Coasian Equilibria in Sequential Auctions*

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Abstract

We study stationary equilibria in a sequential auction setting. A seller runs a sequence of standard first-price or second-price auctions to sell an indivisible object to potential buyers. The seller can commit to the rule of the auction and the reserve price of the current period but not to reserve prices of future periods. We prove the existence of stationary equilibria and establish a uniform Coase conjecture—as the period length goes to zero, the seller's profit from running sequential auctions converges to the profit of running an efficient auction uniformly at any point in time and in any stationary equilibrium.

1 Introduction

Consider the standard first-price or second-price auction setting with a seller, a single indivisible object, and multiple buyers whose values are independently drawn from

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a common distribution. It is well known that, under a regularity condition of the distribution function, the standard auction with a properly designed reserve price maximizes the seller's revenue among all mechanisms (Myerson (1981) and Riley and Samuelson (1981)). One of the key underlying assumptions for this result is the seller's ability to commit to her mechanism, specifically, the ability to withhold the object if no bidders place bids exceeding the reserve price. Of course, this assumption is not always realistic, and it is common practice to hold new auctions for unsold objects. A theoretical investigation into the dynamic aspect of this problem necessitates the use of dynamic games. The literature has advanced in at least two natural directions: reducing the set of mechanisms under consideration, as exemplified by Skreta (2006, 2016) and Doval and Skreta (2022), or characterizing attainable equilibrium revenues within a restricted class of mechanisms, as in Liu, Mierendorff, Shi, and Zhong (2019). It is fair to say that characterizing the seller's optimal revenues and selling mechanisms with limited commitment—without exogenously restricting equilibria, mechanisms, or the horizon of interactions—remains an open question.

It is worth noting that a separate body of economic literature developed much earlier, motivated by a similar question: how does a monopolist operate without commitments? Coase (1972) argues that a price-setting monopolist would lose her monopoly power and prices would drop quickly to her marginal cost if she can frequently adjust prices. This idea, known as the "Coase conjecture," has been confirmed by Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986): it holds in every stationary equilibrium in which the buyer's equilibrium strategy can only condition on the current price offer. In the "gap" case, where the buyer's values are strictly higher than the seller's marginal cost (with additional distributional assumptions), all perfect Bayesian equilibria are stationary. In the "no-gap" case, Ausubel and Deneckere (1989) show that, under some mild distributional assumptions, the seller's full-commitment revenue is achievable via non-Coasian equilibria if she can change prices frequently. This literature offers deep insight into the role of commitment in the classic price theory setting, where the trading mechanism is restricted to posting prices.

Understanding the workings of the Coasian force in an auction setting with a finite number of buyers is clearly relevant for understanding optimal selling mechanisms when the seller lacks commitment power. To fix ideas, consider the following problem. In each time period until the object is sold, the seller posts a reserve price and holds a standard auction (e.g., second-price auction or first-price auction). Each buyer can either wait for a future auction or submit a bid no smaller than the reserve price. Waiting is costly—both the buyers and the seller discount at the same rate. Within a period, the seller is committed to the rules of the auction and the announced reserve price. The seller cannot, however, commit to future reserve prices. The seller's commitment power varies with the period length (or effectively with the discount factor). As the period length shrinks, the seller's commitment power diminishes. This limit is of great theoretical interest. In the gap case, McAfee and Vincent (1997) show that the seller cannot obtain a revenue strictly higher than that from an efficient auction in the limit, thus extending the analysis of Fudenberg et al. (1985). Confirming the Coase conjecture in no-gap auctions is the task of this paper.[1] We establish the following strong form of Coase conjecture: the seller's profit from running sequential auctions with reserve prices converges to the revenue of running an efficient auction as the period length goes to zero, uniformly for all symmetric stationary equilibria and all posteriors, thus generalizing the uniform Coase conjecture of Ausubel and Deneckere (1989).

Liu et al. (2019) study non-stationary equilibria and provide precise conditions under which they improve upon efficient auctions in terms of the seller's revenue in the discounting limit. Their existence proof and construction of non-stationary equilibria using trigger strategies rely on the uniform Coase conjecture established in this paper, as the equilibrium incentive constraints must hold across all histories, not

[1] Milgrom (1987) analyzes a continuous-time version of the problem with explicit restrictions on the strategy space. This exercise is quite different, and the formal discrete-time foundation is still needed, as is well understood from the bargaining problem. McAfee and Vincent (1997) provide an example of a Coasian equilibrium for the no-gap case under the uniform distribution of types, which will be a building block for our general existence proof.

just the null history. The exact payoff bound in Liu et al. (2019) is derived indirectly through an auxiliary mechanism design problem. Notably, they demonstrate that the efficient auction can yield the highest possible revenue in the limit with multiple bidders. However, this alone does not imply the existence of stationary equilibria, the non-existence of non-stationary equilibria in any discrete-time game, nor the uniform convergence established in this paper. Together with the existence result we have established here, it implies that the Coase conjecture can hold for non-Markov equilibria in auctions.

The remainder of paper is organized as follows. Section 2 defines the model and introduces basic assumptions. Section 3 states the result. Section 4 uses an example to confirm the result. Section 5 contains the proof. Section 6 concludes with open questions.

2 Model

A seller (she) wants to sell an indivisible object to n potential buyers (he). Buyer i privately observes his own valuation for the object $v^i \in [0,1]$. We use $(v^i, v^{-i}) \in [0,1]^n$ to denote the vector of the n buyers' valuations, and $v \in [0,1]$ to denote a generic buyer's valuation. Each v^i is drawn independently from a common distribution with full support, c.d.f. $F(\cdot)$, and a continuously differentiable density $f(\cdot)$ such that $f(v) > 0$ for all $v \in (0,1)$. We write $F(v|v \leq x) = F(v)/F(x)$ as the truncated distribution of F on $[0,x]$ for $x > 0$. The seller's reservation value for the object is constant over time and we normalize it to zero.

