Auctions with Limited Commitment*

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May 30, 2018

Abstract

We study the role of limited commitment in a standard auction environment. In each period, the seller can commit to an auction with a reserve price but not to future auctions. We characterize the set of equilibrium profits attainable for the seller as her commitment power vanishes. We show that an immediate sale by efficient auction is optimal when there are at least three buyers. For many natural distributions two buyers is enough. We also give conditions under which the maximal profit is attained through an initial auction with a reserve price, followed by a continuously decreasing price path.

1 Introduction

Auction theory has found many applications ranging from private and public procurement to takeover bidding and electronic commerce. The vast majority of prior work on revenue

*We wish to thank Jeremy Bulow, Yeon-Koo Che, Jacob Goeree, Johannes Hörner, Philippe Jehiel, Navin Kartik, Alessandro Lizzeri, Steven Matthews, Benny Moldovanu, Bernard Salanié, Yuliy Sannikov, Vasiliki Skreta, Andrzej Skrzypacz, Philipp Strack, Alexander Wolitzky, and various seminar and conference audiences for helpful discussions and comments. We also thank the co-editor and five referees for comments that greatly improved the paper. Parts of this paper were written while some of the authors were visiting the University of Bonn, Columbia University, ESSET at the Study Center Gerzensee, Princeton University, and the University of Zürich. We are grateful for the hospitality of the respective institutions. Mierendorff gratefully acknowledges financial support from the Swiss National Science Foundation and the European Research Council (ESEI-249433). Shi gratefully acknowledges financial support by the Social Sciences and Humanities Research Council of Canada.

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maximizing auctions has as its starting point the celebrated work of Myerson (1981) and Riley and Samuelson (1981). Under a regularity condition, the optimal auction format is a standard auction (e.g., a second-price auction or a first-price auction) with a reserve price. Consequently, a revenue-maximizing auction prescribes an inefficient exclusion of some low-valued buyers.

To implement the optimal auction, it is crucial for the seller to be able to commit to withholding an unsold object off the market permanently. If no buyers bid above the reserve price, the seller has to stop auctioning the object even though there is common knowledge of unrealized gains from trade. This assumption, however, is not entirely satisfactory in many applications. For example, in the sale of art and antiques, real estate, and automobiles, aborted auctions are common. If an auction fails, the object is still available and can be sold in the future. Indeed, unsold objects are often re-auctioned or offered for sale later, at a price below the previous reserve price. As such, understanding the role of commitment in an auction setting is of both practical and theoretical relevance. We aim to clarify whether reserve prices can be used to increase profits if the seller cannot credibly rule out having auctions with lower reserve prices in the future.

We consider the classic auction model with one seller, a single indivisible object, and multiple buyers whose values are drawn independently from a common distribution. Different from the classic auction model, if the object is not sold on previous occasions, the seller can sell it again with no predetermined deadline. In each time period until the object is sold, the seller posts a reserve price and holds a second-price auction.[1] Each buyer can either wait for a future auction or submit a bid no smaller than the reserve price. Waiting is costly—both the buyers and the seller discount at the same rate. Within a period, the seller is committed to the rules of the auction and the announced reserve price. The seller cannot, however, commit to future reserve prices. The seller's commitment power varies with the period length (or effectively with the discount factor). If the period length is infinite, the seller has full commitment power. As the period length shrinks, the seller's commitment power diminishes. Within this framework, we analyze the continuous-time limit at which the seller's commitment power vanishes.

The role of commitment has been studied in the durable goods monopoly and Coasian bargaining literature; see, e.g., Coase (1972), Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986). Our model can be viewed as a Coasian bargaining model with multiple buyers. The central question raised by Coase is whether the inability to

[1] Allowing the seller to choose between standard auctions will not change our analysis and results.

commit robs the seller of her monopoly power so that she is forced to behave competitively. In the Coasian bargaining literature, the answer is yes, if we restrict attention to stationary equilibria, confirming Coase's conjecture; without this restriction, however, the seller can retain her monopoly power and achieve approximately the monopoly profit (Ausubel and Deneckere, 1989). We show that the results with multiple buyers are qualitatively different.

First, in our auction setting, the full commitment profit cannot be achieved under limited commitment. In order to attain the full commitment profit, the seller would have to maintain a constant reserve price above her reservation value (Myerson, 1981). With more than one buyer, a constant reserve price is not sequentially rational. Once the initial auction fails, the seller can deviate and end the game by running an efficient auction—that is, by setting a reserve price equal to her reservation value. The efficient auction would yield a positive profit to the seller, so it is profitable for her to deviate.

Second, we show that an immediate sale by an efficient auction maximizes revenue if there are three or more buyers. For many natural distributions two buyers is enough. This is our main result. It means that a modest level of buyer competition would induce the seller to surrender her monopoly power completely—in stark contrast to the Coasian bargaining problem. In other words, it is not very effective for the seller to set reserve prices strictly above her reservation value if there are more than two buyers. The intuition for this result will be discussed in detail in the next section.

Third, with two buyers and for some distributions, the seller may not behave competitively and an immediate sale by an efficient auction is not revenue-maximizing. The equilibrium reserve prices, still constrained by the seller's lack of commitment, must decrease over time and eventually converge to the competitive level. If the monopoly profit function associated with the value distribution is concave, the optimal limit outcome is described by an ordinary differential equation, which allows us to characterize the exact maximal revenue and show that it can be attained through an initial auction with a strictly positive reserve price followed by a sequence of continuously declining reserve prices.

Finally, we extend the model to allow for an uncertain number of buyers and explain why an immediate sale by an efficient auction may not be optimal.

The key idea we employ is to translate the limited-commitment problem into an auxiliary mechanism design problem with full commitment, but with a crucial extra constraint intended to capture limited commitment. In the original limited-commitment problem, at any stage of the game, the seller can always run an efficient auction to end the game, so her continuation value in any equilibrium must be bounded below by the payoff from an efficient auction for the corresponding posterior belief. We impose the same bound as a constraint in the full commitment problem.[2] The value of the auxiliary problem provides an upper bound for the equilibrium payoffs in the original game (in the continuous-time limit). We proceed to solve the auxiliary problem and show that its value and its solution can be approximated by a sequence of equilibrium outcomes of the original game. Therefore, the value of the auxiliary problem is precisely the maximal attainable equilibrium payoff in our original problem, and the solution to the auxiliary problem is precisely the limiting selling strategy that attains this maximal payoff.

1.1 Related Literature

The Coasian bargaining model with a single buyer is a special case of our setup. Coase (1972) argues that a price-setting monopolist completely loses her monopoly power and prices drop quickly to her marginal cost if she can revise prices frequently. Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986) confirm that every stationary equilibrium—stationary in the sense that the buyer's equilibrium strategy can only condition on the current price offer—satisfies the Coase conjecture. Ausubel and Deneckere (1989) show that, if there is "no gap" between the seller's reservation value and minimum valuation of the buyer, there is a continuum of non-stationary "reputational equilibria" in addition to the stationary Coasian equilibria. In these reputational equilibria, the price sequence posted by the seller may start with some arbitrary price which decreases over time, and any deviation from the equilibrium price path by the seller is deterred by the threat to switch to a low-profit Coasian equilibrium path. In the limit as the period length diminishes, these trigger-strategy equilibria allow the seller to achieve any profit between zero and the monopoly profit.[3] In contrast, if there is a "gap" so that the seller's reservation value is strictly below the lowest buyer valuation, as is the case in Fudenberg, Levine, and Tirole (1985), the game is essentially a game with a finite horizon. All equilibria are stationary, so it is impossible to construct trigger-strategy equilibria and achieve a profit strictly higher than what is attained in Coasian equilibria.

