Meetings and Mechanisms*

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Abstract

We analyze a market in which sellers compete for heterogeneous buyers by posting mechanisms. A general meeting technology governs how buyers and sellers meet. We introduce a one-to-one transformation of this meeting technology that helps to clarify and extend many of the existing results in the literature, which has focused on two special cases: urn-ball and bilateral meetings. We show that the optimal mechanism for sellers is to post auctions combined with a reserve price equal to their own valuation and an appropriate fee (or subsidy) which is paid by (or to) all buyers meeting the seller. Even when there are externalities in the meeting process, the equilibrium is efficient. Finally, we analyze the sorting patterns between heterogeneous buyers and sellers and show under which conditions high-value sellers attract more high-value buyers in expectation.

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1 Introduction

Real-life markets display a large degree of heterogeneity in the way in which economic agents meet and trade with each other: for example, in traditional bazaars, meetings between buyers and sellers tend to be bilateral; in real estate markets, multiple buyers may bid on the same house; and in labor markets, a typical vacancy receives a large number of applications but only interviews a subset.[1] Similarly, there is variation over time as the internet has made it easier for agents to meet multiple potential trading partners simultaneously; prominent examples of platforms utilizing this feature include eBay in the product market, Match.com in the dating market, CareerBuilder in the labor market, and Google AdWords in the market for online advertising.

Despite these observations, economic theory is mostly silent on the question how agents in these markets get to meet each other and how this meeting process affects equilibrium outcomes. This silence is most apparent in work that sidesteps a detailed description of the meeting process altogether by assuming a Walrasian equilibrium. Perhaps more surprisingly, the search literature—which aims to analyze trade in the absence of a Walrasian auctioneer—does not provide much more guidance: without much motivation, the vast majority of papers in this literature simply assumes one of two specific meeting technologies: either meetings between agents are one-to-one (bilateral meetings) or they are n-to-1, where n follows a Poisson distribution (urn-ball meetings).[2]

This approach seems restrictive for a number of reasons. First, neither bilateral meetings nor urn-ball meetings are necessarily an adequate description of real-life markets; in many cases, e.g. in the labor market example above, it appears necessary to consider alternatives. Second, assuming a particular meeting technology inevitably affects aggregate outcomes, even if the exact influence is not immediately obvious; examples presumably include equilibrium trading mechanisms, as e.g. auctions are more useful when there are a lot of bidders, or sorting patterns, as e.g. crowding out of high-type agents by low-type agents is a larger concern when meetings are bilateral.

In this paper, we aim to make progress by presenting a unified framework that allows for a wide class of meeting technologies. We do so in an environment in which a continuum of buyers with heterogeneous private valuations and a continuum of sellers try to trade. The

[1] See Geertz (1978) for a characterization of the market interaction at a bazaar, Han and Strange (2014) for empirical evidence on bidding wars in real estate markets, and Wolthoff (2017) for evidence on applications and interviews in the labor market.

[2] Bilateral meetings can be found in e.g. Albrecht and Jovanovic (1986), Moen (1997), Guerrieri et al. (2010), and Menzio and Shi (2011). Urn-ball meetings are used in e.g. Peters (1997), Burdett et al. (2001), Shimer (2005), Albrecht et al. (2014) and Auster and Gottardi (2017). In addition, some papers in the mechanism design literature explore urn-ball meetings in a finite market, making n binomial rather than Poisson, by allowing for entry of buyers into a monopolistic auction (Levin and Smith, 1994).

class of meeting technologies that we consider allows for all sorts of externalities (positive, negative or zero) between agents and includes both the bilateral and the urn-ball meeting technology as special cases. This allows us to not only clarify existing results in the literature but to also analyze which of them carry over to our more general setting.

We start by analyzing the case in which all sellers are homogeneous. We find that each seller cannot do better than posting a second-price auction, combined with a meeting fee to be paid by (or to) each buyer meeting him.[3] The meeting fee determines the endogenous distribution of buyers that the seller attracts, may vary across sellers in equilibrium and ensures that the equilibrium is constrained efficient. Intuitively, in a large market, sellers take buyers' equilibrium payoffs as given, making sellers the residual claimant on any extra surplus that they create and providing them with an incentive to maximize this surplus. Auctions guarantee that the good is allocated efficiently, while the meeting fees price any positive or negative externalities in the meeting process. As a result, all agents receive a payoff equal to their social contribution, which is a crucial requirement for efficiency of the equilibrium.[4] We establish that although other equilibria may exist, these equilibria are payoff-equivalent to the one with auctions and meeting fees, as long as a standard assumption on sellers' out-of-equilibrium beliefs is satisfied: sellers are optimistic in the sense that if there exist multiple solutions to the market utility condition governing their beliefs, then they expect the solution that maximizes their payoff.[5] Subsequently, we strengthen our result by demonstrating that this assumption is in fact redundant under a few weak restrictions on the meeting technology.

In the final part of the paper, we consider the case in which sellers are heterogeneous as well. In this environment, there is scope for sorting and a natural question is under which conditions high-valuation buyers visit higher-valuation sellers (in expectation) and whether this is desirable. We derive conditions on the meeting technology for assortative sorting in meetings which also implies assortative sorting in matching. We further show that our existence, uniqueness and efficiency results carry through for this case.

After this brief outline, we now discuss some of our contributions in more detail. By studying an environment with both private information and competition between sellers, we contribute to the literature that lies at the intersection of search theory and mechanism design. In this work, the number of bidders and their distribution of valuations are equilibrium objects that depend on the mechanism that the seller posted. Following the pioneering

[3]If the meeting fee is positive, it is equivalent in our framework to a participation fee or a bidding fee, as used by e.g. Sothebys.com for certain auctions. For housing auctions in the UK, participating buyers are sometimes required to pay a fee.

[4]For detailed discussions regarding the relation between the division of surplus and efficiency in search models, see, e.g., Mortensen (1982), Hosios (1990), Moen (1997), Albrecht et al. (2014) and Lester et al. (2017).

[5]This assumption is used by e.g. McAfee (1993), Eeckhout and Kircher (2010a,b) and Auster and Gottardi (2017).

work of McAfee (1993), Peters (1997) and Peters and Severinov (1997), this literature has generally focused on urn-ball meetings, with Albrecht et al. (2014) being a recent example. Eeckhout and Kircher (2010b) were the first who emphasized the importance of the meeting technology, but they derive the equilibrium only for a subset of technologies, while we provide results for a wide class. Moreover, they consider two buyer types while we allow for arbitrary distributions of buyer valuations. Lester et al. (2015) provide a full characterization of the equilibrium, but in a simpler environment in which all buyers are ex ante identical. Cai et al. (2017) apply the tools that are developed in this paper to derive conditions on the meeting technology for which the equilibrium features either perfect separation or perfect pooling of different types of buyers, and they relate those conditions to other properties of meeting technologies that have been derived in the literature, like invariance (Lester et al., 2015) and non-rivalry (Eeckhout and Kircher, 2010b).

The equilibrium mechanism that we identify includes both regular auctions (when the meeting fee is zero, e.g. when meetings are urn-ball) and posted prices (when meetings are bilateral) as special cases. In other words, varying the degree of search frictions in our model changes what the optimal mechanism looks like. This interaction contrasts with much of the search literature (with the exception of some of the above papers), which assumes that the trading mechanism (e.g bilateral bargaining) is independent of the frictions. However, it corresponds well with what we observe in real-life. For example, as soon as eBay provided a platform for sellers and buyers to meet, auctions quickly gained popularity for the sale of e.g. second-hand products.[6] Similar changes can be observed in the market for freelance services, where new platforms like Upwork (previously oDesk) or Freelancer enable employers from high-income countries to outsource tasks to contractors from mainly low-income countries (see for a detailed description Agrawal et al., 2015).[7] These online platforms facilitate many-to-one meetings (also for small firms), creating scope for wage mechanisms other than bilateral bargaining. In particular, contractors apply to posted jobs by submitting a cover letter and a bid indicating the compensation that they demand for the job, after which procurers select

[6]Lucking-Reiley (2000) presents various statistics regarding the growing popularity of online auctions in the late 1990s. Einav et al. (2017) argue that in recent years the popularity of auctions on eBay has declined relative to posted prices, which they explain by an increase in the *hassle cost* associated with purchasing in an auction (see Backus et al. (2015) for a particular example of such a cost). However, it is worth emphasizing that their study restricts attention to cases in which a seller sells multiple units of the same product (mostly retail items). They acknowledge that auctions remain the trading mechanism of choice for most sellers with a single unit, which is the case that we consider here. Note further that various other platforms, e.g. Catawiki.com, continue to exclusively use auctions. In order to highlight the role of meeting technologies, we therefore abstract from hassle costs here.

[7]Although still relatively new, these platforms already have a substantial impact on this market. The number of hours worked at Upwork increased by 55% between 2011 and 2012, with the 2012 total wage bill being more than 360 million dollar. A 2014 New York Times article states: "It's also helping to raise the standard of living for workers in developing countries. The rise of these marketplaces will increase global productivity by encouraging better matching between employers and employees." (Korkki, 2014).

one of the applicants. These examples nicely illustrate how a new technology can affect the meeting process and how the market responds by adjusting the wage mechanism accordingly.

Our efficiency result contributes to the literature on directed search. In particular, it extends the result by Albrecht et al. (2014) that all agents earn their marginal contribution to surplus in the special case in which meetings are urn-ball and sellers post regular auctions.[8] In that environment, there are no meeting externalities, so a buyer contributes to surplus only if he has the highest valuation among all buyers meeting a seller. The increase in surplus is the difference between his valuation and the next highest valuation, which is exactly the payoff that he receives in an auction, known as his information rent. In contrast, a seller posting an auction affects surplus in two ways. By providing a new trading place, he creates a surplus equal to the maximum valuation among the buyers that he attracts. However, in the seller's absence, these buyers would have contributed to surplus at other sellers; Albrecht et al. (2014) label this effect a "business stealing externality." In a large market, the probability that two or more of these buyers initially visited the same seller is zero, so the magnitude of this externality is exactly buyers' marginal contribution to surplus or payoff at those other auctions.[9] Hence, a seller's net contribution to surplus is the difference between the highest valuation and this externality. This difference is on average equal to the second-highest valuation and is therefore precisely the payoff that the seller receives from the auction.

Now, return to general meeting technologies. This case is more complicated because now a buyer can also impose positive or negative meeting externalities on meetings between the seller and other buyers, which should be reflected in the equilibrium payoffs. In particular, if buyers create negative (positive) externalities by visiting a seller, then their payoffs should be decreased (increased) relative to the urn-ball case, while the seller's payoffs should be increased (decreased), as he reduces the negative (positive) externalities for other sellers by stealing some of their buyers. We show that an appropriate meeting fee/subsidy, depending on the number and types of buyers that a seller attracts in equilibrium, achieves this goal.[10] As a result, all agents continue to receive their marginal contribution to surplus and efficiency survives.

Finally, we also make a methodological contribution. In particular, we introduce an alternative representation of meeting technologies which keeps the analysis tractable. This

[8] Although we assume a fixed number of sellers to simplify exposition, our results carry over to an environment with free entry of sellers, as in Albrecht et al. (2014), in a straightforward manner.

[9] In other words, removing a single buyer from an auction does not change the *sum* of other agents' payoffs from that auction. As an example, suppose the valuations are 0.4, 0.7, 0.9 and 1. Removing the highest bidder decreases the seller's payoff by $0.9 - 0.7 = 0.2$, but this is exactly what the buyer with value 0.9 gains. This is not the case if two buyers leave: removing 0.9 and 1 reduces the seller's payoff by $0.9 - 0.4 = 0.5$, while the winning buyer gets $0.7 - 0.4 = 0.3$. Hence, in a finite market, efficiency is not obtained.

[10] In other words, the fee can vary across sellers in equilibrium. This fact is a key difference with Lester et al. (2015), where the fee is the same for all sellers as it only depends on exogenous parameters.

representation is the probability ϕ that a seller meets at least one buyer from a given subset, e.g. the set of buyers with a valuation above a certain threshold. This probability depends on two arguments: the total queue length λ that the seller faces as well as the queue of buyers μ belonging to the subset. We show that using ϕ instead of the more standard representation of meeting technologies offers a few important advantages. First, the partial derivatives of ϕ have natural interpretations corresponding to key variables such as a buyer's trading probability and the degree of meeting externalities. Second, expected surplus is linear in ϕ, which makes it straightforward to relate the objective of a planner to properties of ϕ.[11] Finally, the use of ϕ guarantees that the expression for a seller's payoff retains a similar structure as in the seminal work by Myerson (1981), i.e. as the integral of buyers' virtual valuation with respect to the distribution of highest valuations, with the difference that this distribution now also depends on how likely each buyer is to meet a seller which in turn depends on the meeting technology. In other words, the introduction of ϕ adds a lot of generality to the competing mechanism literature at relatively low cost.

After describing the environment in detail in section 2 and the alternative representation of the meeting technology in section 3, we start our analysis in section 4 by solving the problem of a social planner. Section 5 discusses how the planner's solution can be decentralized and provides a characterization of the equilibrium. Finally, section 6 introduces two-sided heterogeneity and discusses sorting patterns. Proofs are relegated to the appendix.

2 Model

Before we provide a precise description of the details of the model, we give a brief overview of the problem here. The model is static and has two stages. First, sellers post a selling mechanism, and then after observing all selling mechanisms, buyers decide which seller to visit, subject to meeting frictions. Our main objectives are to derive which selling mechanisms will be preferred in equilibrium, to characterize the allocation of buyers across sellers, and to establish whether or not the decentralized equilibrium is efficient.

Agents and Preferences. The economy consists of a measure 1 of sellers, indexed by $j \in [0, 1]$, and a measure $\Lambda > 0$ of buyers. Both buyers and sellers are risk-neutral. Each seller possesses a single unit of an indivisible good, for which each buyer has unit demand. Initially, we will assume that all sellers have the same valuation for their good, which we normalize to zero; later, in Section 6, we will consider seller heterogeneity. Buyers have a valuation between 0 and 1, and the buyer value distribution is denoted by $G(x)$ with $0 \leq x \leq 1$ and $G(0) < 1$. Buyers' valuations are private information and the market is anonymous in

[11] Cai et al. (2017) exploit this feature in their work.

the sense that buyers and sellers cannot condition their strategies on the identities of their counterparties.

Mechanisms. In the first stage, each seller posts and commits to a direct anonymous mechanism to attract buyers. The mechanism specifies, for each buyer i, a probability of trade and an expected payment as a function of: (i) the total number n of buyers that successfully meet with the seller; (ii) the valuation x_i that buyer i reports; and (iii) the valuations x_{-i} reported by the $n-1$ other buyers.[12]

Search. We refer to all identical mechanisms as a *submarket*. After observing all submarkets, each buyer chooses the one in which he wishes to attempt to match. Because we consider a large market, we assume that buyers can not coordinate their visiting strategies, such that buyers must use symmetric strategies in equilibrium; this is a standard assumption in the literature (see e.g. Montgomery, 1991; Burdett et al., 2001; Shimer, 2005).

Meeting Technology. Consider a submarket with a measure b of buyers and a measure s of sellers. The meetings between buyers and sellers are frictional and governed by a *meeting technology*, which we model analogous to Eeckhout and Kircher (2010b). The meeting technology is anonymous; it treats all buyers (sellers) in a symmetric way, i.e., independent of their identity. A buyer can meet at most one seller, while a seller may meet multiple buyers. Define $\lambda = b/s$ as the *queue length* in this submarket.[13] The probability of a seller meeting n buyers, $n = 0, 1, 2, \ldots$, is given by $P_n(\lambda)$, which is assumed to be continuously differentiable. Because each buyer can meet at most one seller, $\sum_{n=1}^{\infty} nP_n(\lambda) \leq \lambda$. By an accounting identity, the probability for a buyer to be part of an n-to-1 meeting is $Q_n(\lambda) \equiv nP_n(\lambda)/\lambda$ with $n \geq 1$. Finally, the probability that a buyer fails to meet any seller is $Q_0(\lambda) \equiv 1 - \sum_{n=1}^{\infty} Q_n(\lambda)$. [14]

Strategies. Let M be the set of all direct anonymous mechanisms equipped with some natural σ-algebra $\mathcal{M}$. A seller's strategy is a probability measure δ_s on $(M, \mathcal{M})$. A buyer needs to decide on whether or not to participate in the market, and if yes, which sellers (who are characterized by the mechanisms they post) to visit. To acknowledge that a buyer's strategy depends (only) on his value x and the fact that—due to the lack of coordination—buyers treat all sellers who post the same mechanism symmetrically, we denote his strategy

[12] In line with most of the literature, we abstract from mechanisms that condition on other mechanisms present in the market. See Epstein and Peters (1999) and Peters (2001) for a detailed discussion.

[13] For simplicity, we assume here that a positive measure of buyers and sellers visit the submarket. This need not be the case; e.g. the economy could have a continuum of submarkets with each a measure zero of buyers and sellers. In that case, we could use Radon-Nykodym derivatives to define queue lengths.

[14] It is straightforward to allow buyers to observe only a fraction of the sellers. If the fraction of sellers that a buyer observes is type independent, this will not change our results.

by $\delta_b(x,\cdot)$, a measure on $(M,\mathcal{M})$. If $\delta_b(x,M)<1$, then buyers with value x will choose not to participate in the market with probability $1-\delta_b(x,M)$, in which case their payoff will be zero.[15] The allocation of all buyers with value less or equal to x across posted mechanisms can be formally denoted as a measure $\Psi(x,\cdot)$ on $\mathcal{M}$. Individual strategies and the aggregate allocation satisfy, for any measurable subset $N \in \mathcal{M}$,

$$\Psi(x,N)=\int_0^x \delta_b(y,N)dG(y).$$

Since a buyer can only visit a mechanism if a seller posted it, we require that for each x, the measure $\Psi(x,\cdot)$ is absolutely continuous with respect to δ_s.[16] The Radon-Nikodym derivative $d\Psi(x,\cdot)/d\delta_s$ determines the queue length and queue composition—i.e., how many buyers and what types of buyers—for each mechanism (almost surely) in the support of δ_s. Formally, for (almost every) mechanism ω in the support of δ_s, the queue length $\lambda(\omega)$ and queue composition $F(x,\omega)$ are given by

$$\lambda(\omega)F(x,\omega)=\frac{d\Psi(x,\cdot)}{d\delta_s}. \tag{1}$$

Payoffs. Note that for any mechanism $\omega \in M$, the expected payoff of a seller who posts mechanism ω is completely determined by ω and its queue length $\lambda(\omega)$ and queue composition $F(x,\omega)$. Therefore, we can denote it by $R(\omega,\lambda(\omega),F(x,\omega))$. Similarly, let $V(z,\omega,\lambda(\omega),F(x,\omega))$ denote the expected payoff of a buyer with value z from visiting a submarket with mechanism ω which has queue length $\lambda(\omega)$ and queue composition $F(x,\omega)$.[17]

Market Utility and Beliefs. We now define conditions on buyers' and sellers' strategy (δ_s,δ_b) which need to be satisfied in equilibrium. First, consider the optimality of buyers' strategies. The *market utility function* $U(z)$ is defined to be the maximum utility that a buyer with value z can obtain by visiting a seller or being inactive.

$$U(z)=\max\left(\max_{\omega\in \operatorname{supp}(\delta_s)} V(z,\omega,\lambda(\omega),F(x,\omega)),0\right).$$

[15]We assume that sellers always post a selling mechanism. This is without loss of generality, since sellers can stay inactive by posting a sufficiently inattractive selling mechanism, e.g. a reserve price above 1.

[16]This rules out the scenario in which a zero measure of sellers attracts a positive measure of buyers. This restriction is natural and can be justified by the optimal choices of buyers and sellers (see below).

[17]$R(\omega,\lambda(\omega),F(x,\omega))$ can be calculated as $\sum_{n=1}^{\infty}P_n(\lambda)R_n(\omega,F(x,\omega))$, where $R_n(\omega,F(x,\omega))$ denotes the expected payoff of the seller when n buyers arrive. $V(z,\omega,\lambda(\omega),F(x,\omega))$ can be calculated in a similar way.

where $\lambda(\omega)$ and $F(x,\omega)$ follow from equation (1). Of course, optimality of buyers' choices requires that buyers choose the mechanism that yields the highest payoff. Formally, we have

$$V(x,\omega,\lambda(\omega),F(x,\omega)) \leq U(x) \quad \text{with equality if } \omega \text{ is in the support of } \delta_b(x,\cdot).$$

Next we consider the optimality of sellers' strategies. All posted mechanisms should generate the same expected payoff π^* and there should be no profitable deviations. That is,

$$\pi(\omega,\lambda(\omega),F(x,\omega)) \leq \pi^* \quad \text{with equality if } \omega \text{ is in the support of } \delta_s$$

A seller considering a deviation to a mechanism $\widetilde{\omega}$ not in the support of δ_s needs to form beliefs regarding the queue $(\lambda(\widetilde{\omega}),F(x,\widetilde{\omega}))$ that he will be able to attract. We call a queue $(\widetilde{\lambda},\widetilde{F}(x))$ *compatible* with the mechanism $\widetilde{w}$ and the market utility function $U(x)$ if for any z,

$$V(z,\widetilde{\omega},\widetilde{\lambda},\widetilde{F}(x)) \leq U(z) \quad \text{with equality if } z \text{ is in the support of } F(x). \tag{2}$$

Of course, for any mechanism ω in the support of δ_s, $(\lambda(\omega),F(x,\omega))$ is compatible with mechanism ω and the market utility function because of the optimal search behavior of buyers. The literature usually assumes that when posting $\widetilde{w}$, the seller will expect the most favorable queue among all queues that are compatible with $\widetilde{\omega}$ and the market utility function (see, for example, McAfee, 1993; Eeckhout and Kircher, 2010a,b). That is,

$$(\lambda(\widetilde{\omega}),F(x,\widetilde{\omega})) = \underset{\widetilde{\lambda},\widetilde{F}(x)}{\arg\max}\ R(\widetilde{\omega},\widetilde{\lambda},\widetilde{F}(x)) \tag{3}$$

where the choice of $(\widetilde{\lambda},\widetilde{F}(x))$ is subject to the constraint of equation (2). Initially, we will adopt this convention, but later we will show that—with some mild restrictions on the meeting technology—this assumption is unnecessary: when $\widetilde{\omega}$ is (without loss of generality) an auction with entry fee, these restrictions imply that there is only one possible queue compatible with $\widetilde{\omega}$ and the market utility function.

Equilibrium Definition. We can now define an equilibrium as follows.

Definition 1. *A directed search equilibrium is a pair (δ_s,δ_b) of strategies with the following properties:*

1. *Each $\widetilde{\omega}$ in the support of δ_s maximizes $\pi(\omega,\lambda(\omega),F(x,\omega))$, where, depending on whether or not ω belongs to the support of δ_s, $\lambda(\omega)$ and $F(x,\omega)$ are given by equations* (1) *and* (3)*, respectively.*

2. *For each buyer type z, every $\widetilde{\omega}$ in the support of $\delta_b(z,\cdot)$ maximizes $V(z,w,\lambda(\omega),F(x,\omega))$. If for any mechanism ω in the support of δ_s the buyer value $V(z,\omega,\lambda(\omega),F(x,\omega))$ is negative, then buyers with value z will choose inactivity and $\delta_b(z,M)=0$.*

3. *Aggregating queues across sellers does not exceed the total measure of buyers of each type.*

3 Alternative Representation

In this section, we present a transformation of the meeting technology that greatly simplifies the analysis. In particular, we introduce a new function $\phi(\mu,\lambda)$ with $0\leq\mu\leq\lambda$, which is defined as

$$\phi(\mu,\lambda)=1-\sum_{n=0}^{\infty}P_n(\lambda)\left(1-\frac{\mu}{\lambda}\right)^n. \tag{4}$$

To understand this function, consider a submarket in which sellers face queues of length λ. Suppose now that a fraction μ/λ of the buyers in the submarket has an arbitrary characteristic, e.g. we color them "blue." Since the meeting technology treats different buyers symmetrically, $\phi(\mu,\lambda)$ then represents the probability that a seller meets at least one blue buyer.

In many situations, by choosing "blue buyers" as buyers with valuations above some level, the function ϕ allows us to study competing mechanisms with general meeting functions in a way that is both more tractable and more intuitive than with $P_n(\lambda)$, $n=0,1,\ldots$. The following Proposition establishes that the function ϕ is an equivalent way of characterizing frictions in the market. That is, no information is lost by considering ϕ instead of P_n.

Proposition 1. *There is a one-to-one relationship between $\phi(\mu,\lambda)$ and $P_n(\lambda)$, $n=0,1,2,\ldots$.*

Proof. See appendix A.1. □

To develop intuition for $\phi(\mu,\lambda)$, consider a submarket in which a measure μ of buyers has high valuations, while the remaining measure $\lambda-\mu$ has low valuations. If $\Delta\lambda$ more buyers visit this submarket, then the probability that the seller meets at least one *incumbent* high-value buyer becomes $\phi(\mu,\lambda+\Delta\lambda)$. Therefore, $\phi_\lambda(\mu,\lambda)$ measures the effect of the new entrants on the meeting probabilities between sellers and incumbent high-value buyers: $\phi_\lambda(\mu,\lambda)<0$ (resp. >0) represents negative (resp. positive) meeting externalities. In the special case of $\phi_\lambda(\mu,\lambda)=0$, there is no meeting externalities among buyers.

For future reference, note that

$$\phi_\mu(\mu,\lambda) = \sum_{n=1}^{\infty} Q_n(\lambda)\left(1-\frac{\mu}{\lambda}\right)^{n-1}. \tag{5}$$

That is, $\phi_\mu(\mu,\lambda)$ is the probability for a buyer to be part of a meeting in which all other buyers (if any) have low valuations. In this case, if the buyer has a high valuation, then he increases social surplus directly, since the good would have been allocated to a low-value buyer in his absence. It is easy to see that $\phi_\mu(\mu,\lambda)$ is decreasing in μ, implying that $\phi(\mu,\lambda)$ is concave in μ, which holds strictly if and only if $P_0(\lambda)+P_1(\lambda)<1$.[18] Two special cases of equation (5) are worth mentioning: i) $\phi_\mu(0,\lambda)=1-Q_0(\lambda)$, i.e. the probability that a buyer meets a seller, and ii) $\phi_\mu(\lambda,\lambda)=Q_1(\lambda)$, i.e. the probability that a buyer meets a seller without other buyers.

Examples of Meeting Technologies.

1. *Bilateral.* With bilateral meeting technologies, each seller meets at most one buyer, i.e., $P_0(\lambda)+P_1(\lambda)=1$ with $P_1(\lambda)$ strictly concave. In this case, $\phi(\mu,\lambda)=P_1(\lambda)\mu/\lambda$.[19]

2. *Invariant.* Invariant meeting technologies are defined by the absence of meeting externalities, i.e. $\phi_\lambda(\mu,\lambda)=0$ for any $0\le\mu\le\lambda$.[20] One example is the urn-ball meeting technology, which specifies that the number of buyers meeting a seller follows a Poisson distribution with a mean equal to the queue length λ. That is, $P_n(\lambda)=e^{-\lambda}\lambda^n/n!$, which yields $\phi(\mu,\lambda)=1-e^{-\mu}$.

3. *Non-Rival.* Eeckhout and Kircher (2010b) define a meeting technology to be non-rival if $Q_0(\lambda)=q$, where q is a constant, i.e., the probability that a buyer successfully meets a seller is not affected by the presence of other buyers. From equation (5), we can see that non-rival meeting technologies can also be defined by the condition $\phi_\mu(0,\lambda)=1-q$ for any λ, or equivalently $\phi_{\mu\lambda}(0,\lambda)=0$.

   Note that non-rival meeting technologies are very general because any meeting technology can be approximated arbitrarily closely by a non-rival meeting technology in the following sense. Start with any meeting technology, e.g. the bilateral technology. If some buyers fail to meet sellers, we let them meet with an arbitrary small measure of

[18] For each $n\ge0$, $-(1-\mu/\lambda)^n$ is increasing and concave in μ, and it is strictly concave in μ if and only if $n\ge2$. Therefore, $\phi(\mu,\lambda)$ is strictly concave in μ if and only if there exists at least one $n\ge2$ such that $P_n(\lambda)>0$.

[19] To keep the exposition concise, we omit the (straightforward) derivation of $\phi(\mu,\lambda)$ for each example.

[20] Lester et al. (2015) first introduced invariant meeting technologies in terms of $P_n(\lambda)$. Cai et al. (2017) show that their definition is equivalent to $\phi_\lambda(\mu,\lambda)=0$.

sellers who were set aside initially according to a non-rival meeting technology, like urn-ball. The meeting technology obtained from the above two-stage process is non-rival since every buyer will meet a seller for sure ($Q_0(\lambda) = 0$). By making the measure of sellers in the second step close to zero, the resulting technology can be made arbitrarily close to the original one, while remaining non-rival.[21]

4 Planner's Problem

Given the above environment, the problem of a social planner consists of two parts. First, the planner must allocate buyers and sellers to submarkets. That is, he must determine the queue length and composition (i.e., the buyer value distribution) for each seller. Second, the planner must specify the allocation of the good after meetings have taken place. We focus on the first part below, since the second part is trivial: the planner will always allocate the good to the buyer with the highest value.

Surplus. We start by deriving total surplus and agents' marginal contribution to this surplus in a submarket with queue length λ and a queue composition $F(x)$. Proposition 2 presents the results, suppressing the argument $(\lambda(1 - F(z)), \lambda)$ in the function ϕ and its partial derivatives to enhance readability.

Proposition 2. *Consider a submarket with a measure 1 of sellers and a measure λ of buyers whose values are distributed according to $F(x)$. Total surplus then equals*

$$S(\lambda, F) = \int_0^1 \phi dz. \tag{6}$$

The marginal contribution to surplus by a buyer with valuation z equals

$$T(x, \lambda, F) = \int_0^x \phi_\mu dz + \int_0^1 \phi_\lambda dz. \tag{7}$$

A seller's marginal contribution to surplus equals

$$R(\lambda, F) = \int_0^1 (\phi - \lambda(1 - F(z))\phi_\mu - \lambda\phi_\lambda) dz. \tag{8}$$

Proof. See appendix A.2. □

When there is no risk of confusion, we will suppress the arguments λ and F from the functions $S(\lambda, F), T(x, \lambda, F)$, and $R(\lambda, F)$. The first term of $T(x)$ reflects a buyer's direct

[21] Cai et al. (2017) introduce an example of such a technology.

contribution to surplus when he has the highest value in an n-to-1 meeting, i.e., the difference between the highest and the second highest buyer values. The second term of $T(x)$ represents positive or negative search externalities that the buyer may impose on other buyers; it does not depend on x, because the meeting friction treats all buyers symmetrically. In particular, if a buyer makes it easier for other buyers to meet a seller ($\phi_\lambda \geq 0$), he increases total surplus through a positive meeting externality, even if he does not have the highest value among these buyers. Similar logic applies to a negative meeting externality ($\phi_\lambda \leq 0$). Finally, since total surplus exhibits constant returns to scale, Euler's homogeneous function theorem implies that a seller's marginal contribution equals $R = S - \lambda \int_0^1 T(x)dF(x)$.

Participation. The above expressions allow us to now address the planner's participation decisions. In particular, the following Lemma characterizes under which conditions the planner wants either all buyers or all sellers to be active.

Lemma 1. *If $\phi_\lambda(\mu, \lambda) \geq 0$ (≤ 0 resp.) for all $0 < \mu < \lambda$, then the planner will require all buyers (sellers resp.) to be active in the market.*

Proof. See appendix A.3. □

Intuitively, as long as buyers do not negatively affect the meeting rate of other buyers, they should be included in the market. In contrast, if they do negatively affect other buyers, then the planner will include as many sellers as possible in order to mitigate this negative externality.

Allocation. Next, we consider the allocation of buyers to different submarkets. To simplify notation and deliver an upper bound on the number of submarkets, assume that the number of different buyer types is finite. To be precise, suppose that there are n buyer types with values $x_1, x_2, \ldots, x_n$, satisfying $x_1 < x_2 < \cdots < x_n$, and measures $b_1, b_2, \ldots, b_n$, respectively.

Consider now a submarket i in which there is a positive measure of sellers, such that the queue length is well defined. Let the queue in this submarket be $(\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i)$, where λ_j^i is the number of buyers with value x_j per seller. Then, by Proposition 2, total surplus per seller in this submarket can be written as

$$S\left(\sum_{j=1}^{n} \lambda_j^i, F^i\right) = \sum_{j=1}^{n} (x_j - x_{j-1})\phi(\lambda_j^i + \cdots + \lambda_n^i, \lambda_1^i + \ldots + \lambda_n^i), \tag{9}$$

where $x_0 \equiv 0$, $\sum_{j=1}^{n} \lambda_j^i$ is the queue length, and F^i is the buyer value distribution in the submarket, describing that $x = x_j$ with probability $\lambda_j^i / \sum_{j=1}^{n} \lambda_j^i$. For distributions with discrete support, we will often slightly abuse notation and write $S(\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i)$ instead of $S(\sum_{j=1}^{n} \lambda_j^i, F^i)$ as it is more convenient.

To understand equation (9), start from $n = 1$. In this case, all buyers are homogeneous and a surplus of x_1 is generated whenever a seller meets at least one buyer, i.e. surplus is simply $x_1\phi(\lambda_1^i, \lambda_1^i)$. When $n = 2$ and some buyers have a higher value x_2, the additional surplus is $x_2 - x_1$. This surplus is realized when sellers meet at least one buyer with value x_2. Hence, total surplus is $x_1\phi(\lambda_1^i + \lambda_2^i, \lambda_1^i + \lambda_2^i) + (x_2 - x_1)\phi(\lambda_2^i, \lambda_1^i + \lambda_2^i)$. For general n, the interpretation is the same.

Suppose now that the planner creates k submarkets with positive seller measures $\alpha_1, \dots, \alpha_k$, respectively, and potentially an additional submarket with no sellers but only buyers. Of course, this additional submarket generates no surplus but could play a role in reducing possible meeting externalities. The planner's problem is thus

$$\mathcal{S}(b_1, \dots, b_n) = \sup_{\alpha_1,\dots,\alpha_k,\lambda_1,\dots,\lambda_k} \sum_{i=1}^{k} \alpha_i S(\lambda_1^i, \lambda_2^i, \dots, \lambda_n^i) \tag{10}$$

subject to the standard accounting constraint

$$\sum_{i=1}^{k} \alpha_i = 1, \tag{11}$$

for sellers, and

$$\sum_{i=1}^{k} \alpha_i \lambda_j^i \leq b_j. \tag{12}$$

for each buyer type $j = 1, 2, \dots, n$. Note that in equation (12) we have an inequality rather than an equality. The reason is that the planner may require some buyers not to visit any seller and thus be inactive.[22]

We define an *idle* submarket as a market that either contains only buyers or only sellers and an *active* submarket as a market where both buyers and sellers are present. Of course, the planner will never prefer coexistence of two idle markets, one for buyers and one for sellers. The following Proposition limits the number of submarkets.

Proposition 3. *The planner's problem can be solved by opening at most $n + 1$ submarkets, including one potentially idle submarket.*

Proof. See appendix A.4. □

The intuition behind Proposition 3 is the following. By equation (10), total surplus is a convex combination of the surpluses generated by individual submarkets. The planner

[22]In the k submarkets with positive seller measure, the queue length is finite and well defined. In contrast, the queue length is infinite and not formally defined in a submarket with only buyers and no sellers.

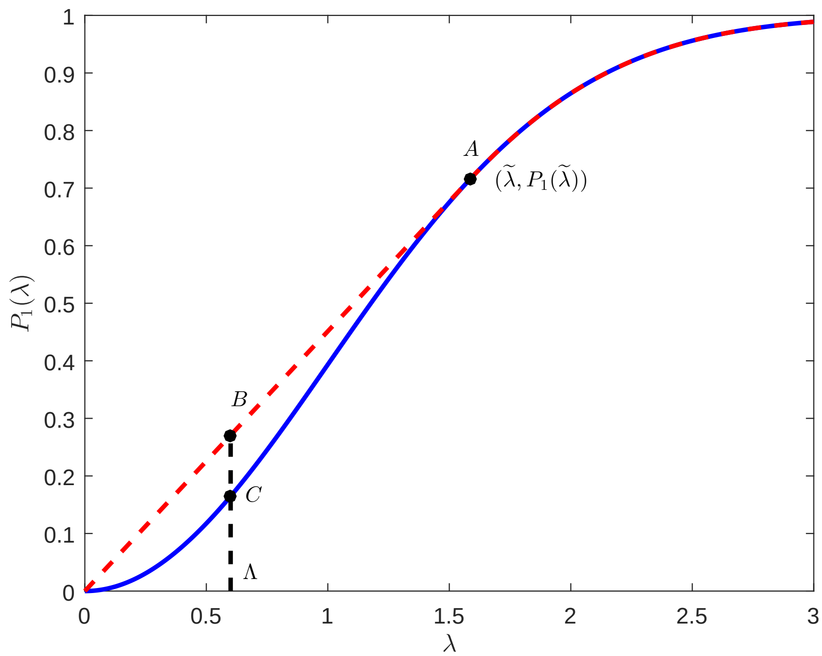

Figure 1: Illustration of Proposition 3

chooses the number of submarkets to find the maximum value that such convex combinations can reach, which simply corresponds to finding the concave hull of the individual submarket surplus function S as presented in equation (9). As a result of this correspondence, the Fenchel-Bunt Theorem provides an upper bound for the number of submarkets needed to solve the planner's problem.[23]

Illustration. As an illustration, consider the simple case in which all buyers are homogeneous and have value 1, $P_0(\lambda) = e^{-\lambda^2/2}$, and $P_1(\lambda) = 1 - P_0(\lambda)$. It is easy to see that $P_1(\lambda)$ is not globally concave, as the solid line in Figure 1 indicates. The concave hull of the function $P_1(\lambda)$ is the dashed line, which consists of two parts: a line segment between the origin and $(\widetilde{\lambda}, P_1(\widetilde{\lambda}))$ and the original function $P_1(\lambda)$ from $\widetilde{\lambda}$ onwards. The point $\widetilde{\lambda}$ is characterized by the condition that the slopes of the original line and the tangent line of the function $P_1(\lambda)$ are equal at $\widetilde{\lambda}$. Proposition 3 says that any point on the dashed line is a convex combination of *two* points on the solid line. If the total buyer measure equals $\Lambda < \widetilde{\lambda}$, then the optimal allocation is point B instead of point C, which implies that the planner will keep $1 - \Lambda/\widetilde{\lambda}$

[23]The classical Caratheodory theory states that any point in the convex hull of a set $A \subset \mathbb{R}^n$ can be represented as a convex combination of $n+1$ points of A. The Fenchel-Bunt Theorem states that if the set A is connected, then for the above construction we only need n points instead of $n+1$. Since the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a connected subset in $\mathbb{R}^{n+1}$, the Fenchel-Bunt Theorem implies that we only need $n+1$ points to construct the concave hull of f.

sellers inactive (i.e. create an idle submarket) and send the buyers and the rest of the sellers to a single submarket, where the queue length is $\widetilde{\lambda}$. If the total buyer measure equals $\Lambda \geq \widetilde{\lambda}$, then the optimal allocation is to simply assign all buyers and sellers to the same submarket.

Characterization. Although Proposition 3 shows that the planner can maximize the social surplus by opening no more than $n+1$ submarkets, it provides no characterization of how queues will vary across submarkets. To address this question, we will show below that the planner's solution can be decentralized by sellers posting an auction with an entry fee, and we will characterize how queues of different submarkets vary with respect to the entry fee.

5 Decentralized Market Equilibrium

In this section, we show that the solution to the planner's problem coincides with a directed search equilibrium in which sellers compete with mechanisms. No seller can do better than posting a second-price auction combined with a meeting fee to be paid by each buyer meeting him. A negative meeting fee means that the seller pays a meeting subsidy *to* each buyer.

5.1 Incentive Compatibility and Payoffs

Before analyzing which mechanism sellers wish to post, we derive agents' expected payoffs. While doing this, it becomes clear how helpful our new representation of meeting technologies, ϕ, is; despite being much more general, the analysis remains almost as simple as that of a monopolistic auction.

Payoffs in a Monopolistic Auction. When a monopolistic seller offers a selling mechanism, incentive compatibility requires that buyers' expected utility is intimately connected with their trading probabilities (see Myerson, 1981; Riley and Samuelson, 1981). To see this, consider n buyers who participate in an efficient mechanism—i.e., a mechanism in which the buyer with the highest value trades if and only if his valuation exceeds that of the seller, like a second-price auction with no reserve price but potentially an entry fee. The expected payoff $V_n(x)$ for a buyer with value x from participating in the mechanism equals

$$V_n(x) = V_n(0) + \int_0^x F^{n-1}(z)dz, \tag{13}$$

where $F^{n-1}(x)$ represents the probability that all $n-1$ other buyers have a value below x. Buyers' payoff is increasing and convex in their type x, since $F^{n-1}(x)$ is increasing in x.

Furthermore, the seller's payoff π_n can be written as

$$\pi_n = -nV_n(0) + \int_0^1 \left(z - \frac{1-F(z)}{f(z)} \right) dF^n(z), \tag{14}$$

where $z - (1-F(z))/f(z)$ is the virtual valuation function (Myerson, 1981) and $F^n(z)$ is the distribution of the highest valuation among n buyers.

Payoffs Under Competing Mechanisms. The function ϕ allows us to derive similar results in an environment with competing mechanisms and general meeting technologies. We do this in two steps. First, we prove that the market utility function must be convex and closely related to buyers' trading probabilities. Subsequently, we derive agents' payoffs in a particular submarket and show that they resemble equations (13) and (14).

For the first step, denote the set of mechanisms that buyers of type x visit in equilibrium by $\Omega^b(x)$, pick an arbitrary $\omega^b(x) \in \Omega^b(x)$ and denote by $p(x, \omega^b(x))$ the probability that a buyer of type z trades when visiting mechanism $\omega^b(x)$. Of course, if buyers of type x choose to be inactive, then we set $\omega^b(x) = \emptyset$ and $p(x, \emptyset) = 0$. The following Proposition then establishes the properties of the market utility function.

Proposition 4. *Given any set of mechanisms posted by sellers, $p(x, \omega^b(x))$ is non-decreasing and the market utility function $U(x)$ is convex, satisfying*

$$U(x) = U(0) + \int_0^x p(z, \omega^b(z)) dz.$$

If $U(x)$ is differentiable at point x_0, then $p(x_0, \omega_0)$ is the same for every $\omega_0 \in \Omega^b(x_0)$, i.e., the probability that a buyer of type x_0 trades is the same at each mechanism that he may visit.

Proof. See appendix A.5. □

There are several statements in Proposition 4 but the basic ideas are the same as in the single seller case: (i) because of the incentive compatibility constraint, high-valuation buyers must have a higher chance of obtaining the object, and (ii) buyers' payoff is determined solely by the trading probabilities. The combination of both ideas makes the market utility function convex. As we will see later, this has important consequences for a seller's optimal choice of selling mechanism; in particular, for sellers who face a convex market utility function, the optimal selling mechanism is to post an auction with an entry fee.

The additional feature introduced by competition between sellers is that buyers also need to consider where other buyers will visit. One consequence of this is that if a buyer x mixes over several submarkets, then the probabilities of winning the object in all these submarkets must be the same, for almost all buyer types $x \in [0,1]$ (with respect to the usual Lebesgue measure). For example, suppose that both buyers of type x and $x + \Delta x$ visit mechanisms i and j. Since buyers' utility is an integral of trading probabilities, $U(x + \Delta x) - U(x) = p(x,i)\Delta x = p(x,j)\Delta x$. Therefore, the trading probabilities of buyers of type x should be equal across the different submarkets that they visit.

For the second step, consider a submarket in which sellers post an efficient mechanism. Suppose the submarket attracts a queue λ of buyers whose values are distributed according to $F(x)$. The following Lemma then establishes agents' expected payoffs in this submarket.

Lemma 2. *Consider a submarket with an efficient mechanism, a queue length λ, and a buyer value distribution $F(x)$. The expected payoff for a buyer with valuation z visiting this submarket is*

$$V(x) = V(0) + \int_0^x \phi_\mu(\lambda(1 - F(z)), \lambda)dz. \tag{15}$$

The expected payoff of a seller in the submarket is

$$\pi = -\lambda V(0) + \int_0^1 \left(z - \frac{1 - F(z)}{f(z)}\right) d\big(1 - \phi(\lambda(1 - F(z)), \lambda)\big). \tag{16}$$

Furthermore, the set $\{x \,|\, V(x) = U(x)\}$ is always an interval.

Proof. See appendix A.6. □

The interpretation of equation (15) is similar to equation (13). By equation (5), the term $\phi_\mu(\lambda(1 - F(z)), \lambda)$ in equation (15) is the probability that a buyer with valuation z meets a seller and has the highest valuation among all buyers who arrived at the seller. Hence, for efficient mechanisms, it is simply the trading probability of the buyer. On the seller side, equation (16) is similar to equation (14). In a standard auction with n bidders, a seller's expected payoff equals the virtual valuation function integrated against the distribution of the highest valuation among n buyers, which is simply $F^n(z)$. In our setting, the probability that the highest valuation equals x depends on the meeting technology and is given by $1 - \phi(\lambda(1 - F(z)), \lambda)$, i.e., the probability that there are no buyers with valuations above z.

The intuition for the last claim is the following. Suppose there is a gap (x_1, x_2) in the support of the queues in a submarket, i.e., no buyers with values between x_1 and x_2 attempt to visit the submarket. If a buyer with value $x \in (x_1, x_2)$ then chooses to visit this submarket, his payoff will be a weighted average of $U(x_1)$ and $U(x_2)$, as follows from equation (15). Since the market utility function is convex, this weighted average will lie above the market utility function. This leads to a contradiction.

One may have expected that allowing for general meeting technologies would severely complicate the payoff functions in (competing) auction theory. We have shown here that our alternative representation of the meeting technology ϕ avoids such complications. In particular, agents' expected payoffs retain the same structure but simply depend on transformations of ϕ instead of transformations of F.

Example. To better understand the above results, consider a bilateral meeting technology with $P_0(\lambda)$ strictly convex. Suppose that the measures of sellers and buyers are both equal to 1. Almost every buyer has value x_0, i.e., buyers with values other than x_0 have measure 0. As in Proposition 4, we do not consider optimality of seller behavior and take the posted mechanisms as given; in particular, suppose half of the sellers posts a second price auction with reserve price 0 (market A), while the other sellers post a second price auction with reserve price or entry fee r, satisfying $0 < r < x_0$ (market B).[24]

In order to solve for buyers' optimal strategy, suppose that market tightness in markets A and B are equal to λ^A and λ^B, respectively. Except in the corner solution in which all buyers with value x_0 visit market A, buyers with value x_0 must then be indifferent between visiting market A and B. That is,

$$Q_1(\lambda^A)x_0 = Q_1(\lambda^B)(x_0 - r)$$

subject to the buyer availability constraint that $\lambda^A + \lambda^B = 2$.[25] The above equation implies $Q_1(\lambda^A) < Q_1(\lambda^B)$. Therefore, using the notation from Proposition 4, we have $p(x_0, A) = Q_1(\lambda^A) < Q_1(\lambda^B) = p(x_0, B)$.

Next, we consider buyers with values other than x_0, even though they have measure 0. If their value x satisfies $r < x < x_0$, then visiting market A will result in a payoff of $Q_1(\lambda^A)x$ and visiting market B will result in a payoff of $Q_1(\lambda^B)(x - r)$. Since $x < x_0$ and $Q_1(\lambda^B)r = x_0(Q_1(\lambda^B) - Q_1(\lambda^A))$, it follows that

$$Q_1(\lambda^A)x > Q_1(\lambda^B)(x - r).$$

Hence, buyers with valuation x will visit market A only. Therefore, $U(x) = Q_1(\lambda^A)x$ for $x < x_0$, and $p(x) = Q_1(\lambda^A)$. If $x > x_0$, a similar logic applies, implying

$$Q_1(\lambda^A)x < Q_1(\lambda^B)(x - r),$$

[24] For bilateral meeting technologies, a reserve price and an entry fee are equivalent. This is not true in general.

[25] Note that $Q_1(\lambda) = \phi_\mu(\lambda, \lambda)$ by equation (5).

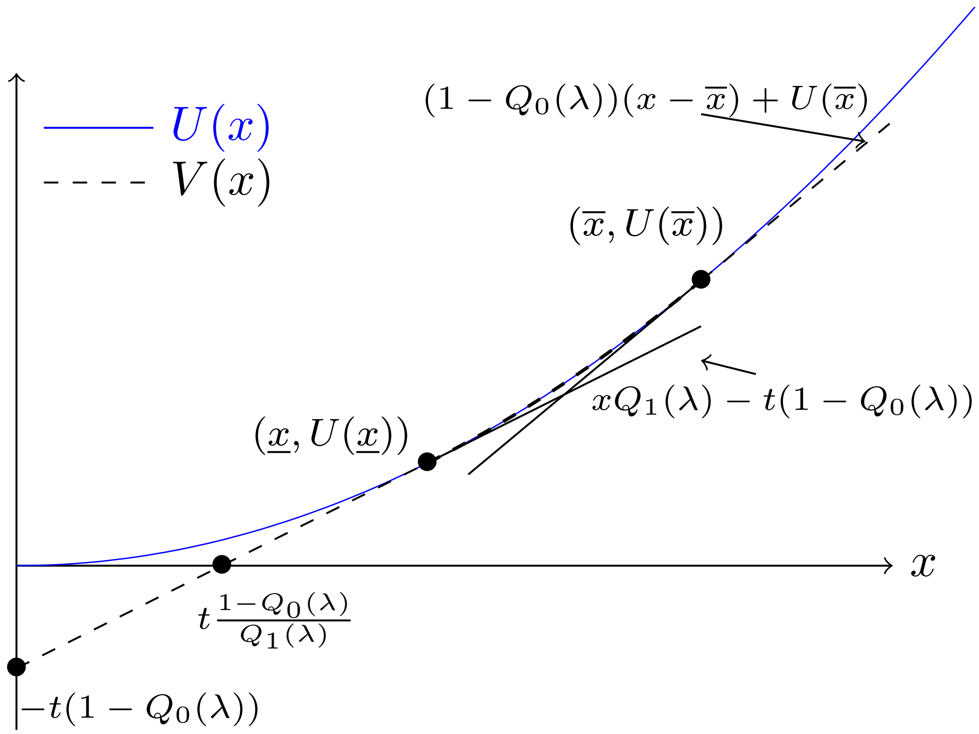

Figure 2: Supporting lines

such that buyers with valuation x will visit market B only. Therefore, $U(x) = Q_1(\lambda^B)(x - r)$ for $x > x_0$ and $p(x) = Q_1(\lambda^B)$.

In sum,

$$p(x) = \begin{cases} Q_1(\lambda^A) \text{ if } x < x_0 \\ Q_1(\lambda^A) \text{ if } x = x_0 \text{ and } x \text{ visits market } A \\ Q_1(\lambda^B) \text{ if } x = x_0 \text{ and } x \text{ visits market } B \\ Q_1(\lambda^B) \text{ if } x > x_0 \end{cases}$$

and

$$U(x) = \begin{cases} Q_1(\lambda^A)x \text{ if } x \leq x_0 \\ Q_1(\lambda^B)(x - r) \text{ if } x \geq x_0 \end{cases}$$

Hence, for any x, we have $U(x) = U(0) + \int_0^x p(z)dz$, where it does not matter whether we set $p(x_0)$ equal to $Q_1(\lambda^A)$ or $Q_1(\lambda^B)$. $U(x)$ is differentiable everywhere except at x_0.

Payoffs Under Auctions and Fees. Lemma 2 established payoffs under general efficient mechanisms. If we focus on a submarket in which sellers post auctions with entry fees, then a more specific expression for buyers' payoffs can be derived, as we establish in the following Lemma.

Lemma 3. *Consider a submarket in which the posted mechanism is an auction with entry fee t, the queue length is λ, and the lowest and the highest buyer type are $\underline{x}$ and $\overline{x}$, respectively. If a buyer with value x chooses to visit this submarket, then his expected payoff $V(x)$ is*

$$V(x) = \begin{cases} xQ_1(\lambda) - t(1 - Q_0(\lambda)) & \text{if} \quad x < \underline{x}, & \text{(17a)} \\ U(x) & \text{if} \quad \underline{x} \leq x \leq \overline{x}, & \text{(17b)} \\ (1 - Q_0(\lambda))(x - \overline{x}) + U(\overline{x}) & \text{if} \quad \overline{x} < x\ , & \text{(17c)} \end{cases}$$

where (17a) *and* (17c) *are the supporting lines of the (convex) market utility function $U(x)$ at the points $(\underline{x}, U(\underline{x}))$ and $(\overline{x}, U(\overline{x}))$, respectively.*

Proof. See appendix A.7. □

Lemma 3 shows that there is a close connection between λ and $\underline{x}$ and $\overline{x}$ through the supporting lines of the convex function $U(x)$. This observation is almost trivial but instrumental for understanding the relation between the entry fee and the queue in the next subsection.

Figure 2 illustrates Lemma 3. By assumption, buyers with values below $\underline{x}$ or above $\overline{x}$ will not choose to visit this submarket. However, if they were to visit the submarket, their payoff would be given by $V(x)$, which is displayed by the dashed line. For values between $\underline{x}$ and $\overline{x}$, $V(x)$ coincides with $U(x)$. A buyer with value $x > \overline{x}$ will always trade as long as he successfully meets a seller, which happens with probability $1 - Q_0(\lambda) = \phi_\mu(0, \lambda)$. Hence, his payoff is given by the linear function $(1 - Q_0(\lambda))(x - \overline{x}) + U(\overline{x})$. In contrast, a buyer with value $x < \underline{x}$ will only trade if no other buyers meet the same seller, yielding a payoff equal to the linear function $xQ_1(\lambda) - t(1 - Q_0(\lambda))$.

5.2 Efficiency

In a decentralized market, in order to maximize his expected profit, a seller must choose a mechanism to attract a queue and the queue must be compatible with the market utility function. Below, we show that even if sellers can buy queues directly from a hypothetical market for queues (where the prices are given by the market utility function), they cannot do better than in the decentralized environment. In other words, the following two problems are equivalent.

1. *Sellers' Relaxed Problem.* There exists a hypothetical competitive market for queues, where the price for each buyer in the queue is given by the market utility function. Sellers choose a queue length λ and a queue composition F to maximize

$$\pi = \int_0^1 \phi(\lambda(1 - F(z)), \lambda)dz - \lambda \int_0^1 U(z)dF(z), \tag{18}$$

where the first term is total surplus (6) and the second term is the price of the queue.

2. *Sellers' Constrained Problem.* We have already described the seller's (constrained) problem in detail in Section 2. Contrary to sellers' relaxed problem, sellers must post mechanisms to attract queues of buyers. For any mechanism, the corresponding queue must be compatible with the market utility function, which means that it needs to satisfy equation (3). In this case, a seller's profit is again given by equation (18), but now queue length and queue composition depend on the posted mechanism.

Using compatibility as defined in equation (2), we have the following result.

Proposition 5. *Given any convex market utility function, any solution* (λ, F) *to the sellers' relaxed problem is also compatible with an auction with an entry fee in the sellers' constrained problem, where the fee is given by*

$$t = -\frac{\int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda)dz}{1 - Q_0(\lambda)}.$$

Proof. See appendix A.8. □

The intuition behind Proposition 5 is the following. In the sellers' relaxed problem, a seller will "buy" buyers with valuation x until their marginal contribution $T(x)$ to surplus is equal to their marginal cost $U(x)$. Hence, if sellers can post a mechanism which delivers buyers their marginal contribution to surplus, then buyers' payoffs are equal to their market utility and the queue is compatible with the mechanism and the market utility function, as defined by equation (3). Proposition 5 argues that auctions with an entry fee can achieve this. To understand why this is the case, note that a buyer's marginal contribution consists of two parts: (i) a direct effect, representing the fact that the buyer may increase the maximum valuation among the group of buyers meeting the seller, and (ii) an indirect effect, representing the externalities that the buyer may impose by making it easier or harder for the seller to meet other buyers. As is well-known, auctions (without reserve prices or fees) provide buyers with a payoff equal to their direct contribution.[26] Buyers' indirect effect on surplus can then

[26]This is easiest to see in a second-price auction. Suppose that the highest and the second highest value are

be priced by the entry fee. The combination of both instruments then guarantees that buyers' payoff is equal to $T(x)$, which yields the desired result.

There is one remaining issue about Proposition 5: for a given auction with entry fee, there might be multiple queues compatible with the market utility function. Hence, even if a solution to the sellers' relaxed problem is compatible with an auction with entry fee, it is not clear that sellers will expect that solution to be the realized queue. Most of the literature resolves this issue by assuming that a deviating seller (expects that he) can coordinate buyers in such a way that the solution to sellers' relaxed problem becomes the realized queue.[27] If we follow this approach, then by Proposition 5, a seller's relaxed and constrained problem are equivalent in the sense that they achieve the same outcome. That is, the directed search equilibrium is equivalent to a competitive market equilibrium for queues, which also coincides with the socially efficient planner's allocation.

Proposition 6. *The directed search equilibrium is constrained efficient.*

Proof. See appendix A.9. □

Hence, we have shown that despite the potential presence of spillovers in the meeting process, business stealing externalities and agency costs, the competing mechanisms problem reduces to one where sellers can buy queues in a competitive market.

5.3 Characterization

We now provide a characterization of the decentralized equilibrium. To facilitate the exposition, we will assume that the aggregate market-wide buyer value distribution G has full support on $[0, 1]$. This condition is not restrictive since any distribution G can be approximated arbitrarily well by $(1 - \varepsilon)G + \varepsilon U[0, 1]$, where $U[0, 1]$ is the uniform distribution.

We first introduce the following assumption on the set of meeting technologies, which we will impose for the remaining part of this paper.

Assumption 1. $Q_1(\lambda)$ *is strictly decreasing in* λ*.*

This assumption states that in submarkets with longer queues, it is less likely that a buyer turns out to be the only one present in an auction. It is not restrictive in the sense that it is satisfied by all examples of meeting technologies that were listed above.

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x_2 and x_1. Then, the payoff for the highest value buyer is $x_2 - x_1$, which is also his contribution to surplus. Other bidders receive zero and their contributions to the surplus of the auction are also zero. Extension of this result to other auction formats follows from revenue equivalence.

[27]See, for example, Eeckhout and Kircher (2010a,b).

In Proposition 4, we established that buyer optimality implies that the market utility function $U(x)$ is always convex, irrespective of what mechanisms sellers post. A stronger result can be proved with Assumption 1.[28]

Proposition 7. *Under assumption 1, in equilibrium $U(x)$ is strictly convex on the open set $\{x \mid U(x) > 0\}$.*

Proof. See appendix A.10. □

The necessity of the support of G being $[0, 1]$ for the above result can be easily seen. For example, when the meeting technology is urn-ball and the support of G contains a gap (x_1, x_2), then $U(x)$ is linear in (x_1, x_2) by equation (15) in Lemma 2.[29]

Next, we compare the queues of two arbitrary sellers, indexed by $i \in \{a, b\}$, who post an auction with an entry fee t_i, and attract a queue λ^i of buyers, in which the lowest buyer type is $\underline{x}_i$. The following Lemma establishes the relation between $\underline{x}_i$ and λ^i.

Lemma 4. *There is a unique $\underline{x}$ for a given queue length λ. Furthermore, under assumption 1, $\lambda^a > \lambda^b$ implies that $\underline{x}_b \geq \underline{x}_a$.*

Proof. See appendix A.11. □

The intuition behind Lemma 4 can be easily seen from Figure 3. Since the market utility function is convex, the slope of a supporting line at x_2 is larger than that at x_1 if $x_2 > x_1$.

Similarly, a relation between a seller's queue length λ^i and the *highest* buyer type $\overline{x}_i$ that he attracts can be established under the following assumption.

Assumption 2. $1 - Q_0(\lambda)$ *is (weakly) decreasing in λ.*

This assumption says that buyers are (weakly) less likely to meet a seller if the queue length in the submarket increases, which could be interpreted as a form of congestion. Like assumption 1, it is satisfied by all examples of meeting technologies that were listed above. Under this assumption, a lower $\overline{x}_i$ implies a longer queue, as the following Proposition establishes.

Lemma 5. *There is a unique $\overline{x}$ for a given queue length λ. Furthermore, under assumption 2, $\lambda^a > \lambda^b$ implies that $\overline{x}_a \leq \overline{x}_b$. Furthermore, with non-rival meeting technologies, the queues attracted by sellers posting an auction with entry fee always have an upper bound $\overline{x} = 1$.*

[28]Inspection of the proof shows that Proposition 7 in fact only requires a weaker version of Assumption 1: $Q_1(0) > Q_1(\lambda)$ for any $\lambda > 0$.

[29]When the meeting technology is urn-ball or more generally jointly concave (Cai et al., 2017, see), then all sellers and buyers will pool into one market in equilibrium. In this case, there is no distinction between the buyer value distribution in a submarket and in the economy as a whole.

Proof. See appendix A.12. $\square$

This Lemma, which does not require assumption 1, is the counterpart to Lemma 4. As for that Lemma, the intuition behind the result can be seen from Figure 2. Since the slope of a supporting line at $\overline{x}$ is $1 - Q_0(\lambda)$, if $1 - Q_0(\lambda)$ is decreasing, a longer queue implies a flatter supporting line at $\overline{x}$, hence the highest buyer value $\overline{x}$ is smaller.

Together with Lemma 2, the second part of Lemma 5 implies that under non-rival meeting technologies, each queue has a connected support with upper-bound 1 in equilibrium. Because we assumed non-rival meetings, if buyers with valuation 1 were absent, then a deviating buyer with valuation 1 would win for sure, assuming $Q_0(\lambda) = 0$. In other words, compared to other buyers in the queue, a buyer with valuation 1 will enjoy a large information rent if he decides to visit that seller, even higher than his market utility. In equilibrium this cannot happen because buyers with valuation 1 will adjust their visiting probability till the market utility constraint becomes binding again.

When the meeting technology is bilateral, the slopes of the supporting lines at $\underline{x}$ and $\overline{x}$ are the same, since $Q_1(\lambda) = 1 - Q_0(\lambda)$). As a result, $\underline{x}$ must be the same as $\overline{x}$. The above geometric argument therefore simply implies that complete market segmentation arises under bilateral meeting technologies. We discuss this result in more detail in Cai et al. (2017).

The above two propositions relate the queue length to the upper and lower bounds of buyer values in a submarket. If we impose an additional assumption, we can compare queue compositions between any two submarkets. Before we do that, we first introduce a weaker version of it, which, by Lemma 1, implies that all sellers are active in equilibrium.

Assumption 3. *Buyers impose (weakly) negative meeting externalities on each other, i.e., $\phi_\lambda(\mu, \lambda) \leq 0$ for $0 \leq \mu \leq \lambda$.*

Recall that $\phi_\lambda(\mu, \lambda)$ measures the externalities that buyers impose on each other. Consider a queue with μ high-value buyers and $\lambda - \mu$ low-value buyers. Then $\phi_\lambda(\mu, \lambda)\Delta\lambda = \phi_\lambda(\mu, \lambda + \Delta\lambda) - \phi_\lambda(\mu, \lambda)$ is the effect of adding $\Delta\lambda$ low-value buyers on the meeting probability between sellers and high-value buyers. The stronger version that we need to compare queue compositions is the following.

Assumption 4. *$\phi_{\mu\lambda}(\mu, \lambda) \leq 0$ for $0 \leq \mu \leq \lambda$.*

To understand this assumption, which is satisfied by e.g. bilateral and invariant meeting technologies, consider a queue with μ high-value buyers and $\lambda - \mu$ low-value buyers. By equation (5) and the subsequent discussion, $\phi_\mu(\mu, \lambda)$ is then the probability for a buyer to be part of a meeting in which all other buyers (if any) have low valuations, which is also the probability that a high-value buyer wins the auction with positive payoffs. Assumption 4 states that if we add more low-value buyers to the queue, then this probability will not increase. That is, low-value buyers create a weakly negative externality on the winning probability of high-value buyers, and not just on their meeting probability as in assumption 3.

The following proposition shows that assumption 4 implies assumptions 2 and 3 and is closely related to assumption 1.

Proposition 8. *Assumption 4 implies that i) $Q_1(\lambda)$ is weakly decreasing, ii) assumption 2, and iii) assumption 3. Furthermore, if we assume $P_2(\lambda) > 0$ for any $\lambda > 0$, then assumption 4 also implies assumption 1.*

Proof. See appendix A.13. □

Proposition 9 then characterizes our main result regarding the queue composition.

Proposition 9. *Under assumptions 1 and 4,consider two submarkets a and b with queues $(\lambda^a, F^a(x))$ and $\left(\lambda^b, F^b(x)\right)$, respectively. If $\lambda^a > \lambda^b$ and $\underline{x}_b < \overline{x}_a$, then for any $x \in [\underline{x}_b, 1]$,*

$$\lambda^b\left(1 - F^b(x)\right) \geq \lambda^a\left(1 - F^a(x)\right).$$

If the meeting technology is invariant, then $\lambda^a\left(1 - F^a(x)\right) = \lambda^b\left(1 - F^b(x)\right)$ for $x \in [\underline{x}_b, 1]$.

Proof. See appendix A.14. □

Note that by Lemma 4 and 5, $\lambda^a > \lambda^b$ implies that $\underline{x}_a < \underline{x}_b$ and $\overline{x}_a \leq \overline{x}_b$. Only buyers with types belonging to $[\underline{x}_b, \overline{x}_a]$ are active in the two queues. For bilateral meeting technologies, the above Proposition becomes void, because we have $\underline{x}_a = \overline{x}_a < \underline{x}_b = \overline{x}_b$, where the strict inequality is due to the assumption that $\lambda^a > \lambda^b$. For all non-rival meeting technologies, $\overline{x}$ is always 1 by Lemma 5. Hence, the support of F^a contains the support of F^b. This is similar to Proposition 3 in Shimer (2005).

Unfortunately, a complete characterization of the equilibrium queues is not feasible, but progress can be made for special cases. For example, building on the results in this paper, Cai et al. (2017) establish that the equilibrium is perfectly separating (i.e. a separate submarket for each active type of buyer) for all $G(x)$ if and only if the meeting technology is bilateral. In contrast, the equilibrium features pooling of all agents in a single submarket for all $G(x)$ if and only if the meeting technology exhibits joint concavity, i.e. $\phi(\mu, \lambda)$ is concave in (μ, λ). For meeting technologies that are neither bilateral nor jointly concave, the equilibrium number of submarkets will generally depend on $G(x)$.

Uniqueness. Of course, by revenue equivalence, there exist multiple efficient selling mechanisms that give buyers and sellers the same payoffs. Moreover, there can be multiple sets of

α_i and λ^i that maximize total surplus, i.e. satisfy equation (10), and those different allocations can all be decentralized. This is because for some meeting technologies, a combination of a high fee and a short queue can give both buyers and sellers the same payoff as a low fee and a long queue; we provide an example in appendix B.[30] What is important is that any equilibrium is constrained efficient (i.e. satisfies Proposition 6). In any decentralized equilibrium, the total surplus is always $\mathcal{S}(b_1, \ldots, b_n)$ as given by equation (10) and hence the marginal contribution to surplus of a buyer with value x_i is always $\partial \mathcal{S}(b_1, \ldots, b_n)/\partial b_i$. So, there may be multiple equilibria, but total surplus and the marginal contributions to surplus must be the same between different equilibria. Since in any decentralized equilibrium, an agent's private payoff equals his marginal contribution to surplus, all equilibria are payoff-equivalent for both buyers and sellers. In the next section, we deal with the more serious issue that without restrictions on beliefs and or the meeting technology, multiple queues can be compatible with market utility.

5.4 Uniqueness of Beliefs

So far, we have assumed that sellers are optimistic, i.e. if multiple queues are compatible with the market utility function, then they expect the queue that is most favorable. In this subsection, we will explore an alternative. In particular, we will introduce one weak additional restriction on the meeting technology, such that there is a monotonic relation between meeting fees and queue lengths and hence a unique queue that is compatible with the market utility function when sellers post an auction with entry fee.

Assumption 5. *$Q_1(\lambda)/(1 - Q_0(\lambda))$ is (weakly) decreasing in λ.*

If we rewrite $(1 - Q_0(\lambda))/Q_1(\lambda)$ as $1 + \sum_{k=2}^{\infty} Q_k(\lambda)/Q_1(\lambda)$, then the assumption states that with a higher buyer-seller ratio, it is relatively more likely that a buyer will meet competitors in an auction rather than being alone. Like assumption 1 and 2, assumption 5 is not restrictive. For bilateral meeting technologies, it is satisfied automatically; for non-rival meeting technologies, it is implied by assumption 1.

The next proposition gives the uniqueness result by relating the meeting fees to the queue lengths.

Proposition 10. *Under assumptions 1, 2, and 5, for each seller posting an auction with entry fee t, there is a unique queue (λ, F) which is compatible with the market utility function $U(x)$. Furthermore, for two sellers posting entry fees t_a and t_b, $t_a < t_b$ if and only if $\lambda^a > \lambda^b$.*

[30] Barro and Romer (1987) give a nice example that illustrates how sellers can promise utility by either a low price or fewer other buyers: the Paris metro used to sell expensive first-class tickets for wagons which were physically similar to the second-class ones but which were less crowded in equilibrium.

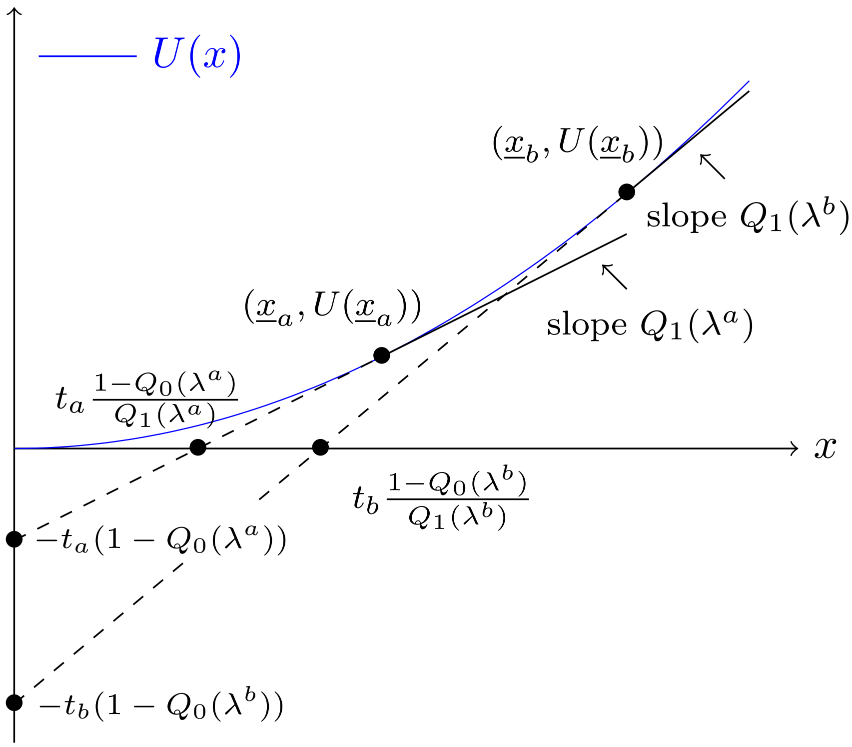

Figure 3: Relation between entry fee and queue length

Proof. See appendix A.15. □

Hence, for any strictly convex market utility function, sellers can adjust entry fees to attract queues of the desired length and composition. Given that a higher entry fee leads to a shorter queue, lemmas 4 and 5 imply that it also leads to a higher lower bound and a higher upper bound of buyer values. Under assumption 4, the meeting fee implies the queue composition by proposition 9.

The intuition behind Proposition 10 readily follows from Figure 3. Consider two different queues a and b. If queue a is longer ($\lambda^a > \lambda^b$), then by Lemma 4 the lowest type $\underline{x}_a$ in queue a is smaller than the lowest type $\underline{x}_b$ in queue b. For bilateral meeting technologies, the intercepts between the supporting lines and the x-axis are $(t_a, 0)$ and $(t_b, 0)$, respectively. Since $\underline{x}_a < \underline{x}_b$, we can easily see from Figure 3 that $t_a < t_b$. For non-rival meeting technologies, assuming $Q_0(\lambda) = 0$, the intercepts between the supporting lines and the y-axis are $(0, -t_a)$ and $(0, -t_b)$, respectively. Since $\underline{x}_a < \underline{x}_b$, we can again easily see from Figure 3 that $t_a < t_b$. A similar logic holds for other meeting technologies. For invariant meeting technologies, we know that there are no entry fees, so the supporting line of $\underline{x}$ must go through the origin, implying that $U(x)$ has a slope of $Q_1(\lambda)$ at $x = 0$ in that case.

Since by Proposition 5, the solution to the sellers' relaxed problem is compatible with the market utility function and an auction with entry fee, Proposition 10 implies that a seller can (and will) always choose an appropriate entry fee such that a solution to the sellers' relaxed problem is the only queue compatible with the auction and the market utility function. Therefore, the solutions to a seller's relaxed and constrained problem coincide. That is, the directed search equilibrium is equivalent to a competitive market equilibrium for queues, which also coincides with the socially efficient planner's allocation.

6 Two-Sided Heterogeneity

In this section, we show that our conclusions on existence, uniqueness and efficiency carry over to an environment in which sellers are heterogeneous. That is, we allow sellers to have different valuations y for the good, satisfying $0 \leq y \leq 1$, and these valuations are sellers' private information. The surplus generated by a seller with value y and a buyer with value x is thus $\max(x - y, 0)$. As before, each seller will post a direct, anonymous mechanism, and we require sellers with the same valuation to use the same (possibly mixed) strategy.

6.1 Market Equilibrium

Proposition 2, which established expressions for (marginal) surplus, can easily be extended to the case with two-sided heterogeneity. In particular, in a submarket in which sellers have value y and attract a queue (λ, F), social surplus is

$$S(y, \lambda, F) = \int_y^1 \phi(\lambda(1 - F(z)), \lambda)\, dz, \tag{19}$$

where, compared to equation (6), the integration starts from the seller's valuation y instead of 0. Similarly, the marginal contribution to surplus of a buyer with valuation x equals,

$$T(x, y, \lambda, F) = \int_y^1 \phi_\lambda(\lambda(1 - F(z)), \lambda)\, dz + \int_{\min\{x,y\}}^x \phi_\mu(\lambda(1 - F(z)), \lambda)\, dz. \tag{20}$$

Note that if $x < y$, then a buyer with value x does not directly contribute to surplus, i.e., the second term on the right hand side of equation (20) is zero. In this case, the buyer's marginal contribution to surplus consists only of spillovers, i.e., the first term on the right hand side of equation (20).

Since Proposition 4 required no restrictions regarding sellers' valuation, it continues to hold when sellers are heterogeneous. Hence, optimality of buyers' choices again implies that the market utility function is always convex. Next, consider a submarket with the posted mechanism being an auction with reserve price y and entry fee t and queue (λ, F). Similar to lemma 2, it is easy to see that the expected payoff for a buyer visiting this submarket is

$$V(x) = \begin{cases} -t(1 - Q_0(\lambda)) & \text{if} \quad x < y, \\ -t(1 - Q_0(\lambda)) + \int_y^x \phi_\mu(\lambda(1 - F(z)), \lambda) dz & \text{if} \quad x > y \end{cases}$$

As in Section 5, we again follow the literature and assume that when multiple queues are compatible with the posted mechanisms, sellers will expect the most favorable queue. The following proposition then establishes that—as in the case of homogeneous sellers—the relaxed and the constrained problem (as defined in Section 5) are equivalent for a seller with value y. Furthermore, a seller with value y can solve its constrained problem by posting a second-price auction with some entry fee and a reserve price y.

Proposition 11. *Given any convex market utility function, any solution (λ, F) to the relaxed problem of a seller with value y is also compatible with an auction with a reserve price y and an entry fee in the sellers' constrained problem, where the fee is given by*

$$t = -\frac{\int_y^1 \phi_\lambda(\lambda(1 - F(z)), \lambda) dz}{1 - Q_0(\lambda)}.$$

Hence, the directed search equilibrium is efficient.

Proof. The proof is the same as that of Proposition 5. □

6.2 Assortative Meetings and Matches

To simplify the analysis, we will impose assumption 3, i.e. buyers impose no or negative meeting externalities on each other, for the remainder of this section. Under this assumption, the social planner will never assign a buyer with value x to a seller with value y if $x < y$. Similarly, sellers will never offer a meeting subsidy in the decentralized market. Therefore, a buyer with value x will never visit a seller with value y if $x < y$. This simplifies the problem, making it no harder than the case with homogeneous sellers.

Next, we will show that the results in sections 5.3 and 5.4 continue to hold in the two-sided heterogeneity environment. First, the following proposition extends Proposition 7, which established conditions for strict convexity of the market utility function, to the case with heterogeneous sellers.

Proposition 12. *Under assumptions 1 and 3, $U(x)$ is strictly convex on the open set $\{x \,|\, U(x) > 0\}$ in equilibrium.*

Proof. Assumption 3 implies that all sellers will post a non-negative entry fee. A buyer with value x will visit a seller with value y only if $y < x$. The rest of the proof is the same as Proposition 7. □

Because the market utility function is again strictly convex and in any submarket, buyer values are always higher than the seller value, Lemma 4, 5 and Proposition 9 continue to hold (with the same proofs) despite the seller heterogeneity.

To relate the types of sellers to the queues that they will attract in equilibrium, we first derive a dual statement of Proposition 4. This statement links the equilibrium payoff of sellers to their selling probability. To do so, denote by $\Omega^s(y)$ the equilibrium set of mechanisms posted by sellers with value y, select an arbitrary mechanism $\omega^s(y) \in \Omega^s(y)$, and denote by $q(\omega^s(y))$ the probability that a seller of type y successfully sells the object by using this mechanism. Finally, let $\pi(y)$ be the equilibrium payoff of a seller (in excess of his own value). We prove the following result.

Proposition 13. *In equilibrium, $q(\omega^s(y))$ is non-increasing in y and $\pi(y)$ is decreasing and convex, satisfying*

$$\pi(y) = \int_y^1 q\left(\omega^s(z)\right) dz.$$

If $\pi(y)$ is differentiable at point y_0, then $q(\omega_0)$ is the same for every $\omega_0 \in \Omega^s(y_0)$, i.e., the probability for sellers of value y to successfully sell their object is the same across all mechanisms that they post.

Proof. See appendix A.16. □

Example. We illustrate Proposition 13 with an example. Suppose there is a measure 1 of buyers, who have values uniformly distributed on $[0, 1]$. There is also a measure 1 of sellers, who almost all have a value 0. Suppose the meeting technology is urn-ball. As Cai et al. (2017) show, this has two implications which simplify the analysis: i) all agents will pool into one market (because ϕ is jointly concave), and ii) the equilibrium meeting fee is zero (because $\phi_\lambda = 0$). By Proposition 4, the market utility function for buyers is

$$U(x) = \int_0^x \phi_\mu(1-z, 1)\, dz = \int_0^x e^{-(1-z)}\, dz = e^{-(1-x)} - e^{-1}. \tag{22}$$

Now, consider a seller with value y, whose optimal mechanism is an auction with reserve price y. Suppose he attracts a queue with length $\lambda(y)$ and composition $F_y(x)$. For any x in the support of $F_y(x)$, $V(x)$ coincides with $U(x)$ around x. Combining equation (15) and (22) then yields

$$\phi_\mu(1-x,1) = \phi_\mu(\lambda(y)(1-F_y(x)),\lambda(y)),$$

Since $\phi_\mu(\mu,\lambda) = e^{-\mu}$, this implies $\lambda(y)(1-F_y(x)) = 1-x$. Suppose the lowest buyer type that the seller attracts is $\underline{x}(y)$. Then F_y is uniform on $[\underline{x}(y),1]$ and the queue length equals $\lambda(y) = 1-\underline{x}(y)$. Buyers with value $\underline{x}(y)$ must obtain the market utility $U(\underline{x}(y))$, which means that $\underline{x}(y)$ has to satisfy $(\underline{x}(y)-y)Q_1(\lambda(y)) = U(\underline{x}(y))$, or equivalently, $y = \underline{x}(y)-(1-e^{-\underline{x}(y)})$.

The probability that a seller with value y successfully sells the good is

$$q(y) = 1-P_0(\lambda(y)) = 1-e^{-(1-\underline{x}(y))}.$$

Next, we consider the profit function $\pi(y)$. It equals

$$\begin{aligned}\pi(y) &= \int_y^1 \phi(\lambda(y)(1-F_y(z)),\lambda(y)) - \lambda(y)\int_y^1 U(z)dF_y(z)\\ &= \int_y^{\underline{x}(y)} \left(1-e^{-(1-\underline{x}(y))}\right)dz + \int_{\underline{x}(y)}^1 \left(1-e^{-(1-z)}\right)dz - \int_{\underline{x}(y)}^1 \left(e^{-(1-z)}-e^{-1}\right)dz,\end{aligned}$$

where in the second line the first two integrals on the right-hand side add up to total surplus and the last integral is the sum of buyers' expected utilities. A straightforward calculation then yields

$$\pi'(y) = \frac{d\pi(y)}{d\underline{x}(y)}\frac{d\underline{x}(y)}{dy} = -(1-e^{-(1-\underline{x}(y))}).$$

Hence, as established in Proposition 13, $\pi'(y) = q(y)$.

Note that Proposition 13 provides a link between seller types and their selling probabilities for an arbitrary set of mechanisms. In equilibrium, a seller attracting a queue λ successfully sells the object with probability $1-P_0(\lambda)$, as buyers' and sellers' strategies are such that there are gains from trade for every meeting. To characterize sorting, we then introduce the following assumption, which states that a seller with a longer queue is more likely to meet at least one buyer.

Assumption 6. $P_0(\lambda)$ *is strictly decreasing in* λ.

For two sellers a and b who have values y_a and y_b, satisfying $y_a < y_b$, according to Proposition 13, it must be that $1-P_0(\lambda^a) \geq 1-P_0(\lambda^b)$, which then implies $\lambda^a \geq \lambda^b$ because of assumption 6. Together with Proposition 9, this implies that for any x in his queue, seller b will attract weakly more buyers with values above x than seller a. We call this the assortative meetings case.

Proposition 14. *(Assortative Meetings) Under assumptions 1, 4, and 6, for any two sellers a and b who have values y_a and y_b, satisfying $y_a < y_b$, and queues $(\lambda^a, F^a(x))$ and $\left(\lambda^b, F^b(x)\right)$, respectively, the following must hold in equilibrium: $\lambda^a \geq \lambda^b$, $\underline{x}_a \leq \underline{x}_b$, and $\overline{x}_a \leq \overline{x}_b$. Furthermore, for any $x \in [\underline{x}_b, 1]$,*

$$\lambda^b \left(1 - F^b(x)\right) \geq \lambda^a \left(1 - F^a(x)\right).$$

If the meeting technology is invariant, then $\lambda^a(1 - F^a(x)) = \lambda^b\left(1 - F^b(x)\right)$ for $x \in [\underline{x}_b, 1]$.

Proof. See the discussion above. □

The intuition for this is the following. For an invariant meeting technology like the urn-ball, Albrecht et al. (2014) show that the buyers with the highest valuation visit all sellers while buyers with lower valuations do not visit sellers with valuations above some threshold.[31] In this case, high-type buyers do not care about how many low-type buyers visit the same seller because they will outbid them anyway in the auction. This gives rise to assortative matching in expectation. Meeting technologies that exhibit congestion strengthen this pattern: high-type buyers then prefer to visit submarkets with short queues, which are created by high-valuation sellers.

Uniqueness of Beliefs. Under assumptions 1, 2, 3 and 5, we can again show that for a given seller posting an auction with reserve price y and an entry fee t, there is a decreasing relation between entry fee and queue length, and for a given queue length, there will be one possible queue compatible with the market utility function. This establishes the uniqueness just as in Proposition 10. The proof is the same as in Proposition 10, except for some minor differences like the intersection point between the supporting line at $\underline{x}$ and the x-axis is $y + t(1 - Q_0(\lambda))/Q_1(\lambda)$ instead of $t(1 - Q_0(\lambda))/Q_1(\lambda)$. These changes should be clear from the context.

7 Conclusion

In this paper, we introduced a new function ϕ which makes the analysis of general meeting technologies tractable. Using this function, we show that in a large economy, despite the presence of private information and possible search externalities, the directed search equilibrium is equivalent to a competitive equilibrium where the commodities are buyer types and the prices are the market utilities. A seller can attract a desired queue by posting an auction with entry fee or subsidy. Furthermore, we introduced conditions on the meeting technology

[31] They use the same logic as McAfee (1993).

such that for any given market utility function, the queue attracted by an auction with fee is unique. This is necessary to establish the equivalence between the two equilibria. Finally, we allowed for seller heterogeneity and derived conditions on the meeting technology that generate assortative meetings which on turn implies assortative matching. Of course, assortative matching could also arise if meetings are random. New datasets by Davis and de la Parra (2017) and Algan et al. (2017), who observe all the applicants for a vacancy, make it possible to distinguish between sorting in the meeting and sorting in the matching stage.

Appendix A Proofs

A.1 Proof of Proposition 1

For a given sequence $P_n(\lambda)$, equation (4) defines the function ϕ immediately. For the reverse relationship, let $m(x,\lambda) \equiv \sum_{n=0}^{\infty} P_n(\lambda)\, x^n = 1 - \phi(\lambda(1-x), \lambda)$ be the probability-generating function of $P_n(\lambda)$. Given ϕ, the probability functions $P_n(\lambda)$, $n = 0, 1, 2, \ldots$, are then uniquely determined by

$$P_n(\lambda) = \frac{1}{n!}\frac{\partial^n}{\partial x^n} m(x,\lambda)\bigg|_{z=0} = \frac{(-\lambda)^n}{n!}\frac{\partial^n}{\partial \mu^n}\left(1 - \phi(\mu,\lambda)\right)\bigg|_{\mu=\lambda}.$$

□

A.2 Proof of Proposition 2

When a seller meets $n \geq 1$ buyers, the surplus z from the meeting is distributed according to $F^n(z)$. Hence, the expected surplus per seller in the submarket is

$$S(\lambda, F) = \sum_{n=1}^{\infty} P_n(\lambda) \int_0^1 z\, dF^n(z) = \int_0^1 \left(1 - \sum_{n=0}^{\infty} P_n(\lambda)\, F^n(z)\right) dz,$$

where we use the Dominated Convergence Theorem to interchange integration with summation. The rightmost integrand equals $\phi(\lambda(1 - F(z)), \lambda)$, so the result follows.

Next, we calculate $T(x)$, the marginal contribution to surplus of a buyer with value x. First, we increase the measure of buyers with value x by ε and denote the new queue length and buyer value distribution as λ' and F' respectively. That is, $\lambda' = \lambda + \varepsilon$, while $\lambda'(1 - F'(z)) = \lambda(1 - F(z))$ for $z > x$ and $\lambda'(1 - F'(z)) = \lambda(1 - F(z)) + \varepsilon$ for $z \leq x$. Thus the average contribution to surplus by buyers with value x is

$$\frac{S(\lambda', F') - S(\lambda, F)}{\varepsilon} = \frac{1}{\varepsilon}\int_0^x \left[\phi(\lambda(1-F(z)) + \varepsilon, \lambda + \varepsilon) - \phi(\lambda(1-F(z)), \lambda)\right] dz$$
$$+ \frac{1}{\varepsilon}\int_x^1 \left[\phi(\lambda(1-F(z)), \lambda + \varepsilon) - \phi(\lambda(1-F(z)), \lambda)\right] dz$$

Let $\varepsilon \to 0$, then the above equation converges to

$$T(x) = \int_0^1 \phi_\lambda\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dz + \int_0^x \phi_\mu\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dz. \tag{23}$$

Since total surplus is homogeneous of degree one in the measures of sellers and buyers of each type, the expression for R follows from Euler's theorem, i.e., $R = S(\lambda, F) - \lambda \int_0^1 T(z) dF(z)$. To complete the proof, note that

$$\begin{aligned}
\int_0^1 T(z)dF(z) &= \int_0^1 \phi_\lambda\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dz + \int_0^1 \int_0^x \phi_\mu\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dz\, dF(x) \\
&= \int_0^1 \phi_\lambda\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dz + \int_0^1 \int_0^1 1_{z \le x} \phi_\mu\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dF(x)\, dz \\
&= \int_0^1 \phi_\lambda\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dz + \int_0^1 \left(1 - F(z)\right) \phi_\mu\left(\lambda\left(1 - F\left(z\right)\right), \lambda\right)\, dz
\end{aligned}$$

where in deriving the second equality above we used Fubini's theorem to change the order of integration. □

A.3 Proof of Lemma 1

Assume $\phi_\lambda(\mu, \lambda) \ge 0$. By equation (7) we have $T(0) \ge 0$. Hence, buyers' marginal contribution to surplus is always non-negative in this case.