Time is discrete and the period length is denoted by Δ. In each period $t = 0, \Delta, 2\Delta, \ldots$, the seller runs a second-price auction with a reserve price. The case of first-price auction or a mixture of first-price and second-price auctions over time can be treated in the same way with a period-by-period payoff-equivalence argument. To simplify notation, we often do not explicitly specify the dependence of the game on Δ. The timing within period t is as follows. First, the seller publicly announces a reserve price p_t for the auction run in period t, and invites all buyers to submit a valid bid, which is restricted to the interval $[p_t, 1]$. After observing p_t, all buyers decide simultaneously either to bid or to wait. If at least one valid bid is submitted, the winner and the payment are determined according to the rules of the second-price auction and the game ends. If no valid bid is submitted, the game proceeds to the next period. Both the seller and the buyers are risk-neutral and have a common discount rate $r > 0$. This implies a discount factor per period equal to $\delta = e^{-r\Delta} < 1$. If buyer i wins in period t and has to make a payment π^i, then his payoff is $e^{-rt}(v^i - \pi^i)$, and the seller's payoff is $e^{-rt}\pi^i$.

We assume that the seller has limited commitment power. She can commit to the reserve price that she announces for the current period: if a valid bid is placed, then the object is sold according to the rules of the announced auction and she cannot renege. She cannot commit, however, to future reserve prices: if the object was not sold in a period, the seller can always run another auction with a new reserve price in the next period. She cannot promise to stop auctioning an unsold object, or commit to a predetermined sequence of reserve prices.

We denote by $h_t = (p_0, p_\Delta, \ldots, p_{t-\Delta})$ the public history at the beginning of $t > 0$ if no bidder has placed a valid bid up to t, and write $h_0 = \emptyset$ for the history at which the seller chooses the first reserve price.[2] Let H_t be the set of such histories. A (behavior) strategy for the seller specifies a Borel-measurable function $p_t : H_t \to P[0, 1]$ for each $t = 0, \Delta, 2\Delta, \ldots$, where $P[0, 1]$ is the space of Borel probability measures endowed with the weak* topology.[3] A (behavior) strategy for buyer i specifies a function $b_t^i : H_t \times [0, 1] \times [0, 1] \to P[0, 1]$ for each $t = 0, \Delta, 2\Delta, \ldots$, where we assume that $b_t^i(h_t, p_t, v^i)$ is Borel-measurable in v^i, for all $h_t \in H_t$, and all $p_t \in [0, 1]$, and that $\operatorname{supp} b_t^i(h_t, p_t, v^i) \subset \{0\} \cup [p_t, 1]$, where "0" denotes no bid or an invalid bid.

We consider perfect Bayesian equilibria (PBE), and we will focus on symmetric weak Markov (or stationary) equilibria. Weak-Markov equilibria are defined as

[2] We do not have to consider other histories because the game ends if someone places a valid bid.

[3] We slightly abuse notation by using p_t both for the seller's strategy and the announced reserve price at a given history.

follows:

Definition 1. An equilibrium $(p, b) \in \mathcal{E}(\Delta)$ is a *weak-Markov (or stationary) equilibrium* if the buyers' strategies depend only on the reserve price announced for the current period.

3 Existence and Uniform Coase Conjecture

Following Ausubel and Deneckere (1989), we impose the following assumption. It is not needed for existence but is used to extend Coasian conjecture to our auction setting.

Assumption 1. *There exist constants* $0 < M \leq 1 \leq L < \infty$ *and* $\alpha > 0$ *such that* $Mv^\alpha \leq F(v) \leq Lv^\alpha$ *for all* $v \in [0, 1]$.

In any equilibrium of the discrete-time game, all buyers play pure strategies that are characterized by history-dependent cutoffs. This is captured by the following Lemma which establishes the "skimming property," an auction analog of a result by Fudenberg et al. (1985). Its proof is standard, and thus omitted.

Lemma 1. *[Skimming Property] Let* $(p, b) \in \mathcal{E}(\Delta)$. *Then, for each* $t = 0, \Delta, 2\Delta, \ldots$, *there exists a function* $\beta_t : H_t \times [0, 1] \to [0, 1]$ *such that every bidder with valuation above* $\beta_t(h_t, p_t)$ *places a valid bid and every bidder with valuation below* $\beta_t(h_t, p_t)$ *waits if the seller announces reserve price* p_t *at history* h_t.

We now state the result:

Proposition 1.

1. *(Existence) A stationary equilibrium exists for every* $r > 0$ *and* $\Delta > 0$.

2. *(Uniform Coase Conjecture) Suppose Assumption 1 holds. For every* $\varepsilon > 0$, *there exists* $\Delta_\varepsilon > 0$ *such that for all* $\Delta < \Delta_\varepsilon$, *all* $x \in [0, 1]$, *and every symmetric stationary equilibrium* (p, b) *of the game with period length* Δ *and a*

truncated distribution $F(v|v \leq x)$ on $[0,x]$, the seller's profit associated with this equilibrium, $\Pi^{\Delta}(p,b|x)$, is bounded above by $(1+\varepsilon)\,\Pi^{E}(x)$, where $\Pi^{E}(x)$ is the seller's profit from the efficient auction under this truncated distribution.

The second part of the proposition implies that the seller's profit in every symmetric stationary equilibrium converges to the profit of the efficient auction as $\Delta \to 0$.[4] Note that the payoff bound is independent of the equilibrium being considered. Given any equilibrium (p,b) on F, its restriction to any continuation game with a truncated distribution $F(v|v \leq x)$ is also an equilibrium for the truncated distribution, where the result applies. Therefore, the payoff bound applies to continuation payoffs across all histories of the same original equilibrium (p,b) on F. As a result, the uniform Coase conjecture is confirmed: $\Pi^{\Delta}(p,b|x)/\Pi^{E}(x) \to 1$ uniformly for all $x \in (0,1]$.