Our auction framework was first introduced by Milgrom (1987) and subsequently studied by McAfee and Vincent (1997). These papers restrict attention to stationary equilibria—

[2]For a given auxiliary mechanism, the seller knows exactly which set of types are left at each moment in time, if the mechanism is carried out. Consequently, she can compute the posterior beliefs as well as her continuation payoff from the given mechanism.

[3]Wolitzky (2010) analyzes a Coasian bargaining model in which the seller cannot commit to delivery. In his model, the full commitment profit is achievable even in discrete time because there is always a no-trade equilibrium which yields zero profit.

explicitly by assumption in Milgrom (1987), and implicitly in McAfee and Vincent (1997) by focusing on the gap case. As in the bargaining model, stationarity implies that the seller behaves competitively as the period length converges to zero.

As in Ausubel and Deneckere (1989), we drop the stationarity restriction and look for the highest profit attainable for the seller among all possible equilibria. A natural idea is to replicate Ausubel and Deneckere's (1989) trigger strategy equilibrium construction with the stationary equilibrium as off-path punishment. With one buyer, Ausubel and Deneckere (1989) are able to attain the full commitment profit in the limit because off-path punishment is very harsh as a stationary equilibrium yields zero profit for the seller. In contrast, with multiple buyers, the only known target—the full commitment profit—is not attainable because the seller attains a positive profit in stationary equilibria (see the companion paper Liu, Mierendorff, and Shi (2018)).[4] Hence, different from Ausubel and Deneckere (1989), we first have to characterize the maximal profit attainable among all equilibria and investigate whether strategies more complicated than the simple trigger strategy can yield a higher profit. Therefore, our main methodological contribution is to define and solve an auxiliary mechanism design problem that characterizes the maximal profit and provides a candidate solution to the original problem.

Several other papers have analyzed auctions or mechanism design with limited commitment. Skreta (2006, 2016) considers a general mechanism design framework but assumes a finite horizon. She shows that the optimal mechanism is a sequence of standard auctions with reserve prices.[5] In contrast, we restrict attention to auction mechanisms in each period and characterize the full set of equilibrium profits as the commitment power vanishes.

An alternative approach to modeling limited commitment is to assume that the seller cannot commit to trading rules even for the present period. McAdams and Schwarz (2007) consider an extensive form game in which the seller can solicit multiple rounds of offers from buyers. In Vartiainen (2013), a mechanism is a pure communication device that permits the seller to receive messages from buyers. Akbarpour and Li (2018) ask which mechanisms are credible in the sense that they are immune to manipulations of the extensive form of the mechanism. In contrast to all these papers, we posit that the seller cannot renege on the agreed terms of the trade in the current period. For example, this might be enforced by the legal environment.

[4] McAfee and Vincent (1997) have discussed this issue (p. 248) and suggested that trigger-strategy equilibria are less likely to exist if there is more than one buyer.

[5] Hörner and Samuelson (2011) and Chen (2012) analyze the dynamics of posted prices under limited commitment in a finite horizon model. They assume that the winner is selected randomly when multiple buyers accept the posted price.

The paper is organized as follows. In the next section, we present a heuristic example that illustrates the intuition behind our main result. Section 3 formally introduces the model. Section 4 states the results. Section 5 presents our methodological approach. Section 6 presents the extension to the setting with an unknown number of buyers. In Section 7 we comment on alternative modeling assumptions. Unless noted otherwise, proofs can be found in Appendix A. Omitted proofs can be found in the Supplemental Material.

2 A Heuristic Example

We use a simple example to illustrate the intuition behind our main result. In particular, we investigate the (im)possibility of heuristically constructing a particular class of equilibria in continuous time that can achieve a higher profit than an efficient auction.

Consider n buyers whose values are uniformly distributed on $[0, 1]$. On the equilibrium path the seller posts a reserve price p_t for $t \geq 0$; a buyer bids his true value at t if his value v is above a cutoff v_t, so v_t is the highest type remaining at time t. Following any deviation by the seller from p_t at time t, the continuation equilibrium is payoff equivalent to an efficient auction without reserve price,[6] and the seller's profit is[7]

$$\Pi^E(v_t) = \frac{n-1}{n+1} v_t.$$

Deviations by buyers are undetectable and thus ignored. Note that, given the cutoff strategy and the uniform prior, the seller's posterior at any history is again uniform. Therefore, it is natural to consider equilibria where the seller chooses a price path p_t that declines at a constant rate $a > 0$, that is $p_t = p_0 e^{-at}$ for some $p_0 > 0$.

The Buyers' Incentives Consider the cutoff type v_t at $t > 0$. This buyer type must be indifferent between buying at p_t, and waiting for a period of length dt to accept a lower price p_{t+dt}. The latter leads to discounting and exposes him to the risk of losing if one of his opponents has a valuation between v_{t+dt} and v_t. Therefore, the indifference condition for $dt \to 0$ is:

$$\dot{p}_t = \left[(n-1) \frac{\dot{v}_t}{v_t} - r \right] (v_t - p_t). \tag{2.1}$$

[6] In the one-buyer case, this off-path outcome is obtained by the continuous time limit of Coasian equilibria. With multiple buyers, the profit of Coasian equilibria converges to Π^E even if the initial reserve price does not converge to zero (see McAfee and Vincent, 1997, p. 251).

[7] This is the expected value of the second order-statistic of n uniform random variables on $[0, v_t]$.

On the right-hand side, $-\dot{p}_t dt$ is the gain from a lower price. On the left-hand side $-rdt\,(v_t - p_t)$ is the loss due to discounting and $(n-1)\frac{\dot{v}_t}{v_t}dt\,(v_t - p_t)$ is the expected loss from losing against an opponent.

Inserting $p_t = p_0 e^{-at}$ in the indifference condition (2.1) we obtain

$$p_0 = \rho v_0, \quad \rho = \frac{(n-1)+r/a}{n+r/a}, \quad \text{and} \quad v_t = v_0 e^{-at}. \tag{2.2}$$

The initial reserve price p_0 may be low enough so that a mass of buyer types $[v_0, 1]$ place valid bids at $t = 0$. After this, the price is lowered smoothly, and the probability that two buyers bid in the same auction is zero. Absent competition in the same auction, a winner of an auction at any time $t > 0$ will therefore just pay the current reserve price p_t.

The Seller's Incentive For the seller to follow the equilibrium price path p_t, we need to ensure that the seller's continuation profit at each $t > 0$ is not lower than the profit following a deviation, $\Pi^E(v_t)$. This condition is given by,

$$\int_t^\infty e^{-r(s-t)} p_s \frac{n\,(v_s)^{n-1}}{(v_t)^n}\,(-\dot{v}_s)\,ds \geq \frac{n-1}{n+1} v_t. \tag{2.3}$$

The left-hand side is the expected present value of the seller's profit at $t > 0$ on the presumed equilibrium path: at each moment $s > t$, the transaction price is p_s if the cutoff buyer v_s bids; the cutoff type has a conditional density $n\,(v_s)^{n-1}/\,(v_t)^n$ (i.e., the density of the highest value of the buyers) and the cutoff changes with the speed $-\dot{v}_s$.

Substituting (2.2) into (2.3), we obtain

$$\underbrace{\frac{n-1+r/a}{n+r/a}}_{\text{seller's share } \rho} \times \underbrace{\frac{n}{n+1+r/a} v_t}_{\text{screening surplus } S} \geq \underbrace{\frac{n-1}{n}}_{\text{seller's share } \rho^E} \times \underbrace{\frac{n}{n+1} v_t}_{\text{efficient surplus } S^E} \tag{2.4}$$

The first term (ρ) on the left-hand side is the seller's share of the surplus. As $a \to \infty$, ρ converges to ρ^E, the seller's share in the efficient auction. The second term (S) is the total surplus generated from active screening through a price path that declines at rate a.[8] As $a \to \infty$, S converges to S^E, the efficient surplus.

[8] To understand the formula for S, imagine that the sale event arrives at Poisson rate na since there are n buyers using the cutoff $v_s = v_t e^{-a(s-t)}$ for $s > t$. In addition, the surplus generated from a sale declines at rate $a + r$ because it is discounted at rate r and the marginal type declines at rate a. Together this yields expected discounted surplus: $\int_t^\infty an\,v_t e^{-an(s-t)} e^{-(r+a)(s-t)} ds = \frac{na}{r+(n+1)a} v_t$.

Cost and Benefit of Screening Relative to Efficient Auction Our main interest is to understand when the seller can attain a higher profit from active screening (i.e., $a < \infty$) than from the efficient auction (i.e., $a = \infty$), that is, when it is possible to construct an equilibrium that yields a higher profit than an efficient auction. The relative magnitude of the four terms in (2.4) nicely illustrates the cost and benefit associated with active screening relative to an efficient auction. The cost of screening is the surplus destroyed due to delayed trading, $S - S^E < 0$, an efficiency loss shared between the seller and the buyers. To a first-order approximation the cost for the seller is $(S - S^E)\rho^E \approx -\frac{n-1}{(n+1)^2}\frac{r}{a}v_t$. On the other hand, the seller may benefit from screening because she can extract a larger share of the surplus, $\rho > \rho^E$. This gain can be approximated by $(\rho - \rho^E)S^E \approx \frac{1}{(n+1)n}\frac{r}{a}v_t$.

The net gain from screening relative to the efficient auction, is strictly positive if $\frac{1}{(n+1)n} - \frac{n-1}{(n+1)^2} > 0$, which is equivalent to $n < \sqrt{2} + 1$. Thus, if there are three or more buyers, active screening is less profitable for the seller than the efficient auction. The reverse is true if there are only two buyers. Theorem 2 proves that this observation holds for a large class of distributions and without making any restrictions on the class of equilibrium price paths.

Summary of Intuition We have heuristically illustrated the trade-off between allocation efficiency and rent extraction faced by the seller. How this trade-off is optimally resolved depends on the number of buyers. With a small number of buyers, the seller's share of the surplus is relatively low due to lack of competition. As a result, her share of the efficiency cost of screening is relatively low but she may benefit a lot from screening through higher rent extraction. By contrast, if the number of buyers is high, the seller already extracts a high share of the surplus through buyer competition. Therefore, a larger fraction of the efficiency loss from screening has to be assumed by the seller, but at the same time there is less room for her to benefit from screening. As the number of buyers increases, the cost of screening will start to dominate the benefit of screening, so the seller will screen buyers only if the number of buyers is low.

Maximal Equilibrium Revenue We have explained that an equilibrium with active screening can be constructed only when there are less than three buyers. With two buyers, the constraint (2.4) is

$$\frac{1 + r/a}{2 + r/a} \times \frac{2}{3 + r/a} \geq \frac{1}{3}$$

This constraint is binding if $r/a \in \{0, 1\}$, and slack of $r/a \in (0, 1)$. Hence for $v_0 \in [0, 1]$ and $r/a \in [0, 1]$, (2.2) describes an equilibrium. Which of these equilibria maximizes the seller's revenue?

We will argue below that for any price path that leaves the constraint slack, there exists a price path with $r/a = 1$ that yields higher revenue. Hence we can set $r/a = 1$ and maximize over v_0. The expected profit for the seller is given by

$$\underbrace{2v_0\left(1-v_0\right)p_0 + \left(1-v_0\right)^2\left(v_0 + \frac{1-v_0}{3}\right)}_{\text{expected revenue from the initial auction at } t=0} + \underbrace{\left(v_0\right)^2 \Pi^E(v_0).}_{\text{continuation value}} \tag{2.5}$$

The initial auction yields a revenue of p_0 if a single buyer has a valuation above the cutoff v_0 (with probability $2v_0\left(1-v_0\right)$). If both buyers bid in the initial auction, the revenue is the expectation of the lower valuation which is $v_0 + \frac{1-v_0}{3}$ (with probability $\left(1-v_0\right)^2$). If none of the buyers places a bid at $t = 0$, the binding incentive constraint implies that the expected revenue from future sales is equal to $\Pi^E(v_0)$ (with the remaining probability $\left(v_0\right)^2$).

Maximizing the profit (2.5) yields $v_0 = 2/3$. Together with $r/a = 1$ we obtain:

$$p_t = \frac{4}{9}e^{-rt} \text{ and } v_t = \frac{2}{3}e^{-rt}.$$

The maximal equilibrium profit, is $\frac{31}{81} \approx 0.38$. It is higher than the profit from the efficient auction ($\frac{1}{3} \approx 0.33$) and lower than the profit from the optimal auction with commitment ($\frac{5}{12} \approx 0.42$).

Binding Incentive Constraint for the Seller To see why the seller's revenue is highest when her incentive constraint is binding, we write the seller's profit as

$$\Pi(v_0, a) = n\int_0^1 J(v)f(v)Q(v)dv$$

where $J(v) = v - \frac{1-F(v)}{f(v)}$ is the virtual valuation and $Q(v) = \left(F(v)\right)^{(n-1)} e^{-rT(v)}$ is the expected discounted trading probability of a buyer with valuation v who trades at time $T(v)$.[9] We use $F(v)$ and $f(v)$ to denote the distribution function and the density of the buyers' valuations.

We now argue that the seller's incentive constraint must bind, because otherwise we can modify the solution in a way that some high types trade earlier and low types trade later, which increases the seller's profit for distributions with a concave monopoly profit $v(1 - F(v))$. Consider a pair (v_0, a) for which the seller's incentive constraint is slack, i.e.,

[9] If $v > v_0$ the trading time is $T(v) = 0$. Otherwise, $T(v)$ is given by $v_{T(v)} = v$. Using $v_{T(v)} = v_0e^{-rT(v)}$, we get yields $T(v) = (1/a)\ln(v_0/v)$ and $Q(v) = \left(F(v)\right)^{(n-1)}\left(v/v_0\right)^{(r/a)}$.

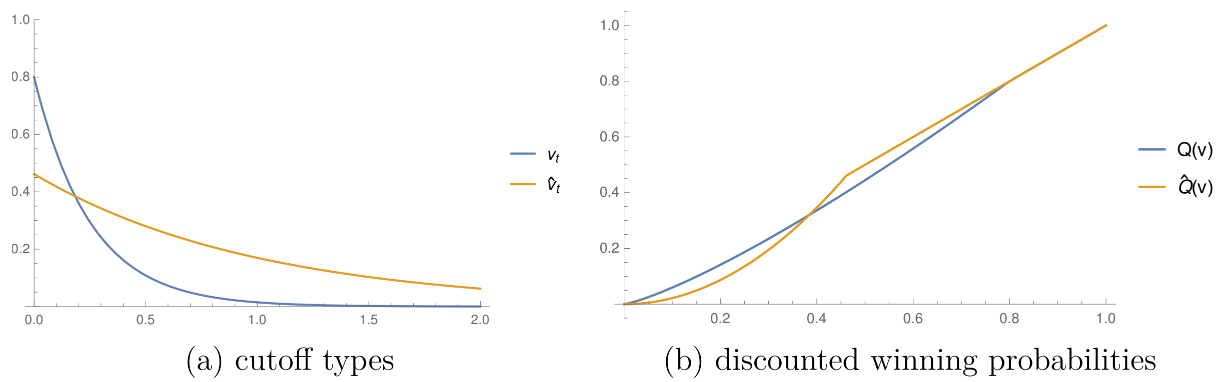

Figure 2.1: Improving profits through mean preserving spreads in trading times. (Parameters: $(v_0, a) = (1, 4)$, $(\hat{v}_0, \hat{a}) = .462, 1)$, $r = 1$)

$r/a < 1$. See Panel (a) in Figure 2.1 for an illustration. We decrease a to $\hat{a} = r$ so that the incentive constraint becomes binding. At the same time we choose $\hat{v}_0 < v_0$ so that buyers with high types trade earlier and buyers with low types trade later. Specifically, we choose $\hat{v}_0$ so that the following condition holds:

$$\int_0^1 Q(v)dv = \int_0^1 \widehat{Q}(v)dv. \tag{2.6}$$

Note that (2.6) implies that $Q(v)$ is a mean-preserving spread of $\widehat{Q}(v)$. We now argue that this implies that $\widehat{Q}$ yields a higher profit for the seller. Using integration by parts, we can rewrite the seller's profit as follows:

$$\Pi(v_0, a) = n \int_0^1 (vf(v) - (1 - F(v)))\, Q(v)dv = n \int_0^1 v(1 - F(v))dQ(v).$$

For the uniform distribution, $v(1 - F(v)) = v(1 - v)$ is concave. Since $Q(v)$ is a mean-preserving spread of $\widehat{Q}(v)$ this implies that

$$\Pi(v_0, a) < \Pi(\hat{v}_0, \hat{a}).$$

Therefore, the alternative pair $(\hat{v}_0, \hat{a})$ yields a higher profit for the seller.

3 Model

We consider the standard auction environment where a seller wants to sell an indivisible object to $n \geq 2$ potential buyers. Buyer i privately observes his own valuation for the object

$v^i \in [0,1]$. Each v^i is drawn independently from a common distribution with c.d.f. $F(\cdot)$, and a twice continuously differentiable density $f(\cdot)$ such that $f(v) > 0$ for all $v \in (0,1)$. The highest order statistic of the n valuations $(v^1, \ldots, v^n)$ is denoted by $v^{(n)}$, its c.d.f. by $F^{(n)}$, and the density by $f^{(n)}$. The seller's reservation value for the object is constant over time and we assume that it is equal to the lowest buyer valuation.[10] In Section 6, we discuss the case that the seller's reservation value is strictly higher than the lowest valuation which introduces uncertainty about the number of serious buyers.

Time is discrete and the period length is denoted by Δ. In each period $t = 0, \Delta, 2\Delta, \ldots$, the seller runs a second-price auction with a reserve price. To simplify notation, we often do not explicitly specify the dependence of the game on Δ. The timing within period t is as follows. First, the seller publicly announces a reserve price p_t for the auction run in period t, and invites all buyers to submit a valid bid, which is restricted to the interval $[p_t, 1]$. After observing p_t, all buyers decide simultaneously either to bid or to wait. If at least one valid bid is submitted, the winner and the payment are determined according to the rules of the second-price auction and the game ends. If no valid bid is submitted, the game proceeds to the next period. Both the seller and the buyers are risk-neutral and have a common discount rate $r > 0$. This implies a discount factor per period equal to $\delta = e^{-r\Delta} < 1$. If buyer i wins in period t and has to make a payment π^i, then his payoff is $e^{-rt}(v^i - \pi^i)$, and the seller's payoff is $e^{-rt}\pi^i$.

We assume that the seller has limited commitment power. She can commit to the reserve price that she announces for the current period: if a valid bid is placed, then the object is sold according to the rules of the announced auction and she cannot renege. She cannot commit, however, to future reserve prices: if the object was not sold in a period, the seller can always run another auction with a new reserve price in the next period. She cannot promise to stop auctioning an unsold object, or commit to a predetermined sequence of reserve prices.

We denote by $h_t = (p_0, p_\Delta, \ldots, p_{t-\Delta})$ the public history at the beginning of $t > 0$ if no buyer has placed a valid bid up to t, and write $h_0 = \emptyset$ for the history at which the seller chooses the first reserve price.[11] Let H_t be the set of such histories. A (behavior) strategy for the seller specifies a Borel-measurable function $p_t : H_t \to P[0,1]$ for each $t = 0, \Delta, 2\Delta, \ldots$, where $P[0,1]$ is the space of Borel probability measures endowed with the weak* topology.[12]

[10] The reservation value can be interpreted as a production cost. Alternatively, if the seller has a constant flow value of using the object, the opportunity cost is the net present value of the seller's stream of flow values.

[11] We do not have to consider other histories because the game ends if someone places a valid bid.

[12] We slightly abuse notation by using p_t both for the seller's strategy and the announced reserve price at a given history.

A (behavior) strategy for buyer i specifies a function $b_t^i : H_t \times [0,1] \times [0,1] \to P(\{\emptyset\} \cup [0,1])$ for each $t = 0, \Delta, 2\Delta, \ldots$, where we assume that $b_t^i(h_t, p_t, v^i)$ is Borel-measurable in v^i, for all $h_t \in H_t$, and all $p_t \in [0,1]$, and that supp $b_t^i(h_t, p_t, v^i) \subset \{\emptyset\} \cup [p_t, 1]$, where "$\emptyset$" denotes no bid or an invalid bid.

We consider perfect Bayesian equilibria (PBE), and we will focus on equilibria that are buyer symmetric.[13] We will not distinguish between strategies that coincide with probability one for all histories. In the rest of the paper, "equilibrium" is used to refer to this class of symmetric perfect Bayesian equilibria. Let $\mathcal{E}(\Delta)$ denote the set of equilibria of the game for given Δ.[14] Let $\Pi^{\Delta}(p, b)$ denote seller's expected revenue in any equilibrium $(p, b) \in \mathcal{E}(\Delta)$. We are interested in the entire set of profits that the seller can achieve in the limit when the period length vanishes. The maximal profit in the limit is

$$\Pi^* := \limsup_{\Delta \to 0} \sup_{(p,b) \in \mathcal{E}(\Delta)} \Pi^{\Delta}(p, b).$$

The minimal profit in the limit is

$$\Pi_* := \liminf_{\Delta \to 0} \inf_{(p,b) \in \mathcal{E}(\Delta)} \Pi^{\Delta}(p, b).$$

The analysis of the continuous-time limit allows us to formulate a tractable optimization problem. We will justify our approach by providing approximations through discrete time equilibria. An alternative approach is to set up the model directly in continuous time. This approach, however, has unresolved conceptual issues regarding the definition of strategies and equilibrium concepts in continuous-time games of perfect monitoring, which are beyond the scope of this paper.[15]

Remark 1 (**Interpretation of the Continuous Time Limit**). We take $\Delta \to 0$ in computing the limiting payoff. This need not be interpreted literally as running auctions frequently in real time. As in the dynamic games literature, this formulation is equivalent to taking $\delta \to 1$ in a discrete-time problem with fixed Δ. The continuous-time limit, however, is more convenient when we consider limiting price paths.

[13] See Fudenberg and Tirole (1991) for the definition of PBE in finite games or Fudenberg, Levine, and Tirole (1985) for infinite games.

[14] We establish equilibrium existence in Proposition 1.(i) (see Appendix A.1).

[15] See Bergin and MacLeod (1993) and Fuchs and Skrzypacz (2010) for related discussions.

4 Results

This section presents the results of the paper. Before we proceed, we introduce a mild assumption on the density function f at zero.

Assumption 1. *The density $f(v)$ is bounded at $v = 0$: $f(0) < \infty$.*

Our analysis goes through without Assumption 1 but we focus the exposition of the paper on the simpler, and arguably more relevant case that the density is bounded. In Section 6, we discuss how our results change if an infinite density (or an atom) at zero is allowed.

Our first theorem formalizes our earlier observation that with limited commitment, the seller's maximal commitment profit, denoted Π^M, is not attainable in any perfect Bayesian equilibrium.[16]

Theorem 1. *Suppose Assumption 1 holds. Then the maximal profit Π^* that the seller can achieve in equilibrium with limited commitment as $\Delta \to 0$, is strictly below the seller's maximal commitment profit Π^M.*

In order to attain Π^M, the seller must maintain a constant reserve price $p^M > 0$ in equilibrium. This is impossible because in all equilibria of our game prices must decline to zero. In fact, for any fixed $\Delta > 0$, as well as in the limit as $\Delta \to 0$, the maximal profit the seller can attain is strictly below the full commitment profit Π^M.

Our primary goal is to characterize Π^* as well as the set of perfect Bayesian equilibrium payoffs for the seller in the limit as $\Delta \to 0$. To do that, we introduce the following assumption:

Assumption 2. $\phi := \lim_{v\to 0} \left(f'(v)v\right)/f(v)$ *exists and* $\phi \in (-1, \infty)$.

Assumption 2 is a mild regularity condition on the lower bound that is imposed for technical reasons.[17] For example, it is satisfied if the density function f is bounded away from 0 and has a bounded derivative. It is also satisfied for a class of distributions which includes densities with $f(0) = 0$ or $f(0) = \infty$ such as the power function distributions

[16] If the virtual surplus $J(v) = v - (1 - F(v))/f(v)$ is increasing, the maximal commitment profit is given by the profit of Myerson's optimal auction. Otherwise, Myerson's optimal auction may involve bunching and is not contained in the class of auction formats that we consider.

[17] It is easy to see (using l'Hospital's rule) that $\phi = \lim_{v\to 0} \left(f(v)\, v\right)/F(v) - 1 \geq 0$ if the limit exists. Assumption 2 rules out the knife-edge cases of $\phi = -1$ and $\phi = \infty$. An example for the knife-edge cases, due to Yuliy Sannikov, is the distribution function $F(v) = v^{(\ln(1/v))^k}$ defined on $[0, 1]$. For this distribution function, $\phi = -1$ if $k = -1/2$, and $\phi = \infty$ if $k = 1/2$. With Assumption 1, we have $\phi \geq 0$ since $\lim_{v\to 0} f'(v)v = 0$ for any density. We state Assumption 2 in its weaker form (i.e., only imposing $\phi > -1$) to make clear which assumption is used for which argument in the proofs.

$F(v) = v^k$ with $k > 0$. To obtain distributions that satisfy both Assumptions 1 and 2, we can restrict to $k \geq 1$.

The next theorem is our main result. It shows that the only equilibrium profit achievable by the seller is the profit of the efficient auction if there are at least three buyers. (If $f(0) = 0$ the result also holds for two buyers.)

Theorem 2. *Suppose Assumptions 1 and 2 hold. If $n > 2$ (or $n > 1$ when $f(0) = 0$), then the profit of the efficient auction is the unique equilibrium profit attainable in the limit:* $\Pi^* = \Pi_* = \Pi^E$.

In the proof of the theorem, we show existence of a sequence of equilibria for which the profit converges to Π^E, and the reserve prices for all $t > 0$ converge to 0 as $\Delta \to 0$.

According to Theorem 2 (and the complementary Theorem 3.(i) below), the optimality of the efficient auction in the limit only depends on the lower tail of the distribution $f(0)$. The intuition is as follows. At any time t, the seller's posterior is a truncation from above of the original distribution. Therefore, the tail of the distribution determines the set of equilibria in subgames which start after sufficiently many periods. Suppose the tail allows multiple equilibria with different profits for the seller in every subgame starting in period $t+\Delta$. Then, it is possible to have multiple equilibria with different profits in any subgame starting at t. By contrast, if the tail pins down a unique continuation equilibrium profit (as $\Delta \to 0$) for all possible histories after sufficiently many periods, then there is a unique equilibrium profit in the whole game. Therefore, the degeneracy of the equilibrium profit set hinges on the properties of the tail of the distribution.

If $n = 2$ and $f(0) > 0$, the efficient auction no longer attains the highest equilibrium revenue.[18] We construct a sequence of equilibria that achieves $\Pi^* > \Pi^E$ and characterize the entire set of limiting profits that the seller can obtain in equilibrium. For the construction of equilibria we need the following additional assumption. It is adopted from Ausubel and Deneckere (1989) who use it to prove the uniform Coase conjecture.[19] We use it when we extend the Coase conjecture to the auction setting (see the companion paper Liu, Mierendorff, and Shi, 2018).

Assumption 3. *There exist constants $0 < M \leq 1 \leq L < \infty$ and $\alpha > 0$ such that $Mv^\alpha \leq F(v) \leq Lv^\alpha$ for all $v \in [0, 1]$.*

[18]Without Assumption 1 this is also possible for $n > 2$, depending on the type distribution. We discuss the case of an infinite density, or an atom at $v = 0$ in Section 6.

[19]This is a standard technical restriction which is satisfied by a large class of distributions.

To obtain a precise characterization of the equilibrium payoff set and the limit price path (as $\Delta \to 0$) that achieves the maximal equilibrium payoff, we need the following additional assumption.

Assumption 4. *The revenue function $v(1-F(v))$ is concave on $[0,1]$.*

Assumption 4 is used to show that the seller's incentive constraint must be binding to attain Π^*. It is only used in the second part of the following Theorem.

Theorem 3. *Suppose Assumptions 1-3 hold, $n=2$ and $f(0)>0$. Then*

(i) the maximal equilibrium profit in the limit is strictly higher than the profit of the efficient auction: $\Pi^* > \Pi_* = \Pi^E$.

(ii) If in addition, Assumption 4 holds, any $\Pi \in \left[\Pi^E, \Pi^*\right]$ is a limit of a sequence of equilibrium payoffs as $\Delta \to 0$.

In part (ii) of Theorem 3, Assumption 4 allows us to show that the seller's incentive constraint must bind in the limit as $\Delta \to 0$ in order to achieve Π^*.[20] The binding constraint, in turn, allows us to identify the optimal cutoff path which is then approximated by discrete time equilibrium outcomes. The optimal cutoff path v_t is described by the following ODE which is derived from the seller's binding incentive constraint (see Section A.1):[21]

$$\dot{v}_t = -\int_0^{v_t} re^{-\int_v^{v_t} g(x)dx} dv, \tag{4.1}$$

where

$$g(v) = \frac{f'(v)}{f(v)} - \frac{\left[v\left(F(v)\right)^{n-1} - 2\int_0^v \left(F(x)\right)^{n-1} dx\right] f(v)}{(n-1)\int_0^v \left[F(v) - F(x)\right]\left(F(x)\right)^{n-2} f(x)\, x dx}. \tag{4.2}$$

We can implement the revenue-maximizing cutoff path v_t and attain Π^* via an initial auction followed by continuously declining reserve prices given by:[22]

$$p_t = v_t + \int_t^\infty e^{-r(s-t)} \left(\frac{F(v_s)}{F(v_t)}\right)^{n-1} \dot{v}_s ds, \quad \forall t > 0. \tag{4.3}$$

[20] In order to achieve this profit, the seller would have to coordinate on a particular equilibrium. This may be possible if she can announce (but cannot commit to) a price price that she plans to use. In the absence of coordination on the revenue-maximizing equilibrium, Theorem 3 characterizes the whole equilibrium payoff set.

[21] This rules out the possibility that the reserve price jumps down at any time $t > 0$, so that a positive measure of types are induced to participate in an auction at the same time. Without Assumption 4 this may not be the case. See Section 5.2.3 for a detailed discussion.

[22] The initial price at $t=0$ is given by $p_0 = v_0^+ + \int_0^\infty e^{-rs}\left(F(v_s)/F(v_0^+)\right)^{n-1} \dot{v}_s ds$, where $v_0^+ = \lim_{t \searrow 0} v_t$. For a derivation see Section 5.2.1.

To understand the role of $g(v)$, consider the class of power function distributions $F(v) = v^k$ for which $g(v)v$ equals to a constant $\overline{\kappa}$:

$$\overline{\kappa} = k - 1 - \frac{nk\,(nk - k - 1)}{nk - k + 1}.$$

Inserting this into (4.1) yields

$$v_t = v_0 e^{-\frac{r}{\overline{\kappa}+1}t}. \tag{4.4}$$

Hence, $g(v)v$ determines the screening speed that achieves the seller's maximal profit. For the uniform example in Section 2, $\overline{\kappa} = 0$ with $n = 2$, so equation (4.4) becomes $v_t = v_0 e^{-rt}$, where $v_0 = 2/3$. The limiting price path $p_t = (4/9)e^{-rt}$ follows from (4.3), yielding the maximal profit $\Pi^* = \frac{31}{81}$.

Relation to the Coase Conjecture. Theorem 2 can be interpreted as a Coase conjecture result, because it predicts that, as $\Delta \to 0$, the seller's profit converges to the competitive level.[23] A related Coase conjecture result is obtained in Milgrom (1987) and McAfee and Vincent (1997), but their result is entirely driven by their stationarity restriction. This restriction is either explicitly assumed (Milgrom, 1987), or implicitly applied by the gap assumption that the seller's reservation value is strictly lower than the lowest buyer valuation (McAfee and Vincent, 1997). In stationary equilibria, all buyers follow stationary bidding strategies which can be interpreted as a *demand curve* faced by the seller. The seller would like to collect the *surplus* below the demand curve as quickly as possible. As $\Delta \to 0$, she can collect the whole surplus by setting more and more finely spaced reserve prices in shorter and shorter intervals. Prices must therefore decline to zero immediately which implies that the demand curve collapses to zero as well, and the Coase conjecture follows. This logic works independent of the type distribution and the number of buyers but crucially relies on stationarity.[24] In contrast, Theorem 2 imposes no stationarity restriction, and shows that limited commitment alone forces the seller to behave competitively if there are at least three buyers. Therefore, Theorem 2 helps clarify the role of limited commitment in the auction setting. With three or more buyers, using reserve prices to screen buyers does not yield a profit in excess of the profit of the efficient auction.

[23] In the bargaining setting ($n = 1$) the Coase conjecture is understood as follows: as $\Delta \to 0$, the seller's initial price offer p_0 must converge to her reservation value 0. As first noted by McAfee and Vincent (1997), however, p_0 can stay positive in the auction setting even though all subsequent reserve prices converge to 0 in the limit. The limiting profit is thus equal to the profit of the efficient auction.

[24] Proposition 1 which establishes the Coase conjecture for stationary equilibria in our auction setting only requires Assumption 3.

The comparison between the profit from an efficient auction and the potential benefits from screening can also help understand the gap case, as analyzed by McAfee and Vincent (1997), where the buyers' type distribution has support $[\varepsilon, 1]$. By posting price $p_t = \varepsilon$, the seller can guarantee herself a profit $\varepsilon > 0$, even with one buyer. In contrast to the no-gap auction case where the lower bound on the seller's profit at time t (i.e., the profit from running the efficient auction at time t) goes to zero as $v_t \to \varepsilon$, here the profit bound ε is a constant independent of v_t. In fact, for v_t sufficiently close to ε, the profit attainable by setting $p_t = \varepsilon$ coincides with the full commitment profit. As a result, the game ends in finite time which implies that all equilibria must be stationary.[25] Hence, in the gap case, the Coase conjecture directly follows from stationarity.

5 Methodology and Overview of Proofs

Our strategy to characterize Π^*, the corresponding limit price path, and the set of limit equilibrium profits for the seller, is to analyze an auxiliary dynamic mechanism design problem. To formulate the problem, we identify basic properties of equilibria of the discrete time game (Section 5.1). These properties are necessary conditions for equilibrium outcomes. We then formulate the same restrictions in continuous time and use them to define the feasible set of mechanisms in the dynamic mechanism design problem (Section 5.2). Necessity of the constraints implies that the value of the auxiliary problem is an upper bound for Π^*. To establish sufficiency, we show that the optimal value of the auxiliary problem is attained by a sequence of discrete time equilibria as period length goes to zero. Therefore, the optimal value of the auxiliary problem is exactly the maximal profit attainable in any equilibrium in the continuous time limit.

5.1 Equilibrium Properties

In any equilibrium of the discrete time game, all buyers play pure strategies that are characterized by history-dependent cutoffs. This is captured by the following Lemma which establishes the "skimming property," an auction analog of a result by Fudenberg, Levine, and Tirole (1985). Its proof is standard and thus omitted.

[25] In the gap case where the last period is endogenous, as well as in a game with an exogenous last period, the equilibrium can be found by backward induction. This implies that it is essentially unique. In both cases reputational equilibria are ruled out by uniqueness.

Lemma 1 (Skimming Property)**.** *Let* $(p,b) \in \mathcal{E}(\Delta)$*. Then, for each* $t = 0, \Delta, 2\Delta, \ldots$*, there exists a function* $\beta_t : H_t \times [0,1] \to [0,1]$ *such that every buyer with valuation above* $\beta_t(h_t, p_t)$ *places a valid bid and every buyer with valuation below* $\beta_t(h_t, p_t)$ *waits if the seller announces reserve price* p_t *at history* h_t*.*

The next lemma shows that randomization by the sender on the equilibrium path is not necessary to attain the maximal profit. This lemma is a new observation that is not trivial. It is used to characterize the maximal profit. In a model with one buyer, this step is not needed since the maximal profit attainable is the full commitment profit. Therefore, the following lemma does not appear in the prior literature on Coasian bargaining.[26]

Lemma 2 (No Need for Randomization)**.** *For every equilibrium* $(p,b) \in \mathcal{E}(\Delta)$*, there exists an equilibrium* $(p',b') \in \mathcal{E}(\Delta)$ *in which the seller does not randomize on the equilibrium path and achieves a profit* $\Pi^\Delta(p',b') \geq \Pi^\Delta(p,b)$*.*

Lemma 1 implies that at any history, the posterior of the seller is given by a truncation of the prior. Lemmas 1 and 2 together imply that for the characterization of Π^*, we can restrict attention to equilibrium allocation rules which are deterministic (up to tie-breaking).[27] Symmetric deterministic equilibrium allocation rules can be described in terms of a trading time function $T : [0,1] \to \{0, \Delta, 2\Delta, \ldots\}$ which must be non-increasing because of Lemma 1. Given that buyers bid truthfully in a second-price auction, in any symmetric equilibrium the object will be allocated at time $T(v^{(n)})$, to the buyer with the highest valuation.

The last lemma in this section shows that the seller can ensure a continuation profit no smaller than the profit of an efficient auction, even though running an efficient auction is not a part of an equilibrium.

Lemma 3 (Lower Bound on Equilibrium Payoff)**.** *Fix any equilibrium* $(p,b) \in \mathcal{E}(\Delta)$ *and any history* h_t*. If the seller announces the reserve price* $p_t = 0$ *at* h_t*, then every buyer bids his true value and the game ends.*

Lemma 3 provides a lower bound for the seller's payoff on and off the equilibrium path which provides a constraint for continuation payoffs in the auxiliary problem introduced below. It also follows from Lemma 3 that $\Pi_* \geq \Pi^E$. See the Supplemental Material for proofs of Lemmas 2 and 3.

[26] Gul, Sonnenschein, and Wilson (1986) show the existence of equilibria without randomization on path whereas Lemma 2 focuses on revenue-maximization.

[27] In the proof of Theorem 3 we show that any payoff in $[\Pi^E, \Pi^*]$ can be achieved in a limit of pure equilibrium outcomes. Therefore, this restriction is also without loss for the *set* of limit profits achievable for the seller.

5.2 The Auxiliary Mechanism Design Problem

In the auction context, limited commitment invalidates the full commitment solution as a target for equilibrium construction, so we have to first find the maximal equilibrium profit in order to characterize the entire set of equilibrium profits for the seller. In this subsection, we set up the auxiliary mechanism design problem with full commitment which we use to characterize the maximal profit, and briefly explain why solving the auxiliary problem constitutes the crucial step in proving the main results.

5.2.1 Mechanisms

The auxiliary mechanism design problem is formulated in continuous time and assumes that the seller has full commitment power. Buyers participate in a direct mechanism and make a single report of their valuations at time zero. The mechanism awards the object to the buyer with the highest reported type (up to tie breaking). It specifies a deterministic and non-increasing trading time function $T : [0, 1] \to [0, \infty]$. If the mechanism awards the object to buyer i, then the allocation takes place at time $T(v^i)$. This is motivated by Lemmas 1 and 2. Moreover, the mechanism specifies a payment for the winning buyer.

The discounted trading probability of a buyer with type v is $e^{-rT(v)}$ if he is the highest buyer and zero otherwise. The (interim) expected discounted winning probability of a buyer is thus $Q(v^i) = (F(v^i))^{n-1}e^{-rT(v)}$, and this is non-decreasing since T is non-increasing. Therefore, any non-increasing trading time function is implementable, and following standard arguments, individual rationality and incentive compatibility constraints for the buyers can be used to express the seller's profit as

$$\int_0^1 J(v)\, e^{-rT(v)} dF^{(n)}(v), \tag{5.1}$$

where $J(v) := v - (1 - F(v))/f(v)$ denotes the virtual valuation. $J(v)$ corresponds to the marginal revenue of a monopolist (see Bulow and Roberts, 1989).

We define cutoff types as

$$v_t := \sup \{v \,|\, T(v) \geq t\}.$$

v_t is the highest type that does not trade before time t. Since all buyers with types $v > v_t$ trade before t, the posterior distribution at t, conditional on the event that the object has not yet been allocated, is given by the truncated distribution $F(v|v \leq v_t)$. Therefore, we call v_t the *posterior at time t*. We denote the posterior distribution functions by

$$F_t(v) := \frac{F(v)}{F(v_t)}, \qquad F_t^{(n)}(v) := \frac{F^n(v)}{F^n(v_t)}.$$

The virtual valuation for the posterior $[0, x]$ is denoted by

$$J(v|v \leq x) := v - \frac{F(v_t|v \leq x) - F(v|v \leq x)}{f(v|v \leq x)} = v - \frac{F(x) - F(v)}{f(v)},$$

and we set $J_t(v) := J(v|v \leq v_t)$, whenever we consider a fixed cutoff path v_t.

Generally, v_t is continuous from the left, and since it is non-increasing, the right limit exists everywhere. We will denote the right limit at t by

$$v_t^+ := \lim_{s \searrow t} v_s.$$

For each t, v_t^+ is the highest type in the posterior after time t if the object is not yet sold.

Any non-increasing trading time function T (with cutoffs v_t) can be implemented by the price path

$$p_t = v_t^+ - \int_0^{v_t^+} e^{-r(T(v)-t)} \left(\frac{F(v)}{F\left(v_t^+\right)} \right)^{n-1} dv. \tag{5.2}$$

This price sequence is derived from the envelope formula which implies that for each $t > 0$ the marginal type v_t^+ is indifferent between bidding at time t and waiting.[28] Consequently, all types above v_t^+ strictly prefer to bid before or at time t, all lower types strictly prefer to wait. It v_t is differentiable, $v_t^+ = v_t$ for all $t > 0$ and (5.2) simplifies to (4.3)

5.2.2 Payoff Floor Constraint

If the seller has full commitment power, the dynamic mechanism design problem of maximizing (5.1) without further constraints, reduces a static problem. The optimal solution is to allocate to the buyer with the highest valuation if his valuation exceeds the optimal reserve price p^M, and otherwise to withhold the object. Formally, in terms of trading times,

[28]The envelope condition for $v \in [v_t^+, v_t]$ is $e^{-rt}\left(\int_{v_t^+}^{v}(v-x)dF^{n-1}(x) + F\left(v_t^+\right)^{n-1}(v-p_t)\right) = \int_0^v Q(x)dx$. Substituting $Q(v)$ and $v = v_t^+$, and rearranging yields (5.2).

this is given by[29]

$$T^M(v) := \begin{cases} 0 & \text{if } v \geq p^M, \\ \infty & \text{if } v < p^M. \end{cases} \tag{5.3}$$

To obtain an auxiliary problem that captures the seller's incentives under limited commitment, we add an additional constraint. Motivated by Lemma 3 we assume that the continuation payoff of the seller must be bounded below by the revenue of an efficient auction for the given posterior at each point in time. To state this "payoff floor constraint" formally, we denote the revenue from an efficient auction for the posterior v_t as

$$\Pi^E(v_t) = \frac{1}{F^{(n)}(v_t)} \int_0^{v_t} J_t(x) dF^{(n)}(x).$$

The seller's continuation payoff from the dynamic mechanism at time t is

$$\frac{1}{F^{(n)}(v_t)} \int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x).$$

Therefore, the payoff floor constraint (PF) is given by (where we have dropped the term $1/F^{(n)}(v_t)$ on both sides):

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) \geq \int_0^{v_t} J_t(x) dF^{(n)}(x), \text{ for all } t \geq 0.$$

The payoff floor constraint introduces a dynamic element into the auxiliary problem that distinguishes it from a standard static mechanism design problem under full commitment.

5.2.3 Auxiliary Problem

To summarize, we can formulate the auxiliary problem as the following dynamic mechanism design problem:

$$\sup_{T:[0,1]\to[0,\infty]} \int_0^1 e^{-rT(x)} J(x) dF^{(n)}(x) \tag{5.4}$$

$$\text{s.t.} \quad T \text{ is non-increasing,} \tag{IC}$$

$$\text{and} \quad \int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) \geq \int_0^{v_t} J_t(x) dF^{(n)}(x), \forall t \geq 0. \tag{PF}$$

We call any $T : [0,1] \to [0,\infty]$ that satisfies (IC) and (PF) a *feasible solution* of the

[29] If $J(v)$ is strictly increasing, p^M is given by $J(p^M) = 0$ and $T^M(v)$ induces the same winning probabilities $Q^M(v)$ as Myerson's optimal auction.

auxiliary problem. We denote the value of the auxiliary problem by V and standard techniques can be used to show that an optimal solution exists (see Proposition 3 in Appendix A). In the following we give an overview how the auxiliary problem is used to prove our main results. We only outline the crucial steps, while the formal analysis is deferred to Appendix A.

Using the Auxiliary Problem to Characterize Equilibrium Profits. We first explain why the auxiliary problem is the correct problem for the characterization of the maximal limit profit achievable in equilibrium, i.e., $\Pi^* = V$. For necessity of the constraints, note that (PF) rules out a deviation by the seller to an efficient auction, which is a necessary condition for an equilibrium. Therefore, V is an upper bound for the seller's maximal profit Π^*, which is formally proved in Proposition 2 in Appendix A. To show that (PF) is sufficient, we use existence of stationary equilibria which we show in the companion paper Liu, Mierendorff, and Shi (2018). If $V = \Pi^E$, existence of equilibrium, together with $\Pi^* \le V = \Pi^E$ implies $\Pi^* = \Pi^E$ because any equilibrium yields a profit of at least Π^E. If $V > \Pi^E$ the construction uses the simple trigger strategy with stationary equilibria as off-path punishment. Here the payoff floor constraint is sufficient since the profit of stationary equilibria converges to the right-hand side of the payoff floor constraint as $\Delta \to 0$ (see Liu, Mierendorff, and Shi (2018)). Therefore, the payoff floor constraint exactly captures limited commitment and the optimal value of the auxiliary problem is exactly the maximum revenue attainable in any equilibrium as the seller's commitment ability vanishes.

Optimal Solution of the Auxiliary Problem. To prove Theorems 2 and 3 we characterize the optimal solution to the auxiliary problem. This involves two main steps. First, we show that concavity of $v(1 - F(v))$ implies that the payoff floor constraint must hold with equality at an optimal solution. Using integration by parts we can rewrite the objective function in (5.4) as

$$\int_0^1 e^{-rT(x)} J(x) dF^{(n)}(x) = n \int_0^1 J(v) f(v) Q(v) dv = n \int_0^1 v(1 - F(v)) dQ(v). \tag{5.5}$$

Consider $T(v)$ and $\widehat{T}(v)$ with associated discounted winning probabilities $Q(v)$ and $\widehat{Q}(v)$. If $Q(v)$ is a mean-preserving spread of $\widehat{Q}(v)$, which means that the trading times for $\widehat{T}(v)$ are more spread out, then $\widehat{T}(v)$ yields a higher profit for the seller.

In Lemma 8 in Appendix A.1.3 we show that when the payoff floor constraint is slack in for some time interval $[a, b]$, we can construct a feasible variation $\hat{T}(v)$ with more spread out trading times for the types with trading times in (a, b) so that ex-ante profit is improved. If instead $v(1 - F(v))$ is convex, we have to construct a variation that concentrates the trading times of the types that trade between a and b, rather than spreading them out. Such a variation, however, is only feasible if the trade is not already concentrated on a single point in time. Therefore, with a non-concave monopoly profit, we cannot rule out that the payoff floor constraint is slack on some time-interval if there is an atom of trade in this interval.[30]

The second main step is to determine when there exists a feasible solution to the binding payoff floor constraint. If (PF) holds with equality we show in that v_t must satisfy

$$\frac{\ddot{v}_t}{\dot{v}_t} + g(v_t)\dot{v}_t + r = 0. \tag{5.6}$$

If $n = 2$ this differential equation has a decreasing solution which we can use in the proof of Theorem 3. Conversely, we show that any solution to the differential equation (5.6) is increasing in a neighborhood of zero if $n > 2$ (or $n > 1$ if $f(0) = 0$) (see Lemma 11 in Appendix A.1.3). This means that the binding payoff floor constraint does not yield a feasible solution of the auxiliary problem. For Theorem 2 we exploit that for any distribution $v(1 - F(v))$ is concave in a neighborhood of zero which implies that (PF) must be binding.[31] Since the binding payoff floor constraint does not yield a feasible solution if $n > 2$, only the efficient auction $T^E(v) \equiv 0$ is left as an optimal solution to the auxiliary problem.

6 Uncertain Number of Buyers

So far we have assumed that the seller knows the number of serious buyers who have values above her cost, that is, all buyers' valuations are weakly above the seller's reservation value $c = 0$. This is a natural assumption if all buyers with values below c know that they will surely lose and thus do not show up in the auction.

What if the seller is uncertain about the number of serious buyers? A possible modeling approach is to assume that there are n potential buyers, but not all of them are interested in buying the object because their values may be lower than c. If we maintain the normalization $c = 0$, v is the net valuation of the buyer. This implies that the support of $F(v)$ is $[\underline{v}, 1]$

[30]We have not been able to rule out this possibility or to construct an example where a solution with this feature is optimal.

[31]To see this note that $(v(1 - F(v))'' = -2f(v) - vf'(v)$. Hence concavity is equivalent to$(vf'(v)/f(v)) > -2$. Remember from Footnote 17 that $\lim_{v\to 0}(vf'(v)/f(v)) > -1$. This implies concavity for v in a neighborhood of zero.

with $\underline{v} < 0$.[32] The number of serious buyers with $v > c$ is uncertain, and it is possible that no buyer has a value above c. Again, the seller can either run an efficient auction to end the game immediately, or set a declining reserve price path p_t to screen buyers. If the seller chooses to screen buyers, she will simultaneously update her belief about buyers' values as well as the number of serious buyers.[33]

As in the main model, we adopt the mechanism design approach and use the auxiliary problem to investigate whether it is possible to have an equilibrium with a positive price path that yields a strictly higher revenue than an efficient auction.[34] Since the seller will never sell the object below her cost, the formulation of the auxiliary problem is exactly the same as before, but we need to keep in mind that now $F(0) > 0$, in contrast to the main model, where $F(0) = 0$. Again, we solve the auxiliary problem by constructing a solution candidate, assuming that the payoff floor constraint binds for all $t > 0$. We can differentiate the binding payoff floor constraint and obtain the ODE in (5.6), where $g(v_t)$ is given by (4.2) as before. If $F(0) > 0$, however, this differential equation has a decreasing and thus feasible solution for all n (see Lemma 12 in Appendix A.6). It follows from Proposition 4 in Appendix A.1.3, that for any n, the value of the auxiliary problem is strictly higher than the revenue of an efficient auction.[35]

Intuitively, if the object remains unsold as time goes by, the seller attaches an increasingly higher probability to the event that the number of serious buyers is small. If the number of serious buyers is indeed small, the expected revenue from an efficient auction will be low, and thus it is possible for the seller to use screening to generate revenue strictly higher than an efficient auction. In other words, because the revenue of the efficient auction can be very low, the threat of reverting to Coasian equilibrium becomes very effective and is sufficient to support equilibria with positive reserve prices.

There are two ways to link the model here to our main model. First, we define distribution function $\tilde{F}$ with support $[0, 1]$ by

$$\tilde{F}(v) = F(v) \text{ for all } v \in [0, 1] .$$

$\tilde{F}$ coincides with F, except that $\tilde{F}$ translates all the probability mass assigned by F to

[32] Up to the normalization this is equivalent to assuming that the support of $F(v)$ is $[0, 1]$ and $c > 0$.

[33] Given v_t, the number of serious buyers is binomially distributed with $B(n, 1 - F(0|v < v_t))$.

[34] We completely solve the auxiliary problem for the case of an uncertain number of buyers but do not extend the approximation by discrete time equilibria.

[35] All lemmas (1–9) and propositions (3–5) used to characterize the optimal solution of the auxiliary problem are unchanged.

negative valuations into an atom at $v = 0$. Since the seller does not sell to buyers with valuations strictly below 0, the two distributions $\tilde{F}$ and F are equivalent from the perspective of the seller's revