for } n = 0, \\ \sigma^{n-1} \frac{1}{1+\lambda} \left(\frac{\lambda}{1+\lambda}\right)^n (1 + (1-\sigma)\lambda) & \text{for } n \in \mathbb{N}_1. \end{cases}$$

Substituting (B.7) into equation (1) and simplifying the result yields equation (B.6). □

We now solve the planner's problem analytically. We show that the outcome depends on the extent to which sellers can meet multiple buyers, as determined by the value of σ. We distinguish between three regions by specifying two cutoff values for σ, i.e. $\sigma_0(x_2)$ and $\sigma_1(x_2)$, defined as

$$\text{(B.8)} \qquad \sigma_0(x_2) \equiv \frac{\sqrt{x_2} - 1}{\sqrt{x_2} + 1} < \frac{\sqrt{x_2}}{\sqrt{x_2} + 1} \equiv \sigma_1(x_2)$$

Low Sigma. We first consider the case in which $\sigma \leq \sigma_0(x_2)$. Using the functional form for $\phi(\mu, \lambda)$ given in (B.6), a straightforward calculation yields

$$\text{(B.9)} \qquad H(\mu, \lambda) = \frac{(1-\sigma)^2}{4\sigma} \frac{(1+\lambda)^3}{(1 + (1-\sigma)\lambda)(1 + \sigma\mu + (1-\sigma)\lambda)} > \frac{(1-\sigma)^2}{4\sigma} \geq \frac{1}{x_2 - 1},$$

where the first inequality follows because the second factor in $H(\mu, \lambda)$ is strictly larger than 1, and the second inequality is implied by $\sigma \leq \sigma_0(x_2)$. Consequently, the second-order condition (8) can never be satisfied in this case, i.e. a submarket (μ, λ) where $0 < \mu < \lambda$ cannot be part of the planner's solution. Instead, perfect separation is obtained: one submarket contains all high-type buyers and another submarket contains all low-type buyers.

The allocation of sellers depends on their marginal contribution to surplus, which equals

$$\text{(B.10)} \qquad R(\mu, \lambda) = \frac{(x_2 - 1)\mu(\sigma\mu + (1-\sigma)\lambda)}{(1 + \sigma\mu + (1-\sigma)\lambda)^2} + \frac{\lambda^2}{(1+\lambda)^2}.$$

If $R(B_2, B_2) \geq 1$, then the planner will allocate all sellers to the submarket with high-type buyers; otherwise both submarkets will be active.

Intermediate Sigma. We now consider the case $\sigma \in (\sigma_0(x_2), \sigma_1(x_2)]$, which is illustrated in Figure 3a. The key object for determining the planner's solution is the marginal contribution to surplus of low-type buyers, i.e.

$$T_1(\mu, \lambda) = \frac{1}{(1+\lambda)^2} - \frac{(x_2-1)(1-\sigma)\mu}{(1+\sigma\mu+(1-\sigma)\lambda)^2}. \tag{B.11}$$

First, we are interested in combinations of μ and λ for which $T_1(\mu, \lambda) = 0$, as this is the minimum requirement for a submarket with high-type buyers to also contains low-type buyers. Straightforward algebra shows that, for any $(x_2-1)^{-1}(1-\sigma) \leq \mu \leq (x_2-1)^{-1}(1-\sigma)^{-1}$, there exists a unique λ such that $T_1(\mu, \lambda) = 0$. The locus of these points is represented by the green curves in Figure 3a and 3b. Low-type buyers' contribution to surplus is negative in points above this curve, which therefore cannot be part of the planner's solution.

Next, we are interested in the combinations of μ and λ for which $T_1(\mu, \lambda) = g(R(\mu, \lambda))$, as this is where the planner is indifferent between keeping low-type buyers in the submarket and sending them (with an optimal number of sellers) to a separate submarket. Using (B.1), (B.9) and (B.11) we can solve for μ as a function of λ, i.e.

$$\mu = \frac{\sqrt{(x_2-1)(1+\lambda)}}{\sqrt{(x_2-1)(1+\lambda)} - 2\sqrt{\sigma}} - \frac{1+(1-\sigma)\lambda}{\sigma}. \tag{B.12}$$

This solution is represented by the solid blue curve AB in Figure 3a; sending some low-type buyers and sellers to form a new submarket is beneficial right but not left of this curve. The end points of the curve, i.e. point $A = (\lambda_A, 0)$ on the horizontal axis and point $B = (\lambda_B, \lambda_B)$ on the diagonal, can be obtained by solving (B.12) for $\mu = 0$ and $\mu = \lambda$, respectively. The latter yields $\lambda_B = \sigma\frac{(1+\sqrt{x_2})}{\sqrt{x_2}-1} - 1$.

For every point on the segment AB, there exists—by the definition of g—a corresponding point $(\lambda_0, 0)$ on the horizontal axis with the same marginal contributions of sellers and low-type buyers, i.e. $R(\mu, \lambda) = R(0, \lambda_0)$ and $T_1(\mu, \lambda) = T_1(0, \lambda_0)$ where λ_0 is given explicitly by the function $\Lambda(\mu, \lambda)$ defined in (B.5) in Appendix B.1. As we move from A to B, this corresponding point moves from A to $C = (\lambda_C, 0)$, where $\lambda_C = \Lambda(\lambda_B, \lambda_B)$. The thresholds $\sigma_0(x_2)$ and $\sigma_1(x_2)$ are determined by the location of point C: as $\sigma \searrow \sigma_0(x_2)$, point C approaches $(0, 0)$; in contrast, when $\sigma \nearrow \sigma_1(x_2)$, then $\lambda_C \nearrow \infty$.

We have thus fully characterized the case where the planner's solution consists of two active submarkets and one of two contains both types of buyers, which is the brown area in Figure 3a. After establishing the boundaries for this case, the other two cases (complete pooling and compete separation) are also determined accordingly.

High Sigma. Finally, we consider the case $\sigma > \sigma_1(x_2)$, which is illustrated in Figure 3b. The analysis is quite similar to the case with intermediate sigma. One key difference, however, is that point B no longer lies on the diagonal but on the green curve where $T_1(\mu, \lambda) = 0$. As a result, another point plays an important role: the intersection between the green curve $T_1(\mu, \lambda) = 0$ and the diagonal. This intersection is point D in Figure 3b. Note that at any point on the segment BD of the green curve, we have $T_1(\mu, \lambda) = 0$ and $R(\mu, \lambda) \geq 1$. After fully characterizing the brown area (two active submarkets where one of two contains both types of buyers) and yellow area (two submarkets where one contains both types of buyers and the other contains only low-type buyers and no sellers or high-type buyers) in Figure 3b, the other two areas (complete pooling and compete separation) are determined accordingly. See the main text for a detailed description.

B.2.2 Fixed Capacity

As an alternative to the stochastic capacity technology, we consider the following fixed capacity meeting technology. The number of buyers that try to reach a particular seller follows a geometric distribution but the maximum number of buyers a seller can meet is 2. Thus, $P_n(\lambda) = \left(\frac{1}{1+\lambda}\right)\left(\frac{\lambda}{1+\lambda}\right)^n$ for $n < 2$, and $P_2(\lambda) = 1 - P_0(\lambda) - P_1(\lambda)$. In this case, the function ϕ is given by

$$\phi(\mu, \lambda) = \frac{\mu(1 + 2\lambda - \mu)}{(1+\lambda)^2} \tag{B.13}$$

By equations (6) and (4), we have

$$R(\mu, \lambda) = \frac{2(x_2 - 1)\lambda^2\mu - (x_2 - 1)(\lambda - 1)\mu^2 + (\lambda + 1)\lambda^2}{(\lambda + 1)^3} \tag{B.14}$$

$$T_1(\mu, \lambda) = \frac{1}{(1+\lambda)^2} - (x_2 - 1)\frac{2\mu(\lambda - \mu)}{(1+\lambda)^3} \tag{B.15}$$

To solve the planner's problem, we first calculate the function $H(\mu, \lambda)$ defined by equation (8), which is given by

$$H(\mu, \lambda) = \frac{(\lambda - \mu)^2 - \mu}{1 + \lambda}$$

This is a quadratic equation in μ. Note that $H(0, \lambda) > 0$ and $H(\lambda, \lambda) < 0$, which implies that the equation $H(\mu, \lambda) = 1/(x_2 - 1)$ has a unique root of μ, which is given by,

$$\mu = h(\lambda) \equiv \frac{1 + 2\lambda}{2} - \frac{1}{2}\sqrt{1 + 4\lambda + 4\frac{1+\lambda}{(x_2 - 1)}} \quad \text{with} \quad \lambda \geq \lambda_A \tag{B.16}$$

where λ_A is the root of $h(\lambda) = 0$ and is given by

$$\lambda_A = \frac{1 + \sqrt{1 + 4(x_2 - 1)}}{2(x_2 - 1)}. \tag{B.17}$$

The curve $h(\lambda)$ is represented by the red curve in Figure 1 where we set $(x_2 - 1) = 1$. The Hessian matrix is negative definitive in the area to left of the red curve; the surplus function $S(\mu, \lambda)$ is locally concave at any point in this area.

Next, we proceed to solve equation (B.3). First, consider the equation $T_1(\mu, \lambda) = 0$. By equation (B.15), for any given μ there exists a unique λ such that $T_1(\mu, \lambda) = 0$ (the converse is false), and the solution is simply

$$\lambda = \mu + \frac{1 + \mu}{2(x_2 - 1)\mu - 1}, \qquad \text{where } \mu > \frac{1}{2(x_2 - 1)} \tag{B.18}$$

The above function is represented by the green curve in Figure 1. Note that the upper branch

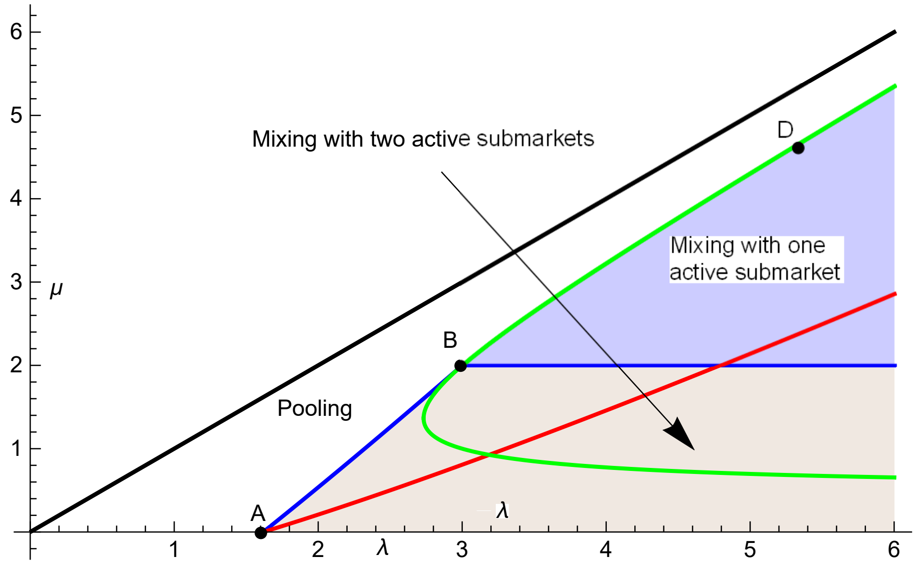

Figure 1: Fixed capacity with $x_2 = 2$

approaches asymptotically to the diagonal line, and the lower branch approaches asymptotically to the horizontal line $\mu = 1/(2(x_2 - 1))$. When $\mu < 1/(2(x_2 - 1))$, $T_1(\mu, \lambda)$ is always positive. A point (μ_0, λ_0) satisfies $T_1(\mu_0, \lambda_0) \geq 0$ if and only if it lies to the left of the green curve.

Next, note that

$$R(0, \lambda) = m(\lambda) - \lambda m'(\lambda) = \left(\frac{\lambda}{1+\lambda}\right)^2, \quad \text{and} \quad T_1(0, \lambda) = m'(\lambda) = \left(\frac{1}{1+\lambda}\right)^2$$

which implies that

$$g(R) = (1 - \sqrt{R})^2 \tag{B.19}$$

We plug equations (B.14), (B.15), and (B.19) into equation (B.3). After some rearrangements, we can find the solution, which is given by the following.

$$\lambda = \frac{1 + (x_2 - 1)\mu + \sqrt{1 + 4(x_2 - 1) + 2(x_2 - 1)\mu}}{2(x_2 - 1)}, \quad \text{where } 0 \leq \mu \leq \mu_B \tag{B.20}$$

and

$$\mu_B = \frac{3 + \sqrt{9 + 16(x_2 - 1)}}{4(x_2 - 1)}$$

The above function, (B.20), is represented by the solid blue curve AB in Figure 1. It intersects with the curve $T_1(\mu, \lambda) = 0$, equation (B.18), at point (λ_B, μ_B), which is represented by Point B in Figure 1. At Point B, $T_1(\mu_B, \lambda_B) = 0$ and $R(\mu_B, \lambda_B) = 1$.

Setting $\mu = 0$ in equation (B.20) gives $\lambda = \frac{1+\sqrt{1+4(x_2-1)}}{2(x_2-1)}$, which is exactly point A from equation (B.17), the same point where $h(\lambda)$ crosses the x-axis. This is no coincidence; it holds for general meeting technologies. To see why, consider point S_1 that lies on the curve AB in Figure 1. It has a corresponding point S_3 on the x-axis. The marginal contributions to surplus of sellers and of the low-type buyers are the same between the two points. Therefore, by Lemma 5, S_1 must lie to the left of the red curve where the determinant of the Hessian matrix is 0. Since S_1 is an arbitrary point on the curve AB, this implies that the entire curve AB must lie to the left of the red curve. As point S_3 moves towards point A, S_2 and hence S_1 also move towards point A. In the end, all three points coincide at point A, which then implies that the blue and the red curve intersect at the same point on the x-axis.

The above analysis has pinned down the boundaries of the relevant regions. The planner's solution is summarized by the following. Note that they satisfy the first-order conditions and are hence optimal by Proposition 2.

Suppose that $(B_1 + B_2, B_2)$ belongs to the blue area. Assume pooling initially, then the marginal contribution of the low-type buyers is negative. Therefore, the planner will move the low-type buyers to a second submarket and the queue in the first submarket will move horizontally to the left till it reaches the green curve BD. At that point, there is one active submarket where the marginal contribution of the low-type buyers is 0 and the marginal contribution of sellers is larger than 1, and one idle submarket with only the low-type buyers.

Suppose $(B_1 + B_2, B_2)$ belongs to the brown area. As we mentioned before, for each point (λ, μ) on the curve AB, there is a corresponding point on the x-axis such that R and T_1 are the same between the two. Formally, the point is given by $(\Lambda(\mu, \lambda), 0)$, where $\Lambda(\mu, \lambda)$ is defined by equation (B.5). As we move from point A to point B, the corresponding point on the x-axis moves from point A to infinity. The convex combinations between points on AB and their corresponding point on x-axis cover the whole brown area. For each point in the brown area, after representing it as a convex combination between a point on AB and its corresponding point on the x-axis, the first-order condition of the planner's problem is satisfied by construction, and we have the optimum: two active submarkets where the queue in the first submarket must lie on the AB curve (for example point S_1) and the second submarket contains some sellers and low-type buyers (for example point S_3).

If $(B_1 + B_2, B_2)$ belongs to the white area, then the optimum is pooling. Note that curve AB divides the area where $T_1(\mu, \lambda) \geq 0$ into two disconnected areas: $T_1(\mu, \lambda) > g(R(\mu, \lambda))$ and $T_1(\mu, \lambda) < g(R(\mu, \lambda))$, with curve AB being the boundary. The white area is the former, and it is not socially beneficial to even move an ε amount of low-type buyers to a second submarket. Thus the optimum is pooling.

B.3 Comparative Statics

B.3.1 Changes in Screening Capacity

Analogous to the stochastic capacity technology case, we assume that the meeting technology is indexed by a parameter σ. To highlight the dependence of ϕ, R, T_1 and $\overline{S}'(B_1)$ on σ, we append it to the arguments of these functions and write $\phi(\mu, \lambda, \sigma)$, etc. We make the following assumption about how ϕ varies with σ.

Assumption 5. *For any μ and λ, $\frac{\partial}{\partial \sigma}\phi(\mu, \lambda, \sigma) \geq 0$, and $\frac{\partial}{\partial \sigma}\phi(\lambda, \lambda, \sigma) = 0$.*

Note that the above assumption holds trivially for the stochastic capacity technology. The first part of this assumption states that a higher σ leads to a higher probability of meeting at least one high-type buyer. The second part states that the probability that a seller meets at least one buyer is independent of σ. In other words, a higher σ makes it easier to identify certain buyers while holding the overall matching rate constant. Because of the second part of the above assumption, we can continue to write $m(\lambda) \equiv \phi(\lambda, \lambda, \sigma)$.

We first consider the optimality of complete separation. As the following proposition establishes, Assumption 5 implies that if—for a given endowment of buyers B_1 and B_2 and a given buyer value dispersion x_2—complete separation is optimal for some σ^b, then it is also optimal for all σ^a with $\sigma^a < \sigma^b$. That is, the parameter range for which complete separation is optimal is shrinking with σ.

Proposition 8. *Under Assumption 5, the area in which complete separation is optimal is shrinking in σ.*

Proof. Because $m(\lambda)$ is independent of σ (Assumption 5), total surplus generated by complete separation is independent of σ. To see this, suppose that the planner allocates α sellers to the submarket of x_2 buyers and the remaining sellers to the submarket of x_1 buyers, then total surplus is $\alpha m(\frac{B_2}{\alpha})(1 + (x_2 - 1)) + (1 - \alpha)m(\frac{B_1}{1-\alpha})$, which is certainly independent of σ. Thus conditional on complete separation, the optimal α, α^*, is also independent of σ.