4 An Example

In this section, we use an example to confirm the Coase conjecture and the uniform version of the Coase conjecture. Let us consider the stationary, linear equilibrium constructed by McAfee and Vincent (1997) for a sequential second-price auction with a uniform distribution of types:

- If buyers' types are in $[0,v_t]$, the seller's reserve price is such that all buyers with a valuation greater than γv_t will bid, so $v_{t+\Delta} = \gamma v_t$.
- A buyer with a valuation v will bid if the reserve price is βv or lower. Therefore, the equilibrium reserve price with support $[0,v_t]$ is $p_t = \beta\gamma v_t$.

Using an argument similar to (1) and (2) in Section 5, it can be verified that γ

[4] Notice that in contrast to the Coase conjecture for one buyer, Proposition 1.(ii) does not show that the initial reserve price p_0 converges to zero. This is in fact not the case in the auction setting, as was noted by McAfee and Vincent (1997). However, reserve prices for $t > 0$ converge to zero which is sufficient for the convergence of equilibrium profits to the profit of an efficient auction—the counterpart of the Coase conjecture in the auction setting.

and β satisfy the two equalities:

$$\beta = 1 - \frac{\delta}{n} \cdot \frac{1-\gamma^n}{1-\delta\gamma^n} = 1 + \frac{1}{n}\left[(1-\delta)\cdot\frac{\gamma}{1-\gamma} - 1\right].$$

Hence, $2\gamma - 1 = \delta\gamma^{n+1}$. The support of buyers' types shrinks by a factor of γ, and as $\Delta \to 0$ or $\delta = e^{-r\Delta} \to 1$, $\gamma \to \gamma_0 < 1$. Therefore, the screening of types, and consequently the decline in reserve prices, occurs arbitrarily quickly. This confirms the Coase conjecture.

We still need to verify the uniform version of the Coase conjecture. To this end, consider the support $[0, x]$ of buyer types. The seller's unconditional equilibrium revenue is

$$R(x) = \max_{y\in[0,x]} \int_y^x zd\left[nxz^{n-1} - (n-1)z^n\right] + \beta yn(x-y)\,y^{n-1} + e^{-r\Delta}R(y),$$

which is a linear equilibrium version of the general problem (1) in Section 5. The seller chooses a reserve price βy, so the cutoff buyer type is y. The first term of the objective function represents the seller's revenue when the second highest bidder's valuation exceeds the cutoff y, where $nxz^{n-1} - (n-1)z^n$ is the distribution function of the second highest value when all values are below x. The second term captures the seller's revenue from selling at the reserve price βy when exactly one buyer's valuation is above y. The third term represents the seller's revenue from the continuation game.

In the linear equilibrium, the solution to the seller's optimization problem is γx, and by the envelope theorem, we have

$$R(x) = \frac{n\beta\gamma^n + (n-1)(1-\gamma^n)}{n+1}x^{n+1}.$$

The seller's equilibrium revenue conditional on reaching the support $[0, x]$ is

$$\Pi^{\Delta}(x) = \frac{R(x)}{x^n} = \frac{n\beta\gamma^n + (n-1)(1-\gamma^n)}{n+1}x.$$

The seller's revenue from running an efficient auction for the same range of buyers' types is

$$\Pi^{E}(x) = \frac{n-1}{n+1}x.$$

Hence,

$$\frac{\Pi^{\Delta}(x)}{\Pi^{E}(x)} = \frac{n}{n-1}\beta\gamma^{n} + (1-\gamma^{n}) = \frac{n-(n+1)\gamma+(2\gamma-1)e^{r\Delta}}{(n-1)(1-\gamma)}.$$

It is readily verified that $\frac{\Pi^{\Delta}(x)}{\Pi^{E}(x)} > 1$ and is *independent* of x. Notice further that $\lim_{\Delta\to 0}\frac{\Pi^{\Delta}(x)}{\Pi^{E}(x)} = 1$. These two observations confirm the uniform Coase conjecture.

Figure 1 numerically demonstrates how the revenue ratio changes with respect to the number of buyers and the discount factor.

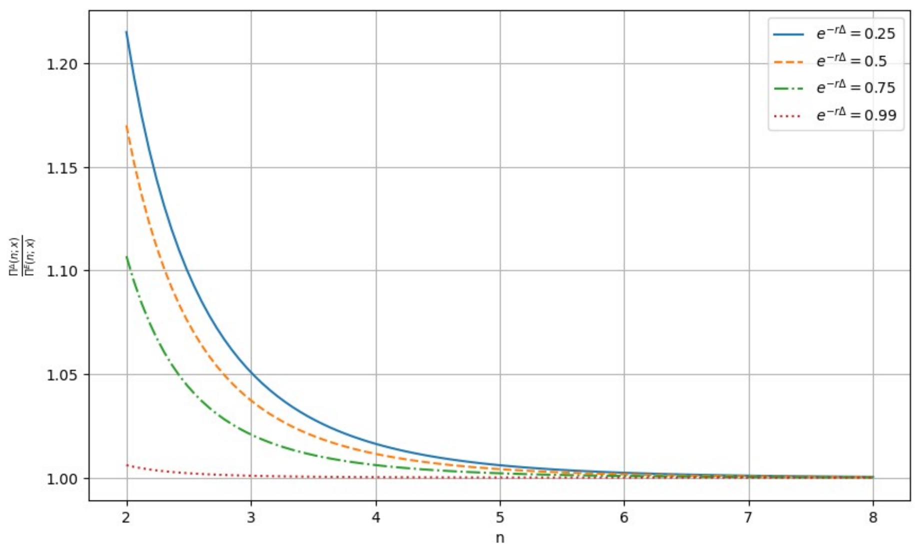

Figure 1: Π^{Δ}/Π^{E} is decreasing in the number of buyers n and the discount factor $e^{-r\Delta}$.