Assume $\phi_\lambda(\mu, \lambda) \le 0$. Since $\phi(\mu, \lambda)$ is concave in μ, $\phi(\mu, \lambda) - \mu\phi_\mu(\mu, \lambda) \ge 0$. Then $\phi_\lambda(\mu, \lambda) \le 0$ implies $R \ge 0$ in equation (8). That is, sellers' marginal contribution to surplus is always non-negative in this case. □

A.4 Proof of Proposition 3

First, define $\psi(a, b, c) = a\phi(b/a, c/a)$ if $a > 0$ and $\psi(0, b, c) = 0$. It is easy to see that ψ is continuous and homogeneous of degree 1.

Suppose that there are k submarkets, and in submarket i the seller measure is $\alpha_i \ge 0$ and the measure of type j buyers is β_j^i. Compared to the formulation of equations (10), (11), and (12), here we do not require that α_i to be strictly positive. Of course, the planner will set at most one α_i to zero, i.e., there will be at most one submarket with only buyers. Define $B^i_j = \beta^i_j + \cdots + \beta^i_n$. Then by equations (9) and (10), the total surplus is

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(x_j - x_{j-1})\psi(\alpha_i, B^i_j, B^i_1).$$

In any submarket i, there must be some buyers or sellers so we have $\alpha_i + B^i_1 = \alpha_i + \beta^i_1 + \cdots + \beta^i_n > 0$. Define $\widetilde{\psi}(b, c) = \psi(1-c, b, c)$ for $0 \le b \le c \le 1$, and for $0 \le z_n \le \cdots \le z_1 \le 1$,

$$\widetilde{S}(z_1, \ldots, z_n) \equiv \sum_{j=1}^{n}(x_j - x_{j-1})\widetilde{\psi}(z_j, z_1). \tag{24}$$

Then the problem of the social planner is,

$$\sup_{\alpha_i, B^i_j} \quad \sum_{i=1}^{k+1}(\alpha_i + B^i_1)\widetilde{S}\left(\frac{B^i_n}{\alpha_i + B^i_1}, \ldots, \frac{B^i_1}{\alpha_i + B^i_1}\right)$$

subject to the constraint of buyer/seller availability.

$$\sum_{i=1}^{k}\alpha_i = 1$$

and for each j,

$$\sum_{i=1}^{k} B^i_j = B_j.$$

where $B_j \equiv b_j + \cdots + b_n$ is the measure of all buyers of types $x_j \ldots, x_n$ in the market. Note that compared to equation (12), we have equality in the buyer availability constraint instead of inequality.

Therefore, similar to equation (10), total surplus is a convex combination of the surpluses generated by individual submarkets.[32] The maximum social surplus as a function of the buyer endowment $(B_n, \ldots, B_1)$ is the concave hull of the function $\widetilde{S}$ of equation (24). The domain of $\widetilde{S}$ is the set $\{(z_1, \ldots, z_n) \,|\, 0 \le z_n \le \cdots \le z_1 \le 1\}$, which is connected, hence by the Fenchel-Bunt Theorem (see Theorem 18 (ii) of Eggleston, 1958), which is an extension of Caratheodory's theorem, it suffices to create $n+1$ submarkets. Furthermore, because function $\widetilde{S}$ is continuous and its domain is compact, the supremum can be reached as a

[32]The sum of the coefficients is $1 + \Lambda$ instead of 1, which can be easily fixed by normalizing the total measure of buyers and sellers to 1.

maximum. □

A.5 Proof of Proposition 4.

The strategy of a buyer with value x is: (i) a probability distribution over the mechanisms to visit and inactivity and (ii) a value to report when the mechanism is not inactivity. Given the mechanisms posted by sellers, suppose that the set of mechanisms that a buyer with valuation x visits is $\Omega^b(x)$, and the probability that the buyer receives the object when visiting seller $\omega \in \Omega^b(x)$ and reporting x by $p(x, \omega)$, with a corresponding expected payment $t(x, \omega)$.

First, we select one element $\omega^b(z) \in \Omega^b(z)$ for each z. Then, by the incentive compatibility constraint (ICC), for any x, z,

$$U(x) \geq xp(z, \omega^b(z)) - t(z, \omega^b(z)), \tag{25}$$

i.e., buyers with valuation x are always better of following their equilibrium strategies than mimicking any other type z. Therefore,

$$U(x) = \max_{z \in [0,1]} xp(z, \omega^b(z)) - t(z, \omega^b(z)).$$

Hence, $U(x)$ is the supreme of a collection of affine functions and must therefore be convex.

Furthermore, we can rewrite equation (25) in the following way.

$$\begin{aligned} U(x) = xp(x, \omega^b(x)) - t(x, \omega^b(x)) &\geq xp(z, \omega^b(z)) - t(z, \omega^b(z)) \\ &= U(z) + p(z, \omega^b(z))(x - z). \end{aligned}$$

So, $p(x, \omega^b(x))$ is the slope of a supporting line for the convex function $U(x)$. Therefore, $p(x, \omega^b(x))$ is a non-decreasing function. Since $U(x)$ is convex, it is absolutely continuous and differentiable almost everywhere. If $U(x)$ is differentiable at x_0, then

$$U'(x_0) = p(x_0, \omega^b(x_0)).$$

Since we have picked $\omega^b(x_0)$ out of $\Omega^b(x_0)$ in an arbitrary way, this implies that for any $\omega_1, \omega_2 \in \Omega^b(x_0)$, we have $p(x_0, \omega_1) = p(x_0, \omega_2) = U'(x_0)$. □

A.6 Proof of Lemma 2

We use $V_n(x)$ to denote the expected payoff of a buyer with value x when n bidders are present in an auction. Taking the expectation with respect to n yields

$$\begin{aligned} V(x) = \sum_{n=1}^{\infty} Q_n(\lambda) V_n(x) &= \sum_{n=1}^{\infty} Q_n(\lambda) \left(V_n(0) + \int_0^x F(z)^{n-1}\, dz \right) \\ &= V(0) + \int_0^x \left(\sum_{n=1}^{\infty} \frac{nP_n(\lambda)}{\lambda} F(z)^{n-1} \right) dz \\ &= V(0) + \int_0^x \left(\sum_{n=1}^{\infty} \frac{nP_n(\lambda)}{\lambda} F(z)^{n-1} \right) dz, \end{aligned}$$

where we have used equation (13) to substitute out $V_n(z)$. Therefore, using equation (5), we have

$$V(x) = V(0) + \int_0^x \phi_\mu(\lambda(1 - F(z)), \lambda) dz.$$

The seller will receive π_n in equation (14) with probability $P_n(\lambda)$. Therefore, for a given λ, the expected profit of a seller is

$$\begin{aligned} \pi = \sum_{n=0}^{\infty} P_n(\lambda)\pi_n &= \sum_{n=0}^{\infty} P_n(\lambda) \left(-nV_n(0) + \int_0^1 \left(z - \frac{1 - F(z)}{f(z)} \right) dF^n(z) \right) \\ &= -\lambda V(0) + \int_0^1 \left(z - \frac{1 - F(z)}{f(z)} \right) d \sum_{n=0}^{\infty} P_n(\lambda) F^n(z) \\ &= -\lambda V(0) + \int_0^1 \left(z - \frac{1 - F(z)}{f(z)} \right) d\big(1 - \phi(\lambda(1 - F(z)), \lambda)\big), \end{aligned}$$

where we interchange integration and summation in the second line.

Finally, suppose that there is a gap (x_1, x_2) in the set $\{x \,|\, V(x) = U(x)\}$, then buyers with values between x_1 and x_2 would earn an expected payoff strictly smaller than their market utilities and will not be present in the submarket, so $F(x) = F(x_1)$ for any $x \in (x_1, x_2)$. By equation (15), the payoff function $V(x)$ is linear between x_1 and x_2. Hence, for a buyer with value x between x_1 and x_2, satisfying $x = \alpha x_1 + (1 - \alpha)x_2$ for some $\alpha \in (0, 1)$, it must be that $V(x) = \alpha V(x_1) + (1 - \alpha)V(x_2) = \alpha U(x_1) + (1 - \alpha)U(x_2) \geq U(\alpha x_1 + (1 - \alpha)x_2) = U(x)$, where in the last inequality we used the fact that the market utility function is always convex (see Proposition 4). We have thus reached a contradiction. $\square$

A.7 Proof of Lemma 3.

By equation (15), for $x < \underline{x}$, we have

$$V(\underline{x}) - V(x) = \int_{x}^{\underline{x}} \phi_\mu(\lambda(1 - F(z)), \lambda) dz.$$

Since $x < \underline{x}$, we have $\phi_\mu(\lambda(1 - F(x)), \lambda) = \phi_\mu(\lambda, \lambda) = Q_1(\lambda)$ by equation (5). Therefore, we obtain equation (17a).

Similarly, by equation (15), for $x > \overline{x}$, we have

$$V(x) - V(\overline{x}) = \int_{\overline{x}}^{x} \phi_\mu(\lambda(1 - F(z)), \lambda) dz.$$

Since $x > \overline{x}$, we have $\phi_\mu(\lambda(1 - F(x)), \lambda) = \phi_\mu(0, \lambda) = 1 - Q_0(\lambda)$ by equation (5). Therefore, we obtain equation (17c). Finally, by Lemma 2 we have $V(x) = U(x)$ for $\underline{x} \leq x \leq \overline{x}$. $\square$

A.8 Proof of Proposition 5

In their relaxed problem, sellers select a queue (λ, F) directly in a hypothetical competitive market. The expected payoff for a seller in this market is the difference between the surplus that he creates and the price of the queue. Suppose that a queue (λ, F) solves sellers' relaxed problem. Then x is in the support of F only if the marginal contribution to this surplus of a buyer with value x equals the market utility $U(x)$, i.e., $U(x) = T(x)$, where $T(x)$ is given by equation (7) in Proposition 2. If a buyer with valuation x is not in the support of F, then $U(x) \geq T(x)$.

If we can find an entry fee t, such that $T(x) = V(x)$, then (λ, F) is is also compatible with an auction with entry fee t in the sellers' constrained problem. Let the entry fee t be given by

$$t = -\frac{\int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda) dz}{1 - Q_0(\lambda)}.$$

By equation (17a), we then have $V(0) = \int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda) dz$. Furthermore, by equation (15), we have

$$V(x) = \int_0^x \phi_\mu(\lambda(1 - F(z)), \lambda) dz + \int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda) dz = T(x).$$

Therefore, any optimal queue chosen by an unrestricted seller who can buy queues directly at prices $U(x)$ is also compatible with an auction with entry fee. $\square$

A.9 Proof of Proposition 6

The sellers' relaxed problem boils down to a competitive market for buyer types. Therefore, the first welfare theorem implies and the equilibrium is efficient. Since the sellers' constrained problem is equivalent to the sellers' relaxed problem, the directed search equilibrium is also efficient. □

A.10 Proof of Proposition 7

Suppose that $U(x)$ is not strictly convex on the open set $\{x \,|\, U(x) > 0\}$, then there exists an interval in which $U(x)$ is a straight line. Denote the interval by (x_1, x_2). We will continue to assume that the relaxed and the constrained problem of a seller coincide and that all sellers in equilibrium post an auction with entry fee.

A queue at a submarket or posted mechanism ω is characterized by (λ, F), and the measure associated with F is denoted by ν_F. Consider all queues (sellers) with $\nu_F(\{x \,|\, x_1 < x < x_2\}) > 0$, i.e., queues with positive measure on the interval (x_1, x_2).

For a seller in the above category, assume that $P_0(\lambda) + P_1(\lambda) < 1$ or, equivalently, $\phi(\mu, \lambda)$ is strictly convex in μ (see footnote 18). Since $\nu_F(\{x \,|\, x_1 < x < x_2\}) > 0$, there exists a pair x_1^* and x_2^* such that $x_1 < x_1^* \leq x_2^* < x_2$, x_1^* and x_2^* belong to the support of F, and $\nu_F(\{x \,|\, x_1^* \leq x \leq x_2^*\}) > 0$. The trading probability $p\,(x, \omega)$ for a buyer $x \in (x_2^*, x_2)$ satisfies $p\,(x, \omega) > p\,(x_1^*, \omega) = U'\,(x_1^*) = U'\,(x)$. Note $p\,(x_1^*, \omega) = U'\,(x_1^*)$ is because i) x_1^* is in the support of F and ii) $U(x)$ is differentiable at x_1^* (see Proposition 4). Therefore, by Proposition 4, the expected payoff for this buyer is $U(x_1^*) + \int_{x_1^*}^{x} p\,(z, i)\; dz > U(x_1^*) + p(x_1^*)(x - x_1^*) = U(x)$. Hence, we have a contradiction.

Therefore, in a submarket with queue (λ, F) if $\nu_F(\{x \,|\, x_1 < x < x_2\}) > 0$, then we must have $P_0(\lambda) + P_1(\lambda) = 1$ in equilibrium. For a buyer who visits this submarket and has value $x^* \in (x_1, x_2)$, their trading probability is $Q_1(\lambda)$ since sellers in this submarket meet at most one buyer, i.e., $P_0(\lambda) + P_1(\lambda) = 1$. Furthermore, the buyer's trading probability is also $U'\,(x)$ by Proposition 4. Therefore, $Q_1(\lambda) = U'(x^*)$, which is the same for all $x \in (x_1, x_2)$. Next consider the seller side. In equilibrium, sellers solve their relaxed maximization problem. The expected profit of sellers in this submarket is $\int_0^1 (P_1(\lambda)x - \lambda U(x))\, dF(x)$ since sellers meet at most one buyer. Consider the set of x's which maximize $P_1(\lambda)x - \lambda U(x)$, i.e., $\arg\max\{P_1(\lambda)x - \lambda U(x) \,|\, x \in [0, 1]\}$, which contains x^* by assumption. Therefore, x is in the support of F only if $P_1(\lambda)x - \lambda U(x) = P_1(\lambda)x^* - \lambda U(x^*)$. Notice that for any $x \in (x_1, x_2)$,

$$xP_1(\lambda) - \lambda U(x) = xP_1(\lambda) - \lambda\,(U(x_1) + Q_1(\lambda)(x - x_1)) = \lambda\,(Q_1(\lambda)x_1 - U(x_1))\,,$$

which is independent of x. Therefore, $[x_1, x_2] \subset \operatorname{Arg\,max}\{P_1(\lambda)x - \lambda U(x) \,|\, x \in [0, 1]\}$.

Therefore, a seller's relaxed problem can also be solved by selecting queue (λ, δ_x) with $x \in (x_1, x_2)$, a queue with length λ and only buyers with value x (δ_x is the Dirac measure at x). Thus sellers prefer queue (λ, δ_x) to queue $(\widetilde{\lambda}, \delta_x)$, which leads to expected profit $(1 - P_0(\widetilde{\lambda}))x - \widetilde{\lambda}U(x)$. Optimality with respect to queue length gives that for any $x \in (x_1, x_2)$,

$$0 = -P_0'(\lambda)x - U(x).$$

With the above equation, the expected profit with queue (λ, δ_x) is thus $(1-P_0(\lambda))x-\lambda U(x) = (P_1(\lambda) + \lambda P_0'(\lambda))x$. Since all queues (λ, δ_x) with $x \in (x_1, x_2)$ belong to the solution of sellers' relaxed maximization problem, they should generate the same expected profit. The only possibility for this to be true is $0 = P_1(\lambda) + \lambda P_0'(\lambda)$, $U(x) = -P_0'(\lambda)x = xP_1(\lambda)/\lambda = xQ_1(\lambda)$, and the optimal expected profit of sellers is zero.

By assumption 1, $Q_1(\lambda) < Q_1(0)$. Hence, by continuity of function Q_1, there exists a λ^* close to zero such that $Q_1(\lambda) < Q_1(\lambda^*)$. The expected profit of picking queue (λ^*, δ_x) with $x \in (x_1, x_2)$ is

$$(1 - P_0(\lambda^*))x - \lambda^* U(x) \geq P_1(\lambda^*)x - \lambda^* U(x) = \lambda^* x \left(Q_1(\lambda^*) - Q_1(\lambda)\right) > 0,$$

which contradicts with the above observation that the optimal expected profit of sellers is zero. Therefore, we have reached a contradiction. $\square$

A.11 Proof of Lemma 4

Note that in equilibrium, $U(1)$ must be strictly positive. Otherwise all sellers will prefer buyers with value 1 in the relaxed problem, but the measure of such buyers is zero. Thus $U(1) = 0$, and hence $U(x) = 0$ for all x, cannot be an equilibrium. Therefore, the set $\{x \,|\, U(x) > 0\}$ is nonempty.

Next, we show that both $\underline{x}_a$ and $\underline{x}_b$ belong to the closure of the set $\{x \,|\, U(x) > 0\}$. Suppose there exists an $x^* > 0$ such that $U(x) = 0$ for $x \leq x^*$ and $U(x) > 0$ for $x > x^*$. Assume $\underline{x}_a < x^*$. Consider the relaxed problem of seller a. The seller strictly prefer buyers of value x^* over buyers of value $\underline{x}_a$ because i) the prices (market utility) of both buyer types are zero, ii) the meeting externalities caused by them are the same, and iii) buyers of x^* lead to a higher surplus. Therefore, if the seller want buyers of zero price in his queue, he should pick x^* in the relaxed problem. This contradiction implies $\underline{x}_a \geq x^*$. Similarly, $\underline{x}_b \geq x^*$.

By Lemma 3, $Q_1(\lambda^i)$ is the slope of a supporting line (subgradient) for the market utility function at point $(\underline{x}_i, U(\underline{x}_i))$ for $i \in \{a, b\}$. Because both $\underline{x}_a$ and $\underline{x}_b$ belong to the closure of the set $\{x \,|\, U(x) > 0\}$, on which $U(x)$ is strictly convex by Proposition 7, the subgradient determines point $\underline{x}_i$ uniquely (see, for example, Theorem 24.1 of Rockafellar (1970)).

Furthermore, by assumption 1, $\lambda^a > \lambda^b$ implies $Q_1(\lambda_b) > Q_1(\lambda_a)$. Since $U(x)$ is assumed to be strictly convex, $Q_1(\lambda_b) > Q_1(\lambda_a)$ implies that $\underline{x}_b \geq \underline{x}_a$. $\square$

A.12 Proof of Lemma 5

By Lemma 3, $1 - Q_0(\lambda^i)$ is the slope of a supporting line (subgradient) for the market utility function at point $(\overline{x}_i, U(\overline{x}_i))$ for $i \in \{a, b\}$. Similar to the proof of Lemma 4, the subgradient determines point $\overline{x}_i$ uniquely when $U(x)$ is strictly convex. Futhermore, by by assumption 2, $\lambda^a > \lambda^b$ implies $1 - Q_0(\lambda^a) \leq 1 - Q_0(\lambda^b)$, which implies $\overline{x}_a \leq \overline{x}_b$ by the strict convexity of $U(x)$.

For non-rival meeting technologies, $1 - Q_0(\lambda)$ is constant, which implies that the highest type of buyer must be the same across all submarkets. Since buyers with value 1 (the highest value) must visit all submarkets (otherwise no buyer will be active in the market), in all submarkets the highest buyer value is 1. $\square$

A.13 Proof of Proposition 8

By equation (5), $1 - Q_0(\lambda) = \phi_\mu(0, \lambda)$. Therefore, $-Q_0'(\lambda) = \phi_{\mu\lambda}(0, \lambda) \leq 0$. Therefore, assumption 4 implies assumption 2.

By equation (4), we have $\phi(0, \lambda) = 0$ for any λ, which implies that $\phi_\lambda(0, \lambda) = 0$ for any λ. Assumption 4 then implies $\phi_\lambda(\mu, \lambda) \leq \phi_\lambda(0, \lambda) = 0$, i.e., assumption 4 implies assumption 3.

By the definition of ϕ (see equation (4)), we have

$$\begin{aligned}
\phi_{\mu\lambda}(\mu, \lambda) &= \sum_0^\infty Q_{n+1}'(\lambda)(1 - \frac{\mu}{\lambda})^n + \sum_1^\infty Q_{n+1}(\lambda)(1 - \frac{\mu}{\lambda})^{n-1} n \frac{\mu}{\lambda^2} \\
&= \sum_0^\infty \left[Q_{n+1}'(\lambda) + Q_{n+2}(\lambda)(n+1)\frac{\mu}{\lambda^2} \right] (1 - \frac{\mu}{\lambda})^n
\end{aligned}$$

Evaluating the above equation at $\mu = \lambda$ gives that $\phi_{\mu\lambda}(\lambda, \lambda) = Q_1'(\lambda) + Q_2(\lambda)/\lambda$. Since $Q_2(\lambda) \geq 0$, assumption 4 implies that $\phi_{\mu,\lambda}(\lambda, \lambda) \leq 0$ and hence $Q_1'(\lambda) \leq 0$, i.e., $Q_1(\lambda)$ is weakly decreasing. If $Q_2(\lambda) > 0$ or equivalently $P_2(\lambda) > 0$, then the same argument implies $Q_1'(\lambda) < 0$, i.e., assumption 1. $\square$

A.14 Proof of Proposition 9

By Lemma 4 and 5, $\lambda^a > \lambda^b$ implies that $\underline{x}_a < \underline{x}_b$ and $\overline{x}_a \leq \overline{x}_b$. If $\overline{x}^a \leq \underline{x}_b$, then for any $x \geq \underline{x}_b$, $\lambda^a (1 - F^a(x)) = 0 \leq \lambda^b \left(1 - F^b(x)\right)$. In the following, we will thus assume $\underline{x}_b < \overline{x}^a$. Therefore, we have $\underline{x}_a < \underline{x}_b < \overline{x}^a \leq \overline{x}^b$. Note that in this case we have $P_0(\lambda^i) + P_1(\lambda^i) < 1$

for $i \in \{a, b\}$, because if $P_0(\lambda^i) + P_1(\lambda^i) = 1$, then we have $\underline{x}_i = \overline{x}_i$ by Lemma 4 and 5 (see also Figure 2), which contradicts the above inequality: $\underline{x}_a < \underline{x}_b < \overline{x}^a \leq \overline{x}^b$.

Note that only buyers with types $x \in [\underline{x}_b, \overline{x}_a]$ are active in both queues. By equation (15), for almost all $x \in [\underline{x}_b, \overline{x}_a]$, we have $\phi_\mu(\lambda^a(1 - F^a(x)), \lambda^a) = U'(x) = \phi_\mu\left(\lambda^b\left(1 - F^b(x)\right), \lambda^b\right)$. Since $F^a(x)$ or $F^b(x)$ are right continuous, for all $x \in [\underline{x}_b, \overline{x}_a)$, we have

$$\phi_\mu(\lambda^a(1 - F^a(x)), \lambda^a) = \phi_\mu\left(\lambda^b\left(1 - F^b(x)\right), \lambda^b\right). \tag{26}$$

We then prove the Proposition by contradiction. Suppose that $\lambda^b\left(1 - F^b(x)\right) < \lambda^a(1 - F^a(x))$ for some $x \in [\underline{x}_b, \overline{x}_a)$. This implies

$$\phi_\mu(\lambda^a(1 - F^a(x)), \lambda^a) < \phi_\mu\left(\lambda^b\left(1 - F^b(x)\right), \lambda^a\right) \leq \phi_\mu\left(\lambda^b\left(1 - F^b(x)\right), \lambda^b\right),$$

where the first inequality is because $\phi(\mu, \lambda^a)$ is strictly concave in μ $((P_0(\lambda^a) + P_1(\lambda^a) < 1))$ and the second is because of assumption 4. The above inequality is at odds with equation (26). Hence, we have reached a contradiction. Thus for any $x \in [\underline{x}_b, \overline{x}_a]$, $\lambda^a(1 - F^a(x)) \leq \lambda^b\left(1 - F^b(x)\right)$.

For $x \geq \overline{x}_a$, $\lambda^a(1 - F^a(x)) = 0 \leq \lambda^b\left(1 - F^b(x)\right)$.

In the special case of invariant meeting technologies, $\overline{x}^a = \overline{x}^b = 1$ by Lemma 5. Moreover, for invariant meeting technologies $\phi(\mu, \lambda)$ is strictly concave in μ and does not depend on λ. Hence, by equation (26), we have $\lambda^a(1 - F^a(x)) = \lambda^b\left(1 - F^b(x)\right)$. $\square$

A.15 Proof of Proposition 10

We consider the triple (t, λ, F), which means that queue (λ, F) is compatible with an auction with entry fee t and the market utility function (see Section 2 for more detailed definition). We will prove the proposition in the following three steps.

Step 1: Claim: *In any such triple, λ determines F uniquely.* By Lemma 4 and 5, a queue length λ determines the lowest buyer value $\underline{x}$ and the highest buyer value $\overline{x}$ uniquely because of the strict convexity of $U(x)$. If $P_0(\lambda) + P_1(\lambda) = 1$, then $Q_0(\lambda) + Q_1(\lambda) = 1$ and $\underline{x} = \overline{x}$, as discussed in the main text. Hence, F is simpley the Dirac measure $\delta_{\underline{x}}$. In contrast, if $P_0(\lambda) + P_1(\lambda) < 1$, then $Q_0(\lambda) + Q_1(\lambda) < 1$ and $\underline{x} < \overline{x}$. Consider $x \in (\underline{x}, \overline{x})$. By Lemma 2, $U(x) = V(x) = U(\underline{x}) + \int_{\underline{x}}^{x} \phi_\mu(\lambda(1 - F(z)), \lambda)dz$. Therefore, $\phi_\mu(\lambda(1 - F(x), \lambda) = U'(x)$ almost everywhere, which determines $F(x)$ almost everywhere since $\phi(\mu, \lambda)$ is strictly concave in μ. Furthermore, since F is right-continuous, the above procedure determines F uniquely.

Step 2: Claim: *In any such triple, λ determines t uniquely.* As shown in Lemma 4, the queue length λ uniquely determines the lowest buyer type $\underline{x}$. Furthermore, the supporting line associated with subgradient $Q_1(\lambda)$ can be written as $U(\underline{x}) + Q_1(\lambda)(x - \underline{x})$. By Lemma 3, the supporting line is also given by $xQ_1(\lambda) - t(1 - Q_0(\lambda))$. Therefore, the entry fee t is given by $-(U(\underline{x}) - \underline{x}Q_1(\lambda))/(1 - Q_0(\lambda))$, and therefore uniquely determined by λ.

Step 3: Consider two such triples (t_a, λ^a, F^a) and (t_b, λ^b, F^b). Claim: $t_a < t_b$ *if and only if* $\lambda^a < \lambda^b$. For the remaining part of the proof, we will use the following geometric observation, which can be easily seen from Figure 3.

Lemma. *Consider two supporting lines a and b at point $(\underline{x}_a, U(\underline{x}_a))$ and $(\underline{x}_b, U(\underline{x}_b))$ and with slopes $Q_1(\lambda^a)$ and $Q_1(\lambda^b)$, respectively. If $Q_1(\lambda^a) < Q_1(\lambda^b)$, then the intercept between the supporting line a and the x-axis is strictly smaller than the intercept between the supporting line b and the x-axis. A similar statement holds between the intercepts between the supporting line and the y-axis.*

Proof. First, consider the intercepts between the supporting lines and the x-axis. By the definition of the supporting line,

$$U(\underline{x}_a) > U(\underline{x}_b) + Q_1(\lambda^b)(\underline{x}_a - \underline{x}_b), \tag{27}$$

where the strict inequality is due to the strict convexity of $U(x)$. This implies that

$$\underline{x}_b - \underline{x}_a > \frac{U(\underline{x}_b)}{Q_1(\lambda^b)} - \frac{U(\underline{x}_a)}{Q_1(\lambda^b)} >\geq \frac{U(\underline{x}_b)}{Q_1(\lambda^b)} - \frac{U(\underline{x}_a)}{Q_1(\lambda^a)},$$

where the second inequality follows from $Q_1(\lambda^a) < Q_1(\lambda^b)$. As the intercept between the supporting line and the x-axis is $\underline{x} - U(\underline{x})/Q_1(\lambda)$, the desired result follows.

Next, consider the intercepts between the supporting lines and the y-axis. Equation (27) and $Q_1(\lambda^a) < Q_1(\lambda^b)$ also imply that

$$U(\underline{x}_b) - U(\underline{x}_a) < \underline{x}_b Q_1(\lambda^b) - \underline{x}_a Q_1(\lambda^b) \leq \underline{x}_b Q_1(\lambda^b) - \underline{x}_a Q_1(\lambda^a).$$

As the intercept between the supporting line and the y-axis is $U(\underline{x}) - \underline{x}Q_1(\lambda)$, the desired result follows. □

Since $Q_1(\lambda^a) < Q_1(\lambda^b)$ if and only if $\lambda^a > \lambda^b$ by assumption 1, the above geometric result implies that

$$\lambda^a > \lambda^b \Leftrightarrow t_a \frac{1 - Q_0(\lambda^a)}{Q_1(\lambda^a)} < t_b \frac{1 - Q_0(\lambda^b)}{Q_1(\lambda^b)} \Leftrightarrow t_a(1 - Q_0(\lambda^a)) > t_b(1 - Q_0(\lambda^b)), \tag{28}$$

where we have written the intercepts in terms of entry fees, using Lemma 3. We now distinguish three different cases. First, if $t_a < 0 < t_b$, then the proof is immediate: the second and third inequality of equation (28) hold, hence $\lambda^a > \lambda^b$.

Second, if $0 < t_a < t_b$, then we will prove by contradiction. Assume $\lambda^a \leq \lambda^b$, then we have $(1 - Q_0(\lambda^a))/Q_1(\lambda^a) \leq (1 - Q_0(\lambda^b))/Q_1(\lambda^b)$ by assumption 5. Multiplying this inequality with $t_a < t_b$ gives $t_a(1 - Q_0(\lambda^a))/Q_1(\lambda^a) < t_b(1 - Q_0(\lambda^b))/Q_1(\lambda^b)$, which implies $\lambda^a > \lambda^b$ by equation (28). Hence, we have reached a contradiction.

Finally, if $t_a < t_b < 0$, then again we will prove by contradiction. Assume $\lambda^a \leq \lambda^b$, then we have $1 - Q_0(\lambda^a) \geq 1 - Q_0(\lambda^b)$ by assumption 2. Multiplying this inequality with $t_a < t_b$ gives $t_a(1 - Q_0(\lambda^a)) > t_b(1 - Q_0(\lambda^b))$, which implies $\lambda^a > \lambda^b$ equation (28). Hence, we have again reached a contradiction. Therefore, $t_a < t_b$ if and only if $\lambda^a > \lambda^b$. □

A.16 Proof of Proposition 13

First, denote by $r(\omega)$ the expected revenue of a mechanism ω. Then, the expected value of a seller y (in excess of his own value) who posts ω is $r(\omega) - q(\omega)y$. Then

$$\pi(y) = \max_{\omega \in M} \quad r(\omega) - q(\omega)y$$

where M is the set of all direct mechanisms. Because $\pi(y)$ is the supreme of a collection of linear functions, it is convex. Furthermore, the optimality of $\omega^s(y)$ implies that

$$\pi(y) = r(\omega^s(y)) - q(\omega^s(y))y \geq r(\omega^s(z)) - q(\omega^s(z))y = \pi(z) - q(\omega^s(z))(y - z).$$

Therefore, $-q(\omega^s(z))$ is the slope of a supporting line at point $(z, \pi(z))$ for the convex function π. Note that $\pi(1) = 0$ because the highest buyer type is also 1. The rest of the proof is the same as in Proposition 4. □

Web Appendices (not for publication)

Appendix B Multiple Equilibria

In this appendix, we show that our assumptions on the meeting technologies are not sufficient for the decentralized equilibrium to be unique. We consider a special meeting technology and show that there is a continuum of equilibria, each with a different allocation of buyers and sellers. However, as we noted in the main text, all equilibria are payoff-equivalent.

The meeting technology that we consider is constructed as follows: we take the bilateral meeting technology $P_1(\lambda) = 1 - e^{-\lambda}$, insert a linear segment between Λ_0 and Λ_1, and then

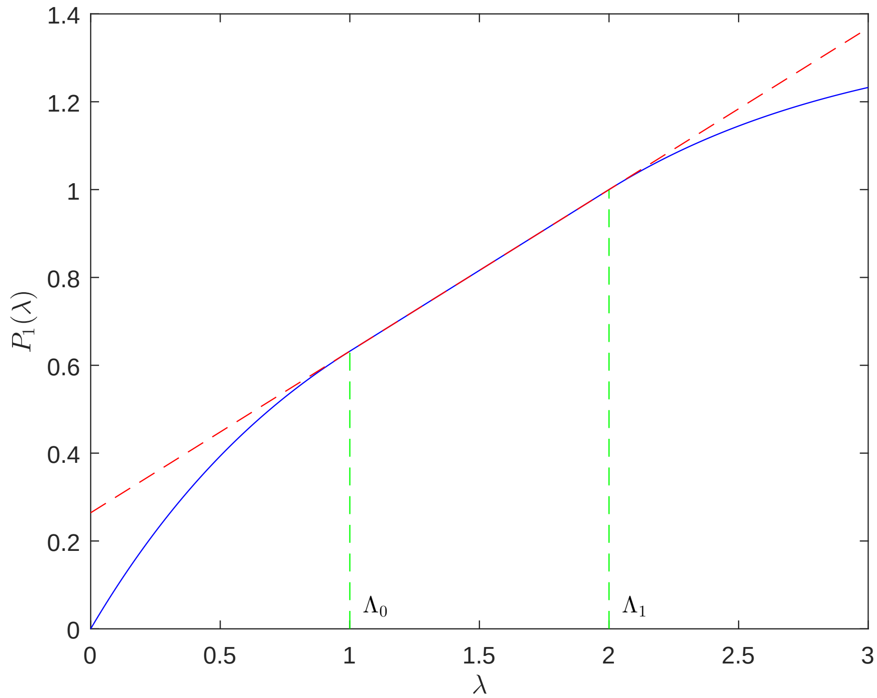

Figure B.1: A weakly concave bilateral meeting technology

properly scale it, as illustrated in Figure B.1 for $\Lambda_0 = 1$ and $\Lambda_1 = 2$. The resulting meeting technology is *weakly* concave instead of *strictly* concave.

Formally, $P_n(\lambda) = 0$ for $n \geq 2$ and

$$P_1(\lambda) = \begin{cases} \frac{1}{1+e^{-\Lambda_0}(\Lambda_1-\Lambda_0)} \left(1 - e^{-\lambda}\right) & \text{if } \lambda \leq \Lambda_0, \\ \frac{1}{1+e^{-\Lambda_0}(\Lambda_1-\Lambda_0)} \left(1 - e^{-\Lambda_0} + e^{-\Lambda_0}(\lambda - \Lambda_0)\right) & \text{if } \Lambda_0 \leq \lambda \leq \Lambda_1, \\ \frac{1}{1+e^{-\Lambda_0}(\Lambda_1-\Lambda_0)} \left(1 - e^{-\lambda+\Lambda_1-\Lambda_0} + e^{-\Lambda_0}(\Lambda_1 - \Lambda_0)\right) & \text{if } \lambda \geq \Lambda_1. \end{cases}$$

It is easy to see that $P_1(\lambda)$ is continuously differentiable and weakly concave and satisfies $\lim_{\lambda\to\infty} P_1(\lambda) = 1$. Note that $Q_1'(\lambda)$ is strictly decreasing, which implies $\phi_\lambda(\mu, \lambda) < 0$. This meeting technology satisfies all the assumptions in the paper. Hence, for any seller posting an auction with entry fee, there exists a unique queue compatible with the market utility function.

Suppose now that there are a measure 1 of sellers with value 0 and a measure $\Lambda \in (\Lambda_0, \Lambda_1)$ of buyers with value 1. Because $P_1(\lambda)$ is concave and buyers are homogeneous, perfect pooling is then an optimal allocation. However, the optimal allocation is not unique. To see this, consider k submarkets with seller measures $\alpha_1, \alpha_2, \ldots, \alpha_k$, and queue lengths $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively. We require $\Lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \Lambda_1$, $\sum_1^k \alpha_i = 1$, and $\sum_1^k \alpha_i\lambda_i = \Lambda$.

One example is $k = 2$, $\alpha_1 = \alpha_2 = 1/2$, and $\lambda_1 = \Lambda - \Delta\Lambda$ and $\lambda_2 = \Lambda + \Delta\Lambda$ with $\Delta\Lambda \leq \min(\Lambda - \Lambda_0, \Lambda_1 - \Lambda)$. Since $P_1(\lambda)$ is linear in $[\Lambda_0, \Lambda_1]$, we have

$$\sum_1^k \alpha_i P_1(\lambda_i) = P_1(\Lambda),$$

which implies that the above allocation with k submarkets generates the same surplus as pooling does. Hence, this allocation is also optimal.

Since the optimal surplus is $P_1(\Lambda)$, the marginal contribution to surplus of a buyer is simply $U = P_1'(\Lambda)$. The marginal contribution of a seller is thus $P_1(\Lambda) - \Lambda P_1'(\Lambda)$.

Next, we consider the decentralized equilibrium. For a seller posting price t or equivalently a second-price auction with entry fee t, the attracted queue length λ must satisfy the market utility condition $Q_1(\lambda)(1-t) = U$. Since $Q_1(\lambda)$ is strictly decreasing in λ, this condition has a unique solution, if a solution exists. Hence, there exits a unique queue compatible with the market utility function. In other words, we have proved Proposition 10 directly for the simplified environment with bilateral meetings and homogeneous buyers.

The pooling allocation can be easily decentralized as an equilibrium by all sellers posting price t given by

$$t = \frac{P_1(\Lambda) - \Lambda U}{P_1(\Lambda)} = 1 - \frac{\Lambda P_1'(\Lambda)}{P_1(\Lambda)}.$$

Next, we show that the more general optimal allocation with n submarkets can also be decentralized as an equilibrium. Suppose that there are α_i sellers posting prive t_i given by

$$t_i = \frac{P_1(\lambda_i) - \lambda_i U}{P_1(\lambda_i)} = 1 - \frac{\lambda_i P_1'(\Lambda)}{P_1(\lambda_i)}$$

It is easy to see that $Q_1(\lambda_i)(1 - t_i) = U$. Hence, the queue attracted by such a seller has length λ_i, such that the equilibrium coincides with the optimal allocation. Hence, we have thus proved that there exists an infinite number of payoff-equivalent equilibria.

Appendix C Multiple Compatible Queues

Below, we show that a violation of assumption 1 or 2 or 5 may lead to the following problem: For a given auction with entry fee, there might be multiple queues compatible with the market utility function.

We will mainly consider the special case where all buyers are homogeneous: there is a measure 1 of sellers and a measure Λ of buyers with value 1. The market utility of buyers is

U. We will assume that U is strictly positive. For a seller posting an auction with entry fee t, a queue with length λ is compatible with the market utility if and only if

$$Q_1(\lambda) - (1 - Q_0(\lambda))t = U,$$

where the first term on the left hand side denotes the auction payoff and the second term is the expected payment of entry fee. Rewriting the above equation gives

$$\frac{Q_1(\lambda)}{1 - Q_0(\lambda)} - \frac{U}{1 - Q_0(\lambda)} = t. \tag{29}$$

By assumption 2 and 5, the left hand side of the above equation is weakly decreasing in λ. Next we will show that with assumption 1, it is strictly decreasing in λ.

Suppose not, then there exists λ^* such that

$$\frac{d}{d\lambda}\left(\frac{Q_1(\lambda)}{1 - Q_0(\lambda)}\right)|_{\lambda=\lambda^*} = 0$$
$$\frac{d}{d\lambda}(1 - Q_0(\lambda))|_{\lambda=\lambda^*} = 0$$

which then implies that $Q_1'(\lambda^*) = 0$, contradicting with assumption 1.

Therefore, with assumption 1, 2, and 5, the left hand side of equation 29 is strictly decreasing in λ. For a given auction with entry fee, there is a unique queue compatible with the market utility. Furthermore, a higher entry fee implies a shorter queue. In the following, we will show that a violation of each of the above three assumptions will lead to the multiplicity problem.

C.1 $Q_1(\lambda)$ not strictly decreasing

We consider the following meeting technology introduced by Lester et al. (2015).

> *Pairwise Urn-Ball.* This technology is a variation on the urn-ball technology. Buyers first form pairs, after which each pair is randomly assigned to a seller in the submarket. That is, $P_n(\lambda) = 0$ for $n \in \{1, 3, 5, \ldots\}$ and $P_n(\lambda) = e^{-\lambda/2}\frac{(\lambda/2)^{n/2}}{(n/2)!}$ for $n \in \{0, 2, 4, \ldots\}$, which implies $\phi(\mu, \lambda) = 1 - e^{-\mu\left(1-\frac{1}{2}\frac{\mu}{\lambda}\right)}$.

Since $Q_1(\lambda) = 0$, which is not *not* strictly decreasing, assumption 1 fails. However, since $Q_0(\lambda) = 0$, assumption 2 and 5 hold (in this case $Q_1(\lambda)/(1 - Q_0(\lambda))$ is weakly decreasing).

Consider first the case of homogeneous buyers. Since $P_0(\lambda) = e^{-\lambda/2}$, it is strictly convex in λ. Since buyers are homogeneous and $1 - P_0(\lambda)$ is strictly concave, the social planner will pool all buyers and sellers into one market. The social surplus is $1 - P_0(\Lambda)$, and a buyer's marginal contribution to surplus is $-P_0'(\Lambda)$. For the decentralized equilibrium, equation (29) becomes $-U = t$. Therefore, when sellers post entry fee $-U$ (entry subsidy), any queue is compatible with the market utility.

This example can be easily extended to the case of heterogeneous buyers. Consider a measure 1 of sellers with value 0, and a measure Λ of buyers with value distribution $G(x)$ and $0 \leq x \leq 1$. Cai et al. (2017) show that for this meeting technology, it is socially optimal for all sellers to post an auction with entry fee t^*, which is given by

$$t^* = \int_0^1 \phi_\lambda(\Lambda(1 - G(y)), \Lambda) dy.$$

Since all sellers post the same mechanism, buyers will randomize over sellers with equal probabilities. By Lemma 2, the market utility function in this case is

$$U(x) = t^* + \int_0^x \phi_\mu(\Lambda(1 - G(z)), \Lambda) dz.$$

The marginal contribution of a buyer with value x equals $U(x)$ by Proposition 2. An auction with entry fee t^* thus solves the sellers' relaxed problem and sellers achieve the highest possible profit among all queues. Therefore, no seller will deviate and the above constitutes an equilibrium.

Next, we show, however, that multiple queues are compatible with the market utility function $U(x)$ and the auction with entry fee t^*. By Lemma 2, the queue (λ, F) that the seller attracts is compatible with the market utility function if and only if

$$U(x) = t^* + \int_0^x \phi_\mu(\lambda(1 - F(z)), \lambda) dz.$$

which implies that

$$\phi_\mu(\lambda(1 - F(x)), \lambda) = \phi_\mu(\Lambda(1 - G(x)), \Lambda)$$

Since $\phi_\mu(\lambda(1 - F), \lambda) = F(x)e^{-\lambda(1-F(x))\left(1-\frac{1-F(x)}{2}\right)}$, the above equation becomes,

$$F(x)e^{-\lambda(1-F(x))\left(1-\frac{1-F(x)}{2}\right)} = G(x)e^{-\Lambda(1-G(x))\left(1-\frac{1-G(x)}{2}\right)}.$$

For any $\lambda > 0$, at $x = 1$, the above equation solves for $F(x) = 1$; at $x = 0$, it solves for $F(x) = 0$. Because it is monotonic in $F(x)$, the above equation has a solution for $F(x)$ for every x. Since the RHS is monotonic in x, $F(x)$ solving the equation will be automatically monotonically increasing and thus it is a distribution function. Thus for any $\lambda > 0$, we can solve for a cumulative distribution $F(x)$.

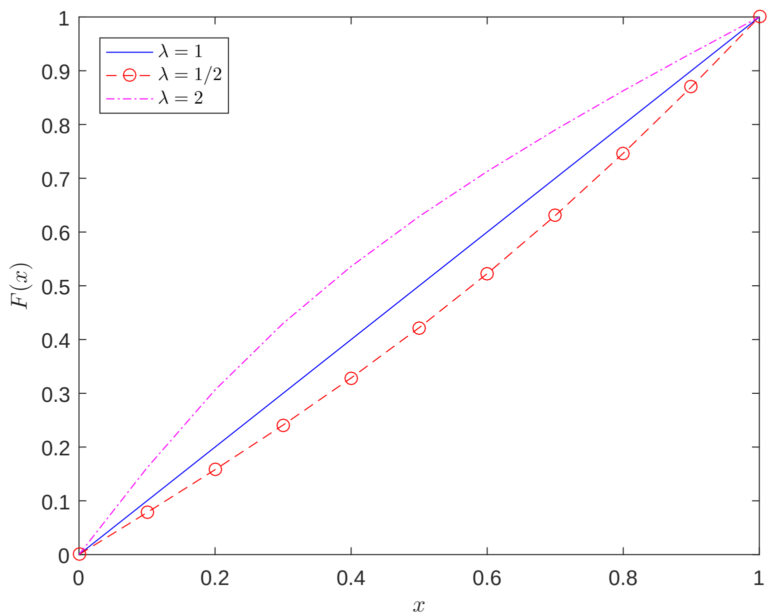

Figure C.1: Multiplicity problem with pairwise urn-ball

Therefore, there are infinitely many queues compatible with the market utility function. In Figure C.1, we set $\Lambda = 1$ and for the buyer value distribution G, we take the uniform distribution, $U[0,1]$. The solution for F is then plotted for $\lambda \in \left\{\frac{1}{2}, 1, 2\right\}$.

C.2 $1 - Q_0(\lambda)$ is not weakly decreasing in λ

Consider the following meeting technology.

> *Minimum Demand.* This technology consists of two rounds. In the first round, the b buyers in the submarket are allocated to the s sellers according to the urn-ball technology. In the second round, each seller draws a minimum demand requirement and operates only if the number of buyers that arrive weakly exceeds this minimum. We assume that the minimum demand requirements follows a geometric distribution, such that the minimum is weakly less than $n \in \mathbb{N}_1$ with probability $1-(1-\psi)^n$ for $0 < \psi < 1$. Hence, $P_n(\lambda) = e^{-\lambda}\frac{\lambda^n}{n!}\left(1-(1-\psi)^n\right)$ for $n \geq 1$ and $P_0\left(\lambda\right) = 1 - \sum_{n=1}^{\infty} P_n\left(\lambda\right) = e^{-\psi\lambda}$, which implies that $\phi\left(\mu, \lambda\right) = 1 - e^{-\mu} - e^{-\psi\lambda} + e^{-\lambda\psi-\mu(1-\psi)}$.

Note that $Q_1(\lambda) = \psi e^{-\lambda}$, which is strictly decreasing. However, $1 - Q_0(\lambda) = \phi_\mu(0, \lambda) = 1-(1-\psi)e^{-\psi\lambda}$, which is strictly increasing in λ, which violates assumption 2. Also $Q_1(\lambda)/(1-Q_0(\lambda))$ is strictly decreasing in λ, hence satisfying assumption 4.

There is a measure 1 of sellers with value 0, and a measure Λ of homogeneous buyers with value 1. Since $P_0(\lambda) = e^{-\psi\lambda}$, it is strictly convex in λ. Since buyers are homogeneous

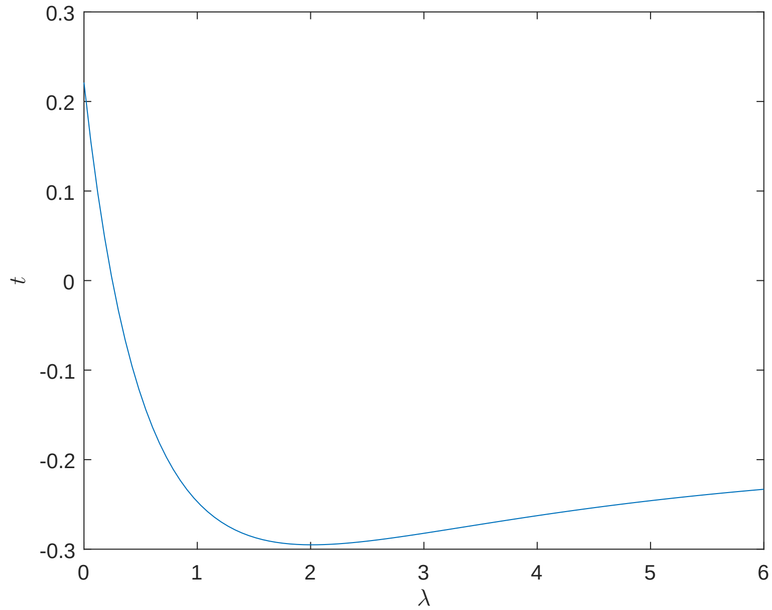

Figure C.2: Multiplicity problem with minimum demand

and $1 - P_0(\lambda)$ is strictly concave, the social planner will pool all buyers and sellers into one market. The social surplus is $1 - P_0(\Lambda)$, and a buyer's marginal contribution to surplus is $-P_0'(\Lambda)$.

We set $\Lambda = 1$. Thus at the planner's solution, the marginal contribution to surplus of a buyer should be $-P_0'(\Lambda)$. We set $U = -P_0'(\Lambda)$, and plot the left hand side of equation (29) in figure C.2. It is easy to see that there is another λ (other than Λ) satisfying equation (29).

C.3 $Q_1(\lambda)/(1 - Q_0(\lambda))$ is not weakly decreasing in λ

Consider the following meeting technology.

> Formally, $P_0(\lambda) = e^{-\lambda}$, $P_1(\lambda) = 1 - e^{-\lambda} - \frac{\lambda^2}{2}e^{-\lambda}$, $P_2(\lambda) = \frac{\lambda^2}{2}e^{-\lambda}$, and $P_n(\lambda) = 0$ for $n \geq 3$.

We can prove that both $Q_1(\lambda)$ and $1 - Q_0(\lambda)$ are strictly decreasing, but $Q_1(\lambda)/(1 - Q_0(\lambda))$ is not monotone, hence violating assumption 5.

Again consider the case of homogeneous buyers. Since $P_0(\lambda)$ is strictly convex, the social planner will pool all buyers and sellers into one market.

We set $\Lambda = 3$. Thus at the planner's solution, the marginal contribution to surplus of a buyer should be $-P_0'(\Lambda)$. We set $U = -P_0'(\Lambda)$, and plot the left hand side of equation (29) in figure C.3. It is easy to see that there is another λ (other than Λ) satisfying equation (29).

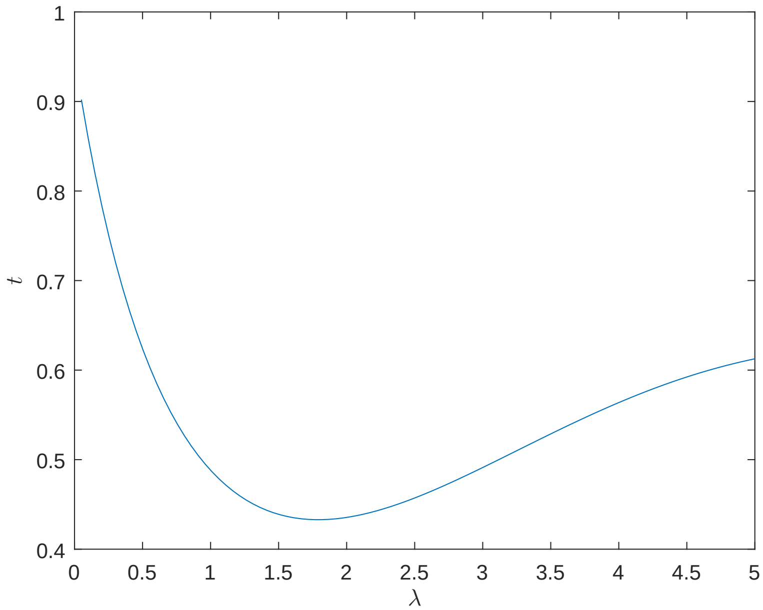

Figure C.3: Multiplicity problem

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