Next, consider a general allocation with L submarkets. When $\sigma = \sigma^b$, by assumption we have $\alpha^* m(\frac{B_2}{\alpha^*})(1+(x_2-1))+(1-\alpha^*)m(\frac{B_1}{1-\alpha^*}) \geq \sum_{\ell=1}^{L} \alpha^\ell S(\mu^\ell, \lambda^\ell, \sigma^b)$, where $S(\mu^\ell, \lambda^\ell, \sigma^b)$ is given by equation (3) and now depends also on σ. Since for any μ and λ, $\phi(\mu, \lambda, \sigma^b) \geq \phi(\mu, \lambda, \sigma^a)$, we have $S(\mu, \lambda, \sigma^b) \geq S(\mu, \lambda, \sigma^a)$. Therefore, $\alpha^* m(\frac{B_2}{\alpha^*})(1 + (x_2 - 1)) + (1 - \alpha^*)m(\frac{B_1}{1-\alpha^*}) \geq \sum_{\ell=1}^{L} \alpha^\ell S(\mu^\ell, \lambda^\ell, \sigma^b) \geq \sum_{\ell=1}^{L} \alpha^\ell S(\mu^\ell, \lambda^\ell, \sigma^a)$. Thus complete separation is also optimal for $\sigma = \sigma^a$. □

Next, we consider the case of complete pooling. We are interested in the following question: if—for a given endowment of buyers B_1 and B_2 and a given buyer value dispersion x_2—pooling is optimal for some σ, then under what conditions will pooling continue to be optimal for $\sigma + \Delta\sigma$?

By Proposition 2, pooling is optimal at a given σ if and only if $\overline{S}'(B_1, \sigma) \geq 0$. If this inequality is strict, then by continuity with respect to σ, it continues to hold for $\sigma + \Delta$. Hence, the more complicated case is the one in which $\overline{S}'(B_1, \sigma) = 0$; pooling then continues to be optimal for $\sigma + \Delta\sigma$ if and only if $\overline{S}'(B_1, \sigma + \Delta\sigma) \geq 0$, which is equivalent to[1]

$$\text{(B.21)} \qquad \frac{\partial T_1(\mu, \lambda, \sigma)}{\partial \sigma} \geq g'(R(B_2, B_1 + B_2, \sigma)) \frac{\partial R(\mu, \lambda, \sigma)}{\partial \sigma}.$$

By equations (4) and (6), we have

$$\frac{\partial T_1(\mu, \lambda, \sigma)}{\partial \sigma} = (x_2 - 1)\phi_{\lambda\sigma} \quad \text{and} \quad \frac{\partial R(\mu, \lambda, \sigma)}{\partial \sigma} = (x_2 - 1)\left(\phi_\sigma - \mu\phi_{\mu\sigma} - \lambda\phi_{\lambda\sigma}\right).$$

Moreover, by equation (B.2), we have $g'(R(B_2, B_1 + B_2, \sigma)) = -1/\Lambda(\mu, \lambda, \sigma)$, where Λ is defined in B.5, and $1/\infty = 0$ by convention. As a result, we can rewrite (B.21) as

$$\text{(B.22)} \qquad \frac{1}{\Lambda(\mu, \lambda, \sigma) - \lambda}\left(\phi_\sigma - \mu\phi_{\mu\sigma}\right) \geq -\phi_{\lambda\sigma},$$

[1]Note that the function g is independent of σ because $m(\lambda)$ is independent of σ.

which leads to the following result regarding the parameter range in which complete pooling is optimal.

Proposition 9. *Under Assumption 1, 2, 4, and 5, the area in which complete pooling is optimal is expanding with σ if and only if* (B.22) *holds for all* (μ, λ).

Proof. See above. $\square$

Next we show that (B.22) holds for the stochastic capacity technology, so the result applies.

Since $\phi(\mu, \lambda)$ is given by equation (B.6), equation (B.4) which determines $\Lambda(\lambda z, \lambda, \sigma)$ can be rewritten as

$$\frac{(1+\lambda)(\lambda_0 - \lambda)}{2+\lambda+\lambda_0} = \frac{\lambda z \sigma}{1-\sigma}$$

As a function of λ_0, the supremum of the left-hand side is $1+\lambda$. Thus if $\frac{\lambda z \sigma}{1-\sigma} < 1+\lambda$, the solution to the above equation is given by

$$\Lambda(\lambda z, \lambda, \sigma) = \frac{\lambda\left((1+\lambda)(1-\sigma) + \sigma z(1+\lambda)\right)}{(1+\lambda)(1-\sigma) - \lambda z \sigma}. \tag{B.23}$$

Otherwise $\Lambda(\lambda z, \lambda, \sigma) = \infty$. To verify (B.22), we rewrite the above equation as

$$\frac{1}{\Lambda(\lambda z, \lambda, \sigma) - \lambda} = \frac{(1+\lambda)(1-\sigma) - \lambda z \sigma}{2\sigma z \lambda(1+\lambda)}. \tag{B.24}$$

Note that when $\frac{\lambda z \sigma}{1-\sigma} > 1+\lambda$, the right-hand side of the above equation is still well-defined, and it is negative (an underestimate of the true value, which is zero in this case).

Next, by equation (B.6) direct computation gives

$$\phi_\sigma(\lambda z, \lambda, \sigma) - \lambda z \phi_{\mu\sigma}(\lambda z, \lambda, \sigma) = z^2\lambda^2 \frac{1 + (1+\sigma - z\sigma)\lambda}{(1+(1-\sigma(1-z))\lambda)^3} > 0$$

$$\phi_{\lambda\sigma}(\lambda z, \lambda, \sigma) = z\lambda \frac{1 + ((2-\sigma)z - (1-\sigma))\lambda}{(1+(1-\sigma(1-z))\lambda)^3}.$$

From the above equations we can see that $\phi_\sigma - \lambda z \phi_{\mu\sigma}$ is always strictly positive, but the sign of $\phi_{\lambda\sigma}$ is indeterminate. A sufficient condition for (B.22) is thus we plug the right-hand side of equation (B.24) into (B.22) irrespective of whether $\frac{\lambda z \sigma}{1-\sigma} > 1+\lambda$, which then gives

$$\frac{(\phi_\sigma - \mu\phi_{\mu\sigma})}{\Lambda(\lambda z, \lambda, \sigma) - \lambda} + \phi_{\lambda\sigma} = \lambda z \frac{1 + \sigma + (\lambda + \lambda\sigma(z-1))^2 + \lambda((2-\sigma)(1+z\sigma) + \sigma^2)}{2(\lambda+1)\sigma(1+(1-\sigma(1-z))\lambda)^3}$$

The right-hand side is always positive. We have thus proved (B.22) for this meeting technology.

B.3.2 Changes in the Dispersion of Buyer Values

To highlight the dependence of $\overline{S}'(B_1)$, R and T_1 on x_2, we append it to the arguments of these functions and write $\overline{S}'(B_1, x_2)$, etc.

Consider complete pooling first. As established in equation (15), complete pooling is socially optimal if and only if $\overline{S}'(B_1, x_2) \geq 0$. Pooling continues to be optimal for x_2' with $x_2' < x_2$ if and only if $\overline{S}'(B_1, x_2') \geq 0$. Thus, for the pooling area to shrink as x_2 increases, we need that $\overline{S}'(B_1, x_2)$, as a function of x_2, crosses the x-axis at most once and from above. As is well-known in the literature, a sufficient condition for this is that $\frac{\partial}{\partial x_2}\overline{S}'(B_1, x_2) < 0$ if $\overline{S}'(B_1, x_2) = 0$; a necessary condition is that $\frac{\partial}{\partial x_2}\overline{S}'(B_1, x_2) \leq 0$ if $\overline{S}'(B_1, x_2) = 0$ (note the difference between the strict and weak inequality). By equation (14),

$$
\begin{aligned}
\frac{\partial}{\partial x_2}\overline{S}'(B_1, x_2) &= \frac{\partial}{\partial x_2}T_1(B_2, B_2 + B_1, x_2) - g'(R(B_2, B_2 + B_1, x_2))\frac{\partial}{\partial x_2}R(B_2, B_2 + B_1, x_2) \\
&= \phi_\lambda + \frac{1}{\Lambda(B_2, B_1 + B_2)}\left(\phi - B_2\phi_\mu - (B_1 + B_2)\phi_\lambda\right), \qquad \text{(B.25)}
\end{aligned}
$$

where we have suppressed argument $(B_2, B_1 + B_2)$ from function ϕ, and function Λ is defined by equation (B.5). It turns out that equation (B.5) implies that the above equation is always strictly negative, and we have the following result.

Proposition 10. *Under Assumption 1, 2, and 4, the area in which complete pooling is optimal is shrinking in x_2.*

Note that Assumption 4 holds for the stochastic capacity technology, so the result applies.

Proof. If $R(B_2, B_2 + B_1, x_2) \geq 1$, then $g(R(B_2, B_2 + B_1, x_2)) = 0$. $\overline{S}'(B_1, x_2) = 0$ then implies $T_1(B_2, B_2 + B_1, x_2) = g(R(B_2, B_2 + B_1, x_2)) = 0$ and $\Lambda(B_2, B_1 + B_2) = \infty$. Since $T_1(B_2, B_2 + B_1, x_2) = m'(B_2 + B_1) + (x_2 - 1)\phi_\lambda(B_2, B_1 + B_2)$, we have $\phi_\lambda(B_2, B_1 + B_2) < 0$. Thus (B.25) is strictly negative.

Next, if $R(B_2, B_2 + B_1, x_2) \in (0, 1)$, then $g(R(B_2, B_2 + B_1, x_2)) \in (0, 1)$. Thus $\overline{S}'(B_1, x_2) = 0$ implies that $\Lambda(B_2, B_1 + B_2)$ is defined by first row of equation (B.5) and we have $\phi_\lambda(B_2, B_1 + B_2) < 0$. It is easy to see that (B.25) is strictly negative if and only if the following holds

$$
\begin{aligned}
&\frac{-\phi_\lambda}{\Lambda}\left(\Lambda - (B_1 + B_2) - \frac{\phi - B_2\phi_\mu}{-\phi_\lambda}\right) \\
&= \frac{-\phi_\lambda}{\Lambda}\left(\Lambda - (B_1 + B_2) - \frac{m(\Lambda) - m(B_1 + B_2) - (\Lambda - (B_1 + B_2))m'(\Lambda)}{m'(B_1 + B_2) - m'(\Lambda)}\right) \\
&= \frac{-\phi_\lambda}{\Lambda}\left(\frac{(\Lambda - (B_1 + B_2))\, m'(B_1 + B_2) - (m(\Lambda) - m(B_1 + B_2))}{m'(B_1 + B_2) - m'(\Lambda)}\right) < 0.
\end{aligned}
$$

where we have suppressed the arguments of Λ and ϕ. For the equality in the second line we used equation (B.4), and the last inequality is because m is strictly concave and $\Lambda > B_1 + B_2$. □

Next consider complete separation. The logic is similar to the case for pooling. By equation (15), complete separation is socially optimal if and only if $\overline{S}'(0, x_2) \leq 0$. For the area of complete separation to expand with x_2, we need that $\overline{S}'(0, x_2)$, as a function of x_2, crosses the x-axis at most once and from above.

Assume $\overline{S}'(0, x_2) = 0$. Let λ_H (resp. λ_L) be the queue length in the submarket of high-type (resp. low-type) buyers at the optimum. These queue lengths are determined by two equations.

First, sellers' marginal contribution to surplus must be the same between the two submarkets, i.e.

$$x_2\left(m(\lambda_H) - \lambda_H m'(\lambda_H)\right) = m(\lambda_L) - \lambda_L m'(\lambda_L), \tag{B.26}$$

Second, summing the number of sellers across the two submarkets must yield the total measure of sellers, i.e.

$$\frac{B_2}{\lambda_H} + \frac{B_1}{\lambda_L} = 1. \tag{B.27}$$

Next, consider the marginal contribution to surplus of low-type buyers in the two submarkets. Since $\overline{S}'(0, x_2) = 0$, we have

$$m'(\lambda_H) + (x_2 - 1)\phi_\lambda(\lambda_H, \lambda_H) = m'(\lambda_L) \tag{B.28}$$

where the left- and the right-hand denotes the marginal contribution of a low-type buyer in the submarket with high-type and low-type buyers, respectively.[2]

Now suppose that x_2 increases to x_2'. We want to rule out the possibility that $\overline{S}'(0, x_2') = 0$. Suppose otherwise. Then at x_2', the new queue lengths λ_H' and λ_L' also satisfy equations (B.26), (B.27), and (B.28). Note that equations (B.27) and (B.28) are special cases of equation (6) and (4), respectively. As before, we can combine equations (B.27) and (B.28) to eliminate x_2 and the resulting equation is simply (B.4), where the correspondence is $\mu = \lambda = \lambda_H$ and $\lambda_0 = \lambda_L$. Thus we have $\lambda_L = \Lambda(\lambda_H, \lambda_H)$ and $\lambda_L' = \Lambda(\lambda_H', \lambda_H')$, where $\Lambda(\mu, \lambda)$ is defined by equation (B.5) and is independent of x_2.

Conditional on full separation, the allocation of sellers is completely determined by equation (B.26). When x_2 increases to x_2', more sellers will vist the submarket with high-type buyers, which then implies that $\lambda_H' < \lambda_H < \lambda_L < \lambda_L'$. To rule out that λ_H' and λ_L' are also a solution to equation (B.28), a sufficient and necessary condition is simply that $\Lambda(\mu, \mu)$ is weakly increasing in μ, which then implies that if $\lambda_H' < \lambda_H$, then $\lambda_L' = \Lambda(\lambda_H', \lambda_H') \leq \Lambda(\lambda_H, \lambda_H) = \lambda_L$, which contradicts the above assertion that $\lambda_L' > \lambda_L$. This leads to the following assumption.

Assumption 6. *$\Lambda(\mu, \mu)$, which is defined by equation* (B.5)*, is (weakly) increasing in μ.*

In equation (B.23), we give an explicit expression for $\Lambda(\mu, \lambda)$ for the stochastic capacity technology, which verifies that the above assumption is satisfied.

We then have the following result.

Proposition 11. *Under Assumption 1, 2, 4, and 6, the area in which full separation is optimal is expanding with x_2.*

Proof. As we explained before in Proposition 10, Assumption 6 ensures that $\overline{S}'(0, x_2)$ crosses the x-axis at most once. We now prove that if that is the case, $\overline{S}'(0, x_2)$ must cross the x-axis from above. Suppose otherwise. Then $\overline{S}'(0, x_2) = 0$ for some x_2, and $\overline{S}'(0, x_2') > 0$ for all $x_2' > x_2$. As we mentioned before, more sellers will flow into the submarket of x_2 buyers as we increase x_2, and there exists some x_2* such that the solution to equations (B.26) and (B.27) is

[2] Note that although the submarket with high-type buyers does not contain low-type buyers, we can still calculate the effect on surplus of a marginal increase in the number of low-type buyers.

$\lambda_H = B_2$ and $\lambda_L = \infty$. If we increase x_2 further, then λ_H will stay constant and the right-hand side of (B.28) will start decreasing linearly. So $\overline{S}'(0, x_2)$ can not stay positive for sufficiently large x_2, and we have a contradiction. $\square$

C Uniqueness of Beliefs

The efficiency result in Section 4 assumes that sellers are optimistic. Without this assumption, it is not clear how sellers should evaluate the expected payoff of deviations if multiple queues are compatible with market utility. Here, we show that such a scenario is rather special in the sense that—under mild restrictions on the meeting technology—the solution to the market utility condition is in fact unique, rendering the optimism assumption redundant.

Uniqueness. The following proposition then presents our result regarding uniqueness of the beliefs for a seller posting a second-price auction with a reserve price. To avoid the situation where the optimal response of low-type buyers is indeterminate, we assume that their market utility is strictly positive.

Proposition 12. *Assume that $U_1 > 0$. Under assumptions 1 and 3, for each seller posting a second-price auction with a reserve price r, there is a unique queue (μ, λ) compatible with the market utility function. Furthermore, for two sellers posting reserve prices r^a and r^b, it holds that $\lambda^a > \lambda^b$ if and only if $r^a < r^b$.*

Proof. Our proof consists of two parts: i) $r^a < r^b \Leftrightarrow \lambda^a > \lambda^b$, and ii) the queue length λ determines the whole queue uniquely. Denote by V_k^i the expected payoff of x_k buyers from visiting queue i where $k = 1, 2$ and $i = a, b$.

For i), we first prove $r^a < r^b \Rightarrow \lambda^a > \lambda^b$. Suppose otherwise that $\lambda^a \leq \lambda^b$. We distinguish between two cases, $r^a < 1$ and $r^a \geq 1$. First, consider the case $r^a \geq 1$. Then x_1 buyers will not visit the two sellers since their market utility is strictly positive, so that both queues contain only x_2 buyers, and $V_2^a = Q_1(\lambda^a)(x_2 - r^a) > Q_1(\lambda^b)(x_2 - r^a) \geq Q_1(\lambda^b)(x_2 - r^b) = V_2^b$. We then reach a contradiction. Consider next $r^a < 1$. Then by a similar logic, we have

$$V_1^a = Q_1(\lambda^a)(1 - r^a) > \max(Q_1(\lambda^b)(1 - r^b), 0) = V_1^b$$

Thus x_1 buyers strictly prefer queue a, which implies that queue b does not contain x_1 buyers. Note that $V_2^a \geq Q_1(\lambda^a)(x_2 - r^a) > Q_1(\lambda^b)(x_2 - r^b) = V_2^b$, where the first inequality is because queue a may contain x_1 buyers and an x_2 buyer may enjoy a positive payoff even when he is not the only buyer showing up, the second inequality follows the same logic as above, and the last equality is because queue b only contains x_2 buyers. Therefore, x_2 buyers also strictly prefer queue a and we reach a contradiction again. The other direction is proved similarly. Thus $r^a < r^b \Leftrightarrow \lambda^a > \lambda^b$.

For ii), suppose otherwise that there are two different queues a and b with the same length λ that are compatible with the auction with reserve price r. Without loss of generality, set $0 \leq \mu^a < \mu^b \leq \lambda$. Note that $V_1^a = V_1^b = Q_1(\lambda)(x - r) \equiv V_1$ and the expected payoff of an x_2 buyer from queue i is $V_2^i = V_1 + \phi_\mu(\mu^i, \lambda)(x_2 - 1)$ (see the proof of Proposition 5 for the derivation of this equation). If $P_0(\lambda) + P_1(\lambda) < 1$, then $\phi(\mu, \lambda)$ is strictly concave in μ, which implies that $V_2^a > V_2^b$ which gives the desired contradiction. If $P_0(\lambda) + P_1(\lambda) = 1$, then

$\phi_\mu(\mu, \lambda) = Q_1(\lambda)$, independent of μ, and $V_2^a = V_2^b$. Note that since $Q_1(\lambda)(x_2 - 1) = (U_2 - U_1)$, and $r = 1 - U_1/Q_1(\lambda)$, both λ and r are uniquely determined for given market utilities U_1 and U_2. Thus it is a knife-edge case, and our statement is true for all r expect one special value. But note i) this knife-edge phenomenon only occurs because buyer types are discrete, and ii) even in our discrete buyer type framework, this knife-edge reserve price r will never be adopted by sellers because by either increasing or decreasing the reserve price, sellers can obtain a strictly higher profit. □

If both sellers attract low-type buyers, then the expected payoffs for low-type buyers from visiting any of the two sellers must be the same: $Q_1(\lambda^a)(1-r^a) = Q_1(\lambda^b)(1-r^b)$, which implies that $\lambda^a > \lambda^b$ if and only if $r^a < r^b$, since $Q_1(\lambda)$ is strictly decreasing by Assumption 3. When one seller attracts low-type buyers and the other does not, the latter seller must have posted a high reserve price implying a shorter queue without low-type buyers.

Things are slightly more complicated when sellers post a second-price auction with an entry fee. Below, we introduce one weak additional restriction on the meeting technology, which is sufficient to guarantee that there exists a monotonic relation between meeting fees and queue lengths. This implies that there exists a unique queue that is compatible with the market utility function when sellers post an auction with an entry fee.

Assumption 7. $(1 - Q_0(\lambda))/Q_1(\lambda)$ *is weakly increasing in* λ.

If we rewrite $(1 - Q_0(\lambda))/Q_1(\lambda)$ as $1 + \sum_{k=2}^{\infty} Q_k(\lambda)/Q_1(\lambda)$, then this assumption states that with a higher buyer-seller ratio, it is relatively more likely that a buyer will meet competitors in an auction rather than being alone.

Proposition 13. *Under Assumptions 1, 3, and 7, for each seller posting an auction with entry fee* t*, there is a unique queue* (μ, λ) *compatible with the market utility function. Furthermore, for two sellers posting entry fees* t^a *and* t^b*, it holds that* $\lambda^a > \lambda^b$ *if and only if* $t^a < t^b$*.*

Proof. The proof is similar to that of Proposition 12 and consists of two parts: i) $t^a < t^b \Leftrightarrow \lambda^a > \lambda^b$, and ii) the queue length λ determines the whole queue uniquely. Note that part ii) is exactly the same as that of Proposition 12 so we only need to consider part i).

For i), we first prove $t^a < t^b \Rightarrow \lambda^a > \lambda^b$. Suppose otherwise that $\lambda^a \leq \lambda^b$. We distinguish two cases, $t^a < 0$ and $t^a \geq 0$. First, consider the case $t^a < 0$ (entry subsidy). As before, denote by V_k^i the expected payoff of x_k buyers from a queue i where $k = 1, 2$ and $i = a, b$. Then, we have

$$
\begin{aligned}
V_1^a = Q_1(\lambda^a) - (1 - Q_0(\lambda^a))t^a &> Q_1(\lambda^b) - (1 - Q_0(\lambda^a))t^a \\
&\geq Q_1(\lambda^b) - (1 - Q_0(\lambda^b))t^a \\
&\geq Q_1(\lambda^b) - (1 - Q_0(\lambda^b))t^b = V_1^b
\end{aligned}
$$

where the first inequality is because $Q_1(\lambda)$ is strictly decreasing, the second inequality is because $1 - Q_0(\lambda)$ is decreasing and $t^a < 0$, and the final inequality follows from the assumption that $t^a < t^b$. Thus x_1 buyers strictly prefer queue a and queue b does not contain x_2 buyers. However, $V_2^a \geq Q_1(\lambda^a) - (1 - Q_0(\lambda^a))t^a > Q_1(\lambda^b)x_2 - (1 - Q_0(\lambda^b))t^b = V_2^b$, where the first inequality is because queue a may contain x_1 buyers and an x_2 buyer may enjoy a positive payoff even when he is not the only buyer showing up, the second inequality follows the same logic as above, and the last equality is because queue b only contains x_2 buyers. Thus, x_2 buyers also strictly prefer queue a, and we have a contradiction, which implies that $\lambda^a > \lambda^b$. The other direction is proved similarly.

Next, we consider the case $t^a \geq 0$. Again suppose otherwise that $\lambda^a \leq \lambda^b$. As above, we have

$$\begin{aligned} V_1^a &= Q_1(\lambda^a) - (1 - Q_0(\lambda^a))t^a = Q_1(\lambda^a)\left(1 - \frac{1 - Q_0(\lambda^a)}{Q_1(\lambda^a)}t^a\right) > Q_1(\lambda^b)\left(1 - \frac{1 - Q_0(\lambda^a)}{Q_1(\lambda^a)}t^a\right) \\ &\geq Q_1(\lambda^b)\left(1 - \frac{1 - Q_0(\lambda^b)}{Q_1(\lambda^b)}t^a\right) > Q_1(\lambda^b)\left(1 - \frac{1 - Q_0(\lambda^b)}{Q_1(\lambda^b)}t^b\right) = V_1^b \end{aligned}$$

where the inequality in the first line is because $Q_1(\lambda)$ is strictly decreasing, the first inequality in the second line is because of Assumption 7, and the second inequality in the second line follows from the assumption that $t^a < t^b$. The remaining arguments then follow exactly the same as for the case $t^a < 0$. □

The intuition behind Proposition 13 is similar to that of Proposition 12 and readily follows from the correspondence between the reserve price and entry fee: $t = rQ_1/(1 - Q_0)$. Again, consider two different queues a and b. We have shown in Proposition 12 that $\lambda^a > \lambda^b$ if and only if $r^a < r^b$. Under Assumption 7, the two inequalities jointly lead to $t^a < t^b$.

Hence, we have established that under mild restrictions on the meeting technology, there exists only one queue which is compatible with market utility when sellers post an auction with a reserve price or an entry fee. Consequently, the assumption that sellers are optimistic is redundant for a large class of meeting technologies.

D N Buyer Types

D.1 Surplus: Proof of Lemma 3

For later use, we prove a slightly more general version of equation (23) with a general, possibly continuous, buyer value distribution.

Lemma 9. *Consider a submarket with a measure 1 of sellers and a measure λ of buyers whose values are distributed according to $F(x)$ with support $[0, \overline{x}]$. Total surplus then equals*

$$S(\lambda, F) = \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda)dx. \tag{D.1}$$

Proof. **A direct proof.** When a seller meets $n \geq 1$ buyers, the surplus x from the meeting is distributed according to $F^n(x)$. Thus the expected surplus per seller in the submarket is

$$S = \sum_{n=1}^{\infty} P_n(\lambda) \int_0^{\overline{x}} x\, dF^n(x) = \sum_{n=1}^{\infty} P_n(\lambda) \left(\overline{x} - \int_0^{\overline{x}} F^n(x)\, dx\right) = \sum_{n=1}^{\infty} P_n(\lambda) \left(\int_0^{\overline{x}} 1 - F^n(x)\, dx\right),$$

where for the second equality we used integration by parts. Notice that $F^n(x) = 0$ when $n = 0$. We can add a zero term $P_0(\lambda)\,(\int_0^{\overline{x}} 1 - F^0(x)\, dx)$ to the RHS of the above equation and start the summation from $n = 0$. Therefore,

$$S = \sum_{n=0}^{\infty} P_n(\lambda) \left(\int_0^{\overline{x}} 1 - F^n(x)\, dx \right) = \int_0^{\overline{x}} 1 - \sum_{n=0}^{\infty} P_n(\lambda) F^n(x)\, dx$$
$$= \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) dx$$

where for the second equality in the first line we use the Dominated Convergence Theorem to interchange integration with summation and for the last equality we used the definition of ϕ from equation (1).

An alternative approach. First recall the following fact from probability theory. Suppose z is any random variable with cdf $G(z)$ and $z \in [0, \overline{x}]$. Then the expected value of z can be written as $\mathbb{E}z = \int_0^{\overline{x}} z dG(z) = \int_0^{\overline{x}} 1 - G(z) dz$. This equation is well-known and can be proved by integration by parts. We can use it to directly derive our surplus equation.

(Back to our surplus equation.) Let z be the highest valuation among all buyers that a seller meets. The event $z \geq x$ happens if and only if the seller meets at least one buyer with valuation higher than x, the probability of which is simply $\phi(\lambda(1 - F(x)), \lambda)$ by the construction of ϕ. Therefore, by the above equation we have $S = \mathbb{E}z = \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) dx$ □

In our discrete case, $F(x_j) = \mu_j / \mu_1$ for $j = 1, \ldots, N$. The above equation reduces to (23).

Next, we calculate $T_k(\boldsymbol{\mu})$. Note that a marginal entrant of x_k buyers increases μ_j, $j = 1, \ldots, k$, by the same amount. Therefore,

$$T_k(\boldsymbol{\mu}) = \sum_{j=1}^{k} \frac{\partial S(\boldsymbol{\mu})}{\partial \mu_j} = \sum_{j=1}^{k} (x_j - x_{j-1}) \phi_\mu(\mu_j, \mu_1) + \sum_{j=1}^{N} (x_j - x_{j-1}) \phi_\lambda(\mu_j, \mu_1)$$

Because total surplus function is constant-returns-to-scale, if we add one more seller and λ more buyers to the submarket while keeping the buyer value distribution unchanged, the added surplus is simply $S(\lambda, F)$ in equation (D.1). Thus the effect of adding one more seller only is

$$R = S - \lambda \frac{\partial S(\lambda, F)}{\partial \lambda} = \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) - \lambda \frac{\partial \phi(\lambda(1 - F(x)), \lambda)}{\partial \lambda} dx$$
$$= \int_0^{\overline{x}} \phi(\lambda(1 - F(x)), \lambda) - \lambda(1 - F(x)) \phi_\mu(\lambda(1 - F(x)), \lambda) - \lambda \phi_\lambda(\lambda(1 - F(x)), \lambda) dx$$

which is simply equation (25) in the discrete-value case.

Finally, we consider the Hessian matrix. We denote it by $\mathcal{H}$ and its negative is then $-\mathcal{H}$. Also to save space, define $\phi^k_{\mu\mu} \equiv \phi_{\mu\mu}(\mu_k, \mu_1)$, $\phi^k_{\mu\lambda} \equiv \phi_{\mu\lambda}(\mu_k, \mu_1)$, and $\phi^k_{\lambda\lambda} \equiv \phi_{\lambda\lambda}(\mu_k, \mu_1)$ for $k = 1, \ldots, N$. We compute the Hessian matrix by directly calculating $\pi_{ij} \equiv \partial \pi(\boldsymbol{\mu}) / \partial \mu_i \mu_j$. Thus we have

$$-\mathcal{H} = \begin{pmatrix} P_0''(\mu_1)x_1 - \sum_2^N \phi_{\lambda\lambda}(\mu_k,\mu_1)(x_k - x_{k-1}), & -\phi_{\mu\lambda}^2(x_2 - x_1), & \cdots & , -\phi_{\mu\lambda}^N(x_N - x_{N-1}) \\ -\phi_{\mu\lambda}^2(x_2 - x_1), & -\phi_{\mu\mu}^2(x_2 - x_1), & \cdots & , 0 \\ \vdots & \vdots, & \cdots & , \vdots \\ -\phi_{\mu\lambda}^N(x_N - x_{N-1}), & 0, & \cdots, & -\phi_{\mu\mu}^N(x_N - x_{N-1})) \end{pmatrix}$$

By Sylvester's criterion, $-\mathcal{H}$ is positive semidefinite if and only if the determinants of the following N matrices are positive: the bottom right 1×1 corner, the bottom right 2×2 corner, ..., and $-\mathcal{H}$ itself. It is easy to see that the bottom right $n \times n$ corner with $n < N$ is always diagonal and the diagonal elements are always positive since $\phi(\mu, \lambda)$ is always concave in μ. Therefore, $-\mathcal{H}$ is positive semidefinite if and only if its determinant is positive.

To calculate the determinant of $-\mathcal{H}$, for each $n \geq 2$ we multiply column n by $-\phi_{\mu\lambda}^n / \phi_{\mu\mu}^n$ and add it to column 1. The resulting matrix is

$$\begin{pmatrix} -\pi_{11} + \sum_2^N \frac{(\phi_{\mu\lambda}^k)^2}{\phi_{\mu\mu}^k}(x_k - x_{k-1}), & -\phi_{\mu\lambda}^2(x_2 - x_1), & \cdots & , -\phi_{\mu\lambda}^N(x_N - x_{N-1}) \\ 0, & -\phi_{\mu\mu}^2(x_2 - x_1), & \cdots & , 0 \\ \vdots & \vdots, & \cdots & , \vdots \\ 0, & 0, & \cdots, & -\phi_{\mu\mu}^N(x_N - x_{N-1})) \end{pmatrix}$$

In this way, the matrix $-\mathcal{H}$ becomes upper triangular and its determinant can be easily calculated. The determinant is

$$\det(-\mathcal{H}) = \left(P_0''(\mu_1)x_1 + \sum_2^N \left(\frac{(\phi_{\mu\lambda}^k)^2}{\phi_{\mu\mu}^k} - \phi_{\lambda\lambda}^k \right)(x_k - x_{k-1}) \right) \Pi_2^N(-\phi_{\mu\mu}^k(x_k - x_{k-1}))).$$

Again since $\phi(\mu, \lambda)$ is always concave in μ, $\det(-\mathcal{H}) > 0$ is equivalent to that the first term in the parenthesis at the right hand side is positive. Thus we have derived equation (26).□

D.2 Incentive Compatibility and Payoffs

When a monopolistic seller offers a selling mechanism, incentive compatibility requires that buyers' expected utility is intimately connected with their trading probabilities (see Myerson, 1981; Riley and Samuelson, 1981). This logic can be extended to an environment with competing sellers.[3] In such an environment, the expected payoff that a buyer receives from visiting a submarket is equal to what he would get at a monopolistic seller with a random number of buyers as in Levin and Smith (1994). However, buyers must also choose which submarket to visit and this depends on the posted mechanisms which in turn depends on the meeting technology.

In our analysis, it will sometimes be useful to consider buyers with a value x that is not in the set $\{x_1, \ldots, x_N\}$, who thus have measure zero. To do so, we define an extended version of the market utility function $U(x)$, which represents the highest expected payoff that a buyer with value x can achieve, such that $U_k \equiv U(x_k)$ for each k. Given any set of mechanisms posted by sellers, denote the set of mechanisms that buyers of type x visit by $\Omega^b(x)$, pick an

[3]See Peters (2013) for a similar treatment for an invariant meeting technology.

arbitrary $\omega^b(x) \in \Omega^b(x)$ and denote by $p(x, \omega^b(x))$ the probability that a buyer of type x who visits a mechanism $\omega^b(x)$ trades. Of course, if buyers of type x choose to be inactive, then we set $\omega^b(x) = \emptyset$ and $p(x, \emptyset) = 0$. The following Lemma then establishes the properties of the market utility function. Its proof is closely related to the one in Myerson (1981).

Lemma 10. *Given any set of mechanisms posted by sellers, $p(x, \omega^b(x))$ is non-decreasing and the market utility function $U(x)$ is convex, satisfying*

$$U(x) = U(0) + \int_0^x p(z, \omega^b(z))dz.$$

Proof. The strategy of a buyer with value x is: (i) a probability distribution over the mechanisms to visit and inactivity and (ii) a value to report when the mechanism is not inactivity. Given the mechanisms posted by sellers, suppose that the set of mechanisms that a buyer with valuation x visits is $\Omega^b(x)$, and the probability that the buyer receives the object when visiting seller $\omega \in \Omega^b(x)$ and reporting x by $p(x, \omega)$, with a corresponding expected payment $t(x, \omega)$.

First, we select one element $\omega^b(z) \in \Omega^b(z)$ for each z. Then, by the incentive compatibility constraint (ICC), for any x, z,

$$U(x) \geq xp(z, \omega^b(z)) - t(z, \omega^b(z)), \tag{D.2}$$

i.e. buyers with valuation x are always better of following their equilibrium strategies than mimicking any other type z. Therefore,$U(x) = \max_{z \in [x_1, x_N]} xp(z, \omega^b(z)) - t(z, \omega^b(z))$. Hence, $U(x)$ is the supreme of a collection of affine functions and must therefore be convex.

Furthermore, we can rewrite equation (D.2) in the following way.

$$\begin{aligned} U(x) = xp(x, \omega^b(x)) - t(x, \omega^b(x)) &\geq xp(z, \omega^b(z)) - t(z, \omega^b(z)) \\ &= U(z) + p(z, \omega^b(z))(x - z). \end{aligned}$$

So, $p(x, \omega^b(x))$ is the slope of a supporting line for the convex function $U(x)$. Therefore, $p(x, \omega^b(x))$ is a non-decreasing function, and it equals the derivative of $U(x)$ almost everywhere. The latter then implies the integral representation of $U(x)$ in Lemma 10. □

As the supremum of a collection of convex functions (expected payoffs from individual submarkets), the market utility function $U(x)$ is always convex. Because of incentive compatibility, a higher winning probability is associated with a higher expected payoff.

Buyers only visit sellers who offer them their market utility and the sellers are residual claimants of the output. Competition forces sellers to post an efficient mechanism, i.e. a mechanism in which the buyer with the highest value trades if and only if his valuation exceeds that of the seller. In other words, efficient mechanisms are the cheapest way to offer buyers their market utility.

Consider a submarket with an efficient mechanism and a queue $\boldsymbol{\mu} \equiv (\mu_1, \ldots, \mu_N)$ where the lowest type of buyers visiting the submarket is ι. Consider a buyer with value x strictly between x_{k-1} and x_k with $k \geq \iota + 1$. Since the posted mechanism is efficient, his winning probability is $\phi_\mu(\mu_k, \mu_1)$, which, by equation (2), is the probability that the buyer meets a seller and has the highest value among all buyers who arrived at the seller. As in a monopolistic auction, buyers' expected value is a summation (or with continuous types, an integral) over their winning probabilities. The expected value for the buyer is $V(x) = V_{k-1} + (x - x_{k-1})\phi_\mu(\mu_k, \mu_1)$.

Since $V(x)$ must be continuous, the expected payoff for a buyer with valuation x_k visiting this submarket is $V_k = V_{k-1} + (x_k - x_{k-1})\phi_\mu(\mu_k, \mu_1)$, which then implies

$$V_k = V_\iota + \sum_{j=\iota+1}^{k} (x_j - x_{j-1})\phi_\mu(\mu_j, \mu_1). \tag{D.3}$$

The expected payoff for buyers in equation (D.3) is similar to the corresponding payoff in a monopolistic auction. Equation (D.3) also shows that $\phi_\mu(\mu_k, \mu_1)$ and $\phi_\mu(\mu_{k+1}, \mu_1)$ are subgradients at point x_k for the market utility function $U(x)$, since $V(x)$ lies below $U(x)$ and the slope for $V(x)$ with $x \in (x_{k-1}, x_k)$ is $\phi_\mu(\mu_k, \mu_1)$, and the slope for $V(x)$ with $x \in (x_k, x_{k+1})$ is $\phi_\mu(\mu_{k+1}, \mu_1)$. Two special cases are worth mentioning. Suppose that the lowest and the highest value of buyers who visit the submarket are $\underline{x}$ and $\overline{x}$, respectively. Then $Q_1(\mu_1) = \phi_\mu(\mu_1, \mu_1)$ and $1 - Q_0(\mu_1) = \phi_\mu(0, \mu_1)$ are subgradients at point $\underline{x}$ and $\overline{x}$, respectively. A buyer with value $x > \overline{x}$ will always trade as long as he successfully meets a seller, which happens with probability $1 - Q_0(\mu_1) = \phi_\mu(0, \mu_1)$.

Since the mechanism is assumed to be efficient, the expected seller value is given by

$$\begin{aligned}
\pi &= S(\boldsymbol{\mu}) - \sum_{k=\iota}^{N} (\mu_k - \mu_{k+1}) V_k \\
&= \sum_{k=\iota}^{N} (x_k - x_{k-1})\phi(\mu_k, \mu_1) - \sum_{k=\iota}^{N} (\mu_k - \mu_{k+1}) \left(V_\iota + \sum_{j=\iota+1}^{k} (x_j - x_{j-1})\phi_\mu(\mu_j, \mu_1) \right) \\
&= -\mu_1 V_\iota + \sum_{k=\iota}^{N} x_k(\phi(\mu_k, \mu_1) - \phi(\mu_{k+1}, \mu_1)) - \sum_{j=\iota+1}^{N} \mu_j (x_j - x_{j-1})\phi_\mu(\mu_j, \mu_1).
\end{aligned}$$

where in deriving the last equality we changed the order of summation. Rewriting the above equation yields

$$\pi = -\mu_1 V_\iota + \sum_{j=\iota+1}^{N+1} \left(x_{j-1} - \frac{\mu_j \phi_\mu(\mu_j, \mu_1)(x_j - x_{j-1})}{\phi(\mu_{j-1}, \mu_1) - \phi(\mu_j, \mu_1)} \right) \big(\phi(\mu_{j-1}, \mu_1) - \phi(\mu_j, \mu_1)\big). \tag{D.4}$$

To make the comparison with the classic auction literature more clear, we take the limit of the discrete buyer value distribution so that it converges to a continuous distribution F with density f. Then let $\lambda \equiv \mu_1$ and we have $\mu_j = \lambda(1 - F(x_j))$. Let $x_j = x$ and $x_{j-1} = x - \Delta x$, then the summand in equation (D.4) becomes

$$\left(x - \Delta x - \frac{\lambda(1 - F(x))\phi_\mu(\lambda(1 - F(x)), \lambda)\Delta x}{\phi(\lambda(1 - F(x - \Delta x)), \lambda) - \phi(\lambda(1 - F(x)), \lambda)} \right) \big(\phi(\lambda(1 - F(x - \Delta x)), \lambda) - \phi(\lambda(1 - F(x)), \lambda)\big).$$

Letting $\Delta x \to 0$, the first term becomes

$$x - \Delta x - \frac{\lambda(1 - F(x))\phi_\mu(\lambda(1 - F(x)), \lambda)\Delta x}{\phi_\mu(\lambda(1 - F(x)), \lambda)\lambda f(x)\Delta x} \to x - \frac{1 - F(x)}{f(x)},$$

and the second term is the measure of the distribution function $1 - \phi(\lambda(1 - F(x)), \lambda)$ between

x and $x + \Delta x$. Therefore, we can rewrite equation (D.4) in the more familiar integral form

$$\pi = -\lambda V_\iota + \int_{x_\iota}^{x_N} (x - \frac{1 - F(x)}{f(x)}) d\left(1 - \phi(\lambda(1 - F(x)), \lambda)\right), \tag{D.5}$$

where $\lambda = \mu_1$.

In a standard auction with n bidders, a seller's expected payoff equals the virtual valuation function integrated against the distribution of the highest valuation, see Myerson (1981). Our setting is different in two ways: (i) because the buyer value distribution is discrete, the virtual value function takes a slightly more complicated form, and (ii) the distribution of the highest valuation of bidders depends on the meeting technology and is given by $1 - \phi(\mu_j, \mu_1)$, i.e. the probability that there are no buyers with valuations above x_j.

One may have expected that allowing for general meeting technologies would severely complicate the payoff functions in (competing) auction theory. We have shown here that our alternative representation ϕ avoids such complications. In particular, agents' expected payoffs retain the same structure but simply depend on transformations of ϕ.

D.3 Efficiency

Equivalence. To prove constrained efficiency of equilibrium, we show again that even if sellers can buy queues directly in a hypothetical competitive market, they cannot do better than in the decentralized environment. As in the case of two buyer types (see Proposition 5, the following proposition shows that the relaxed problem and the constrained problem of the sellers are equivalent.

Proposition 14. *Given that the market utility function is convex, any solution $\boldsymbol{\mu}$ to the sellers' relaxed problem is compatible with an auction with an entry fee in the sellers' constrained problem, where the fee is given by*

$$t = -\frac{\sum_{j=1}^{N}(x_j - x_{j-1})\phi_\lambda(\mu_j, \mu_1)}{1 - Q_0(\mu_1)}. \tag{D.6}$$

It is also compatible with an auction with a reserve price in the sellers' constrained problem, where the reserve price is given by

$$r = -\frac{\sum_{j=1}^{N}(x_j - x_{j-1})\phi_\lambda(\mu_j, \mu_1)}{Q_1(\mu_1)}. \tag{D.7}$$

Hence, the directed search equilibrium is constrained efficient for any meeting technology.

Proof. In their relaxed problem, sellers select a queue $\boldsymbol{\mu}$ directly in a hypothetical competitive market. The expected payoff for a seller in this market is the difference between the surplus that he creates and the price of the queue. Suppose that a queue $\boldsymbol{\mu}$ solves sellers' relaxed problem. If queue $\boldsymbol{\mu}$ contains buyers of value x_k, then $T_k(\boldsymbol{\mu}) = U_k$, where $T_k(\boldsymbol{\mu})$ is given by equation (24); if queue $\boldsymbol{\mu}$ does not contain buyers of value x_k ($\mu_k = \mu_{k+1}$), then $T_k(\boldsymbol{\mu}) \leq U_k$.

Note that when a seller posts a second-price auction with entry fee and t is given by equation (21), note that $V_\iota = T_\iota(\boldsymbol{\mu})$, where ι is the lowest buyer value in queue $\boldsymbol{\mu}$. The important observation is that by equation (24) and (D.3), for all $k \geq 1$ we have $V_k = T_k(\boldsymbol{\mu})$, where V_k is the expected payoff of buyers with value x_k from the submarket and is given by equation (D.3). Thus $\boldsymbol{\mu}$ is also compatible with a second-price auction with entry fee t in the sellers' constrained problem.

The case with the reserve price is similar except for one difference. When a seller posts a second-price auction with entry fee t given by equation (21), $V_k = T_k(\boldsymbol{\mu})$ for $k \geq 1$. But when a seller posts a second-price auction with reserve price r given by equation (22), $V_\iota = T_\iota(\boldsymbol{\mu})$ where ι is the lowest buyer value in queue $\boldsymbol{\mu}$. For $k > \iota$, V_k is again given by equation (D.3) and we have $V_k = T_k(\boldsymbol{\mu})$. For $k < \iota$, things are slightly more complicated: For $r < x_k < x_\iota$, $V_k = Q_1(\mu_1)(x_k - r) = V_\iota - \phi_\mu(\mu_1, \mu_1)(x_\iota - x_k)$, which implies $V_k = T_k(\boldsymbol{\mu})$. For $x_k < r$, $V_k = 0$ and $T_k(\boldsymbol{\mu}) < 0$. In this case buyers with value x_k will not visit the submarket. Thus queue $\boldsymbol{\mu}$ is compatible with a second-price auction with reserve price r in the sellers' constrained problem.

The sellers' relaxed problem boils down to a competitive market for buyer types. Therefore, the first welfare theorem applies and the equilibrium is efficient. Since the sellers' constrained problem is equivalent to the sellers' relaxed problem, the directed search equilibrium is also efficient. $\square$

D.4 Queues Across Submarkets: Proof of Proposition 6

Again suppose that at the optimum, there are L active submarkets, and the lowest buyer type is $\underline{x}^\ell$ and the highest buyer type is $\overline{x}^\ell$. Suppose that the marginal contribution to surplus of x_k buyers is T_k^* for $k = 1, 2, \ldots, N$. Thus $T_k^* = \max(\max_{\ell=1,\ldots,L} T_k(\boldsymbol{\mu}^\ell), 0)$, where $T_k(\boldsymbol{\mu}^\ell)$ is the marginal contribution to surplus of x_k buyers in submarket ℓ and is given by equation (24). In the following we write it as T_k^ℓ to simplify notations. Of course, if submarket ℓ contains x_k buyers at the optimum, we must have $T_k^* = T_k^\ell$.

Step 1: Since $\phi(\mu, \lambda)$ is always concave in μ, by equation (24) T_k^ℓ is convex in x_k for each ℓ in the following sense: $\frac{T_2^\ell - T_1^\ell}{x_2 - x_1} \leq \frac{T_3^\ell - T_2^\ell}{x_3 - x_2} \leq \cdots \leq \frac{T_N^\ell - T_{N-1}^\ell}{x_N - x_{N-1}}$ (if buyer types are continuous, then we would have the usual notion of convexity). Since T_k^* is the maximum of a collection of convex functions, it is also convex.

Next, define k_0 as the largest index k such that $T_k^* = 0$, or set $k_0 = 1$ if $T_1^* > 0$. Then we show that T_k^* is strictly convex in the following sense: For $k > k_0$ we have

$$\frac{T_{k+1}^* - T_k^*}{x_{k+1} - x_k} > \frac{T_k^* - T_{k-1}^*}{x_k - x_{k-1}}.$$

To see this, note that since $T_k^* > 0$ for any $i > i_0$, buyers of value x_k must visit some submarket ℓ in which $\mu_k^\ell > \mu_{k+1}^\ell$, i.e. the queue length of buyers with value x_k must be strictly positive in the submarket. In this case, $\phi_\mu(\mu_k^\ell, \mu_1^\ell) < \phi_\mu(\mu_{k+1}^\ell, \mu_1^\ell)$, $T_k^\ell = T_k^*$ and $T_{k+1}^* \geq T_{k+1}^\ell$ and $T_{k-1}^* \geq T_{k-1}^\ell$, which implies that

$$\frac{T_{k+1}^* - T_k^*}{x_{k+1} - x_k} \geq \frac{T_{k+1}^\ell - T_k^\ell}{x_{k+1} - x_k} = \phi_\mu(\mu_{k+1}^\ell, \mu_1^\ell) > \phi_\mu(\mu_k^\ell, \mu_1) = \frac{T_k^\ell - T_{k-1}^\ell}{x_k - x_{k-1}} \geq \frac{T_k^* - T_{k-1}^*}{x_k - x_{k-1}}.$$

Hence we have showed that T_k^* is strictly convex when $i \geq i_0$.

Step 2: Claim: If $\mu_1^a > \mu_1^b$, then $\underline{x}^a \leq \underline{x}^b$ and $\overline{x}^a \leq \overline{x}^b$. To see this, recall that $Q_1(\mu_1) = \phi_\mu(\mu_1, \mu_1)$ by equation (2). By Assumption 3, $\mu_1^a > \mu_1^b$ implies $Q_1(\mu_1^b) > Q_1(\mu_1^a)$. Note that by equation (24), that $Q_1(\mu_1^\ell)$ is the slope of a supporting line (subgradient) for the function

T_k^ℓ and hence the function T_k^* at point $\underline{x}^\ell$ for $\ell \in \{a, b\}$. Because of the strict convexity of T_k^* (see Step 1 of the proof), the subgradient determines point $\underline{x}^\ell$ uniquely, and $Q_1(\mu_1^b) > Q_1(\mu_1^a)$ implies $\underline{x}^b \geq \underline{x}^a$.

Similarly, recall that $1 - Q_0(\mu_1) = \phi_\mu(0, \mu_1)$ by equation (2). By Assumption 3, $\mu_1^a > \mu_1^b$ implies $1 - Q_0(\mu_1^a) > 1 - Q_0(\mu_1^b)$. Note that $1 - Q_0(\mu_1^\ell)$ is a subgradient for the function T_k^ℓ and hence the function T_k^* at point $\overline{x}^\ell$ for $\ell \in \{a, b\}$. As before, the subgradient determines point $\overline{x}^\ell$ uniquely, and $1 - Q_0(\mu_1^a) < 1 - Q_0(\mu_1^b)$ implies $\overline{x}^a \leq \overline{x}^b$.

Step 3: Claim: Suppose a submarket ℓ contains buyers of x_{k_1} and x_{k_2} with $k_2 > k_1 + 1$, then it must also contain buyers in between, i.e. buyers of value x_k with $k_1 < k < k_2$. To see this, suppose otherwise (without loss of generality) that submarket ℓ contains no buyers with values between x_{k_1} and x_{k_2}. Then $\mu_{k_1+1} = \cdots = \mu_{k_2}$, which implies that T_k^ℓ is a linear function between x_{k_1} and x_{k_2}. We also know i) $T_k^\ell = T_k^*$ for $k = k_1$ and k_2, and ii) from Step 1 that T_k^* is a strictly convex function. The above two observations imply that $T_k^\ell > T_k^*$ for $k_1 < k < k_2$, which then leads to a contradiction.

Step 4: If $\overline{x}^a \leq \underline{x}^b$, then the proposition is true automatically. In the following, we will thus assume $\underline{x}^b < \overline{x}^a$. Therefore, we have $\underline{x}^a \leq \underline{x}^b < \overline{x}^a \leq \overline{x}^b$. We consider some x_k with $\underline{x}^b < x_k \leq \overline{x}^a$. By equation (24), $T_k^\ell = T_{k-1}^\ell + \phi_\mu(\mu_k^\ell, \mu_1^\ell)(x_k - x_{k-1})$ for $\ell \in \{a, b\}$. Note that x_k and x_{k-1} buyers visit both submarket a and b by Step 3, then $T_k^\ell = T_k^*$ and $T_{k-1}^\ell = T_{k-1}^*$ for $\ell = a$ or b. Therefore, we have

$$\phi_\mu\left(\mu_k^a, \mu_1^a\right) = \phi_\mu\left(\mu_k^b, \mu_1^b\right). \tag{D.8}$$

We then prove the claim by contradiction. Suppose that $\mu_k^b/\mu_1^b < \mu_k^a/\mu_1^a$ for $\underline{x}^b < x_k \leq \overline{x}^a$. This implies

$$\phi_\mu\left(\mu_k^a, \mu_1^a\right) < \phi_\mu\left(\mu_1^a \frac{\mu_k^b}{\mu_1^b}, \mu_1^a\right) < \phi_\mu\left(\mu_1^b \frac{\mu_k^b}{\mu_1^b}, \mu_1^b\right) = \phi_\mu\left(\mu_k^b, \mu_1^b\right),$$

where the first inequality is because $\phi\left(\mu, \mu_1^a\right)$ is strictly concave in μ and the second is because of Assumption 3. The above inequality is at odds with equation (D.8). Hence, we have reached a contradiction. □

References

Levin, D. and Smith, J. (1994). Equilibrium in auctions with entry. The American Economic Review, pages 585–599.

Myerson, R. B. (1981). Optimal auction design. Mathematics of Operations Research, 6(1):58–73.

Peters, M. (2013). Competing mechanisms. In Vulkan, N., Roth, A. E., and Neeman, Z., editors, The Handbook of Market Design, pages 609–628. Oxford University Press.

Riley, J. G. and Samuelson, W. F. (1981). Optimal auctions. The American Economic Review, 71(3):pp. 381–392.