5 Proof

We adopt Ausubel and Deneckere (1989)'s notation and assume that the types of the bidders are i.i.d. draws from $U[0,1]$. We denote the type of buyer i by q^i. The valuation for each type is given by the function $v(q) := F^{-1}(q)$. Assumption 1 implies that the same condition also holds for $v(q)$ and corresponds to the assumption made in Definition 5.1 in Ausubel and Deneckere (1989). In the following we will use that F is continuous and strictly increasing (as in Ausubel and Deneckere (1989) we could relax this even further to general distribution functions but this is not necessary for the purpose of the present paper).[5] Since the proof of Proposition 1 follows closely the approach of Ausubel and Deneckere (1989), we emphasize the parts of the proof of Ausubel and Deneckere (1989) that need to be modified for the case of $n \geq 2$.

5.1 Proof of Proposition 1 (i)

The strategy for the existence proof is as follows. McAfee and Vincent (1997) has shown that a weak-Markov equilibrium exists under a uniform distribution. For a general distribution, we replace its lower tail on the interval $[0, \bar{q}]$ with a uniform distribution (this strategy was devised by Ausubel and Deneckere (1989) for the one-buyer case). We show that the weak-Markov equilibrium for the uniform part at the lower end of the distribution can be extended to a weak-Markov equilibrium for the entire distribution. The final step is to observe that the equilibrium for the modified distribution converges (via a subsequence) to an equilibrium for the original distribution as the uniform tail of the modified distribution diminishes.

In a weak-Markov equilibrium, the buyers' strategy can be described by a function $P : [0,1] \to [0,1]$. A bidder with type q^i places a valid bid if and only if the announced reserve price is smaller than $P(q^i)$. Given that v is strictly increasing, Lemma 1 implies that P is non-decreasing.

[5] In Ausubel and Deneckere (1989) the valuation is decreasing in the type. We define v to be increasing so that higher types have higher valuations.

Also by Lemma 1, the posterior of the seller at any history is described by the supremum of the support, which we denote by q. If all buyers play according to P, the seller's (unconditional) continuation profit for given q is[6]

$$R(q) := \max_{y \in [0,q]} \int_y^q v(z) d\left[nqz^{n-1} - (n-1)z^n\right] + P(y)\, n\,(q-y) y^{n-1} + e^{-r\Delta} R(y). \quad (1)$$

The first term in the objective function on the right-hand side of (1) represents the seller's revenue when all buyers place bids above the reserve price, with the transaction price being the second-highest buyer value. The second term captures the seller's revenue when exactly one buyer places a bid, where the transaction price is the reserve price. The third term represents the seller's discounted revenue from the continuation game when no buyer places a bid in the current period.

Let $Y(q)$ be the argmax correspondence for the optimization problem of (1) and define $y(q) := \sup Y(q)$. Because the objective satisfies a single-crossing property, $Y(q)$ is increasing and hence single-valued almost everywhere. If $Y(q)$ is single-valued at q the seller announces a reserve price $S(q) = P(y(q))$ if the posterior has upper bound q.

The buyers' indifference condition for the case that $Y(q)$ is single-valued so that the seller does not randomize, is given by:

$$v(q) - P(q) = e^{-r\Delta}\left[v(q) - \frac{(y(q))^{n-1}}{q^{n-1}} S(q) - \frac{1}{q^{n-1}} \int_{y(q)}^q v(x) dx^{n-1}\right]. \quad (2)$$

The left-hand side of Equation (2) is the payoff for the marginal buyer of type q who buys at the reserve price $P(q)$. The right-hand side is the buyer's payoff if he delays the purchase to the next period, when q becomes the highest remaining type: he either buys at the new reserve price if no other buyers place a bid, or competes with the highest-valued bidder below him.

[6]Dividing the RHS by q^n and replacing $R(y)$ by $y^n R(y)$ would yield the conditional continuation profit. The unconditional version is more convenient for the subsequent development.

If the seller randomizes over $Y(q)$ according to some probability measure μ, then

$$v(q) - P(q) = e^{-r\Delta}\left[v(q) - \int_{Y(q)}\left\{\frac{y^{n-1}}{q^{n-1}}P(y) + \frac{1}{q^{n-1}}\int_y^q v(x)dx^{n-1}\right\}d\mu(y)\right], \quad (3)$$

which may require that μ depends on $P(q)$.[7]

We will be looking for left-continuous functions R and P such that (1) and (2) are satisfied. If this is true for all $q \in [0, \bar{q}]$, then we say that (P, R) support a weak-Markov equilibrium on $[0, \bar{q}]$. The goal is to show the existence of a pair (P, R) that supports a weak-Markov equilibrium on $[0, 1]$. As in Ausubel and Deneckere (1989), we can show that the seller's continuation profit is Lipschitz-continuous in q.

Lemma 2. *[cf. Lemma A.2 in Ausubel and Deneckere (1989)] If (P, R) supports a weak-Markov equilibrium on $[0, \bar{q}]$, then R is increasing and Lipschitz continuous satisfying*

$$0 < R(q_1) - R(q_2) \leq n(q_1 - q_2)$$

for all $0 \leq q_2 < q_1 \leq \bar{q}$.

Proof. First, we show monotonicity:

$$\begin{aligned}
R(q_1) &= \int_{y(q_1)}^{q_1} v(z)d\left[nq_1 z^{n-1} - (n-1)z^n\right] + P(y(q_1))\, n\,(q_1 - y(q_1))(y(q_1))^{n-1} + e^{-r\Delta}R(y(q_1)) \\
&\geq \int_{y(q_2)}^{q_1} v(z)d\left[nq_1 z^{n-1} - (n-1)z^n\right] + P(y(q_2))\, n\,(q_1 - y(q_2))(y(q_2))^{n-1} + e^{-r\Delta}R(y(q_2)) \\
&> \int_{y(q_2)}^{q_2} v(z)d\left[nq_2 z^{n-1} - (n-1)z^n\right] + P(y(q_2))\, n\,(q_2 - y(q_2))(y(q_2))^{n-1} + e^{-r\Delta}R(y(q_2)) \\
&= R(q_2).
\end{aligned}$$

The first inequality follows from the optimality of $y(q_1)$. The second inequality follows because $q_2 < q_1$. To show Lipschitz continuity, notice that the revenue from sales to

[7] In the following, we give details for the case that the seller does not randomize and refer to Ausubel and Deneckere (1989) for the discussion of randomization by the seller.

types below q_2 in the continuation starting from q_1 is at most $R(q_2)$ and the revenue from types between q_2 and q_1 is bounded above by $P(q_1)(q_1^n - q_2^n)$.[8] Hence

$$\begin{aligned} R(q_1) - R(q_2) &\leq P(q_1)(q_1^n - q_2^n) \\ &\leq (q_1^n - q_2^n) \\ &\leq n(q_1 - q_2) \end{aligned}$$

This proves the desired result. $\square$

Using this lemma, we can show that an existence result for $[0, \bar{q}]$ can be extended to the whole interval $[0, 1]$.

Lemma 3. *[cf. Lemma A.3 in Ausubel and Deneckere (1989)] Suppose* $(P_{\bar{q}}, R_{\bar{q}})$ *supports a weak-Markov equilibrium on* $[0, \bar{q}]$*, then there exists* (P, R) *which supports a weak-Markov equilibrium on* $[0, 1]$.

Proof. We extend $(R_{\bar{q}}, P_{\bar{q}})$ to some $[0, \bar{q}']$. Define

$$R_{\bar{q}'}(q) = \max_{0 \leq y \leq \min\{\bar{q}, q\}} \int_y^q v(z) d\left[nqz^{n-1} - (n-1)z^n\right] + P_{\bar{q}}(y)\, n\,(q - y)y^{n-1} + e^{-r\Delta} R_{\bar{q}}(y)$$

with $y_{\bar{q}'}(q)$ as the supremum of the argmax correspondence. Moreover, we define $P_{\bar{q}'}(q)$ by

$$v(q) - P_{\bar{q}'}(q) = e^{-r\Delta}\left[v(q) - \frac{(y_{\bar{q}'}(q))^{n-1}}{q^{n-1}} P_{\bar{q}}(y_{\bar{q}'}(q)) - \frac{1}{q^{n-1}} \int_{y_{\bar{q}'}(q)}^q v(x) dx^{n-1}\right].$$

For $\bar{q}' = \min\left\{1, \sqrt[n]{\bar{q}^n + (1 - e^{-r\Delta})R_{\bar{q}}(\bar{q})}\right\}$, the constraint in the maximization in the

[8] Suppose by contradiction that for the posterior $[0, q_1]$, the expected payment that the seller can extract from some type $q \in [q_2, q_1]$ is greater or equal than $P(q_1)$. In order to arrive at a history where the posterior is $[0, q_1]$, the seller must have used reserve price $P(q_1)$ in the previous period. But then all types in $[q, q_1]$ would prefer to bid in the previous period because they expect to make higher payments if they wait. This is a contradiction.

definition of $R_{\bar{q}'}(q)$ is not binding and moreover

$$R_{\bar{q}'}(q) = \max_{0 \le y \le q} \int_y^q v(z) d\left[nqz^{n-1} - (n-1)z^n\right] + P_{\bar{q}'}(q)\, n\, (q-y) y^{n-1} + e^{-r\Delta} R_{\bar{q}'}(y)$$

For $y \in [\bar{q}, q]$ we have

$$\begin{aligned}
&\int_y^q v(z) d\left[nqz^{n-1} - (n-1)z^n\right] + P_{\bar{q}'}(q)\, n\, (q-y) y^{n-1} + e^{-r\Delta} R_{\bar{q}'}(y) \\
\le &\int_y^q 1 d\left[nqz^{n-1} - (n-1)z^n\right] + n\, (q-y) y^{n-1} + e^{-r\Delta} R_{\bar{q}'}(y) \\
= &\, q^n - y^n + e^{-r\Delta} R_{\bar{q}'}(q) \\
\le &\, (1 - e^{-r\Delta}) R_{\bar{q}}(\bar{q}) + e^{-r\Delta} R_{\bar{q}'}(q) \\
\le &\, (1 - e^{-r\Delta}) R_{\bar{q}'}(q) + e^{-r\Delta} R_{\bar{q}'}(q) \\
\le &\, R_{\bar{q}'}(q).
\end{aligned}$$

In the first and second steps, we have used that the payments $v(z)$ and $P_{\bar{q}'}(q)$ are less than or equal to one. In the third step, we have used that $\bar{q}' = \min\left\{1, \sqrt[n]{\bar{q}^n + (1 - e^{-r\Delta}) R_{\bar{q}}(\bar{q})}\right\}$; since $\bar{q} \le y \le q \le \bar{q}'$, this implies $q^n - y^n \le (1 - e^{-r\Delta}) R_{\bar{q}}(\bar{q})$. The fourth step uses $R_{\bar{q}}(\bar{q}) = R_{\bar{q}'}(\bar{q})$ and that $R_{\bar{q}'}$ is increasing. Thus $(P_{\bar{q}'}, R_{\bar{q}'})$ supports a weak-Markov equilibrium on $[0, \bar{q}']$. Since $R_{\bar{q}}(\bar{q}) > 0$, a finite number of repetitions suffices to extend $(P_{\bar{q}}, R_{\bar{q}})$ to the entire interval $[0, 1]$. □

We are now ready to complete the existence proof.

Proof. [Proof of Proposition 1 (i)] As in Ausubel and Deneckere (1989), we consider a sequence of valuation functions

$$\hat{v}_\eta(q) = \begin{cases} v(q), & \text{if } q \ge \frac{1}{\eta} \\ v\left(\frac{1}{\eta}\right) \eta q, & \text{otherwise.} \end{cases}$$

This corresponds to the original distribution except that on the interval $[0, 1/\eta]$, we have made the distribution uniform. McAfee and Vincent (1997) show that there exist $(\tilde{P}_{1/\eta}, \tilde{R}_{1/\eta})$ that support a weak-Markov equilibrium on $[0, 1/\eta]$. Hence, by Lemma 3, for each $\eta = 1, 2, \ldots$, there exists a pair (P_η, R_η) that supports a weak-Markov equilibrium on $[0, 1]$. As in Ausubel and Deneckere (1989), we can assume that P_η converges point-wise for all rationals to some function $\Phi(s)$, $s \in \mathbb{Q} \cap [0, 1]$ and taking left limits we can extend this limit to a non-decreasing, left-continuous function $P : [0, 1] \to [0, 1]$. Also, by Lemma 2, after taking a sub-sequence, we may assume that (R_n) converges uniformly to a continuous function R. We have to show that (P, R) supports a weak-Markov equilibrium for v. But given Lemma 2 and 3, only minor modifications are needed to apply the proof of Theorem 4.2 from Ausubel and Deneckere (1989). $\square$

5.2 Proof of Proposition 1 (ii)

Before we begin with the proof, we note that in contrast to the case of one buyer analyzed by Ausubel and Deneckere (1989), the first reserve price in a continuation game where the seller's posterior is v_t need not converge to zero as $\Delta \to 0$.[9] Nevertheless, we obtain the Coase conjecture because prices fall arbitrarily quickly as $\Delta \to 0$. On the buyer side, the strategy is described by a cutoff for the reserve price. A buyer places a bid if and only if the current reserve price is below the cutoff. The Markov property of the buyer's strategy implies that the cutoff only depends on the buyer's type, it is independent of time and of the history of previous reserve prices. As $\Delta \to 0$, the equilibrium cutoff of a buyer with type v converges to the payment that this type would make in a second-price auction without reserve price. Also reserve prices decline arbitrarily quickly so that the delay of the allocation vanishes for all buyers as $\Delta \to 0$. Therefore, the seller's profit converges to the profit of an efficient auction.

We want to show that the profit of the seller in any weak-Markov equilibrium of a subgame that starts with the posterior $[0, q]$, converges (uniformly over q) to $\Pi^E(q)$

[9] For the uniform distribution, this was already noted by McAfee and Vincent (1997).

as $\Delta \to 0$. The proof consists of two main steps. The first step shows that for any type $\xi \in [0,1]$, any $\Delta > 0$, and any weak-Markov equilibrium supported by some pair (P, R), the expected payment that the seller can extract from type ξ is bounded by $\xi^{n-1}P(\xi)$. We prove this by showing that the expected payment conditional on winning is bounded by $P(\xi)$. The second step is to show that $P(1)$ is bounded above by the expected payment from an efficient auction, and we obtain the uniform bound of the continuation payoff by rescaling the distribution as permitted by Assumption 1.

We begin with the first step.

Lemma 4. *Let (P, R) support a weak-Markov equilibrium in the game for $\Delta > 0$. Suppose that in this equilibrium, type $\xi \in [0,1]$ trades in period t, let the posterior in period t be $q_t \geq \xi$, and denote the marginal type in period t by $q_t^+ \leq \xi$. Then we have*

$$P(\xi) \geq \int_{q_t^+}^{\xi} v(x) \frac{dx^{n-1}}{\xi^{n-1}} + \frac{\left(q_t^+\right)^{n-1}}{\xi^{n-1}} P(q_t^+), \qquad \forall \xi \in [0,1],$$

and hence

$$R(q) \leq \int_0^q P(x)\, dx^n, \qquad \forall q \in (0,1].$$

Proof. For $q_t^+ = \xi$ the RHS of the first inequality becomes $P(q_t^+) = P(\xi)$. Hence it suffices to show that

$$\int_q^{\xi} v(x) dx^{n-1} + q^{n-1}P(q)$$

is increasing in q. For $q > \hat{q}$ we have

$$\begin{aligned}
&\int_q^{\xi} v(x) dx^{n-1} + q^{n-1}P(q) - \int_{\hat{q}}^{\xi} v(x) dx^{n-1} - \hat{q}^{n-1}P(\hat{q}) \\
=&q^{n-1}P(q) - \hat{q}^{n-1}P(\hat{q}) - \int_{\hat{q}}^{q} v(x) dx^{n-1}
\end{aligned}$$

Using (2), we have

$$
\begin{aligned}
& q^{n-1}P(q) - \hat{q}^{n-1}P(\hat{q}) \\
=& \left(1-e^{-r\Delta}\right) q^{n-1}v(q) + e^{-r\Delta}\int_{y(q)}^{q} v(x)dx^{n-1} + e^{-r\Delta}\left(y(q)\right)^{n-1}P(y(q)) \\
& - \left(1-e^{-r\Delta}\right)\hat{q}^{n-1}v(\hat{q}) - e^{-r\Delta}\int_{y(\hat{q})}^{\hat{q}} v(x)dx^{n-1} - e^{-r\Delta}\left(y(\hat{q})\right)^{n-1}P(y(\hat{q})) \\
=& \left(1-e^{-r\Delta}\right)\left(q^{n-1}v(q) - \hat{q}^{n-1}v(\hat{q})\right) + e^{-r\Delta}\left(\left(y(q)\right)^{n-1}P(y(q)) - \left(y(\hat{q})\right)^{n-1}P(y(\hat{q}))\right) \\
& + e^{-r\Delta}\int_{\hat{q}}^{q} v(x)dx^{n-1} - e^{-r\Delta}\int_{y(\hat{q})}^{y(q)} v(x)dx^{n-1}
\end{aligned}
$$

and hence

$$
\begin{aligned}
& q^{n-1}P(q) - \hat{q}^{n-1}P(\hat{q}) - \int_{\hat{q}}^{q} v(x)dx^{n-1} \\
=& \left(1-e^{-r\Delta}\right)\left(q^{n-1}v(q) - \hat{q}^{n-1}v(\hat{q})\right) + e^{-r\Delta}\left(\left(y(q)\right)^{n-1}P(y(q)) - \left(y(\hat{q})\right)^{n-1}P(y(\hat{q}))\right) \\
& - \left(1-e^{-r\Delta}\right)\int_{\hat{q}}^{q} v(x)dx^{n-1} - e^{-r\Delta}\int_{y(\hat{q})}^{y(q)} v(x)dx^{n-1} \\
=& e^{-r\Delta}\left(\left(y(q)\right)^{n-1}P(y(q)) - \left(y(\hat{q})\right)^{n-1}P(y(\hat{q})) - \int_{y(\hat{q})}^{y(q)} v(x)dx^{n-1}\right) \\
& + \left(1-e^{-r\Delta}\right)\int_{\hat{q}}^{q} v'(x)x^{n-1}dx
\end{aligned}
$$

Proceeding inductively, we get

$$
q^{n-1}P(q) - \hat{q}^{n-1}P(\hat{q}) - \int_{\hat{q}}^{q} v(x)dx^{n-1} = \sum_{k=0}^{\infty} e^{-k\Delta}\left(1-e^{-r\Delta}\right)\int_{y^k(\hat{q})}^{y^k(q)} v'(x)x^{n-1}dx > 0,
$$

where $y^k(\cdot)$ denotes the function obtained by applying $y(\cdot)$ k times. This shows the first inequality.

For the second inequality, notice that the RHS of the first inequality is the payment that the seller can extract from type ξ if ξ wins the auction. This is bounded by $P(\xi)$

as the first inequality shows. The seller's profit if the posterior at time t is q, therefore satisfies

$$R(q) \leq \int_0^q e^{-r(T(x)-t)} P(x) dx^n,$$

where $T(x)$ denotes the trading time of type x in the weak-Markov equilibrium. This implies the second inequality. $\square$

For the second step, fix the distribution and the corresponding function v and define $v_x : [0,1] \to [0,1]$ such that for all $x \in (0,1]$,

$$v_x(q) := \frac{v(qx)}{v(x)}.$$

Using Helly's selection theorem, we can extend this definition to $x = 0$, by taking the a.e.-limit of a subsequence of functions v_x. Denote by $\mathcal{E}^{wM}(\Delta, x)$ the weak-Markov equilibria of the game with discount factor Δ and distribution given by v_x where $x \to 0$. Slightly abusing notation we write $(P, R) \in \mathcal{E}^{wM}(\Delta, x)$ for a weak-Markov equilibrium that is supported by functions (P, R). We show that there is an upper bound for $P(1)$ that converges to the expected payment in a second price auction without reserve price as $\Delta \to 0$, and the convergence is uniform over x.

Lemma 5. *Fix $v(\cdot)$. For all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in [0,1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, x)$,*

$$P(1) \leq \int_0^1 v_x(s)\, ds^{n-1} + \varepsilon.$$

Proof. Suppose not. Then there exist sequences $\Delta_m \to 0$ and $x_m \to \bar{x}$ such that for all $m \in \mathbb{N}$, there exist equilibria $(P_m, R_m) \in \mathcal{E}^{wM}(\Delta_m, x_m)$ such that for all m,

$$P_m(1) > \int_0^1 v_{x_m}(s)\, ds^{n-1} + \varepsilon.$$

By a similar argument as in the proof of Theorem 4.2 of Ausubel and Deneckere (1989), we can construct a limiting pair $(\overline{P}, \overline{R})$, where $\overline{P}$ is left-continuous and non-decreasing, P_m converges point-wise to $\overline{P}$ for all rationals, and R_m converges uniformly to $\overline{R}$. Obviously, we have

$$\overline{P}(1) \geq \int_0^1 v_{\bar{x}}(s)\, ds^{n-1} + \varepsilon.$$

Left-continuity implies that there exists $\bar{q} < 1$ such that

$$\overline{P}(\bar{q}) \geq \int_0^{\bar{q}} v_{\bar{x}}(s)\, ds^{n-1} + \frac{\varepsilon}{2}. \tag{4}$$

Using an argument from the proof of Theorem 5.4 in Ausubel and Deneckere (1989), we can show that

$$\overline{R}(1) \geq \int_{\bar{q}}^1 \overline{P}(s)\, ds^n + \Pi^E(\bar{q}) \geq \Pi^E(1) + (1-\bar{q})\frac{\varepsilon}{2},$$

where we have used (4) to show the second inequality. Hence, we have

$$R_m(1) \to \overline{R}(1) \geq \Pi^E(1) + (1-\bar{q})\frac{\varepsilon}{2}. \tag{5}$$

But this implies that there must exist a type $\hat{q} > 0$, a time $t > 0$, and $\bar{m}$ such that for all $m > \bar{m}$,

$$T_m(\hat{q}) \geq t.$$

where $T_m(\cdot)$ is the trading time function in the weak-Markov equilibrium supported by (P_m, R_m). To see this, note that delay for low types is needed to increase the seller's revenue beyond the revenue from an efficient auction.

With this observation, we can conclude the proof using a similar argument as in Case I of the proof of Theorem 5.4 in Ausubel and Deneckere (1989). From Lemma 4 we know that the maximal expected payment conditional on winning that a buyer of type q has to make in equilibrium is given by $P_m(q)$. This implies that

$$R_m(1) \leq \int_{\hat{q}}^{1} P_m(z) dz^n + e^{-rt} R_m(\hat{q}).$$

In the limit we have

$$\overline{R}(1) \leq \int_{\hat{q}}^{1} \overline{P}(z) dz^n + e^{-rt} \overline{R}(\hat{q}). \tag{6}$$

On the other hand, the same argument that we used to obtain (5) yields

$$\overline{R}(1) \geq \int_{0}^{1} \overline{P}(z) dz^n. \tag{7}$$

Combining (6) and (7) we get

$$\int_{0}^{\hat{q}} \overline{P}(z) dz^n \leq e^{-rt} \overline{R}(\hat{q}),$$

which implies

$$\overline{R}(\hat{q}) > \int_{0}^{\hat{q}} \overline{P}(z) dz^n,$$

since $t > 0$. But Lemma 4 implies the opposite inequality which is a contradiction. $\square$

Using this lemma, we can show that for a given $v(\cdot)$, the difference between the continuation profit at $[0, q]$ and $\Pi^E(q)$, divided by $v(q)$ converges uniformly to zero.

Lemma 6. *Fix $v(\cdot)$. For all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in (0, 1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, 1)$,*

$$\frac{R(x)}{x^n} - \Pi^E(v(x)) \leq \varepsilon v(x).$$

Proof. The statement of the lemma is equivalent to the statement that for all $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta \leq \Delta_\varepsilon$, all $x \in (0, 1]$, and all $(P, R) \in \mathcal{E}^{wM}(\Delta, x)$,

$$R(1|v_x) - \Pi^E(1|v_x) \leq \varepsilon. \tag{8}$$

This equivalence holds because truncating and rescaling the function $v(\cdot)$ leads to the following transformations:

$$\frac{R(x|v)}{x^n} = v(x)R(1|v_x),$$
$$\Pi^E(v(x)) = v(x)\Pi^E(1|v_x).$$

To show (8), we combine Lemmas 4 and 5, and use that $P(z|v_x) = v_x(z)P(1|v_{z\cdot x})$ to get for all $x \in (0,1]$,

$$\begin{aligned}
R(1|v_x) &\leq \int_0^1 P(z|v_x)dz^n \\
&= \int_0^1 v_x(z)P(1|v_{x\cdot z})dz^n \\
&\leq \int_0^1 v_x(z)\left(\int_0^1 v_{x\cdot z}(s)ds^{n-1} + \varepsilon\right)dz^n \\
&= \int_0^1 \left(\int_0^1 v_x(sz)ds^{n-1}\right)dz^n + \varepsilon\int_0^1 v_x(z)dz^n \\
&\leq \int_0^1 \left(\int_0^z v_x(s)\frac{ds^{n-1}}{z^{n-1}}\right)dz^n + \varepsilon \\
&= \Pi^E(1|v_x) + \varepsilon
\end{aligned}$$

□

This allows us to complete the proof of Proposition 1 (ii).

Proof of Proposition 1 (ii) Translated into the notation of the paper, Lemma 6 implies that for a given distribution function F, for all $\tilde{\varepsilon} > 0$, there exists $\Delta_{\tilde{\varepsilon}} > 0$ such that for all $\Delta \leq \Delta_{\tilde{\varepsilon}}$, all $v \in [0,1]$, and all weak-Markov equilibria $(p,b) \in \mathcal{E}^{wM}(\Delta)$, we have

$$\Pi^\Delta(p,b|v) \leq \Pi^E(v) + \tilde{\varepsilon}v.$$

Recall that by Assumption 1, there exist $0 < M \leq 1 \leq L < \infty$ and $\alpha > 0$ such that $Mv^\alpha \leq F(v) \leq Lv^\alpha$ for all $v \in [0,1]$. This implies that the rescaled truncated distribution

$$\tilde{F}_x(v) := \frac{F(vx)}{F(x)},$$

for all $v \in [0,1]$ is dominated by a function that is independent of x:

$$\tilde{F}_x(v) \leq \frac{Lv^\alpha x^\alpha}{Mx^\alpha} = \frac{L}{M}v^\alpha.$$

Next, we observe that the revenue of the efficient auction can be written in terms of the rescaled expected value of the second-highest order statistic of the rescaled distribution:

$$\Pi^E(v) = \int_0^1 vs\tilde{F}_v^{(n-1:n)}(s)ds.$$

If we define $\hat{F}(v) := \min\left\{1, \frac{L}{M}v^\alpha\right\}$ and $B := \int_0^1 s\hat{F}^{(n-1:n)}(s)ds$, then given $\tilde{F}_x(v) \leq \frac{L}{M}v^\alpha$ we can apply Theorem 4.4.1 in David and Nagaraja (2003) to obtain $\Pi^E(v) \geq Bv > 0$ for all $v \in [0,1]$. If we chose $\tilde{\varepsilon}$ sufficiently small we have

$$\begin{aligned}
& \tilde{\varepsilon} \leq B\varepsilon, \\
\iff \quad & \tilde{\varepsilon}v \leq B\varepsilon v, \\
\implies \quad & \tilde{\varepsilon}v \leq \varepsilon\Pi^E(v), \\
\iff \quad & \Pi^E(v) + \tilde{\varepsilon}v \leq (1+\varepsilon)\Pi^E(v).
\end{aligned}$$

This implies that

$$\Pi^\Delta(p, b|v) \leq (1+\varepsilon)\Pi^E(v)$$

for all $\Delta \leq \Delta_\varepsilon := \Delta_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon}$ sufficiently small. $\square$

6 Concluding Remarks

The paper establishes the uniform Coase conjecture for no-gap auctions, hence filling a gap in the literature. While the concept of weak-Markov equilibrium in this paper draws from the bargaining literature, a limitation of our analysis is that it focuses on symmetric equilibria. The extent of the restriction imposed by weak-Markov equilibrium is explored in Liu et al. (2019), where, among other things, it is shown that the Coasian force is robust for a broad class of distribution functions. Characterizing the seller's achievable revenues without restricting trading mechanisms in the no-gap case is still an open question, even with only one bidder.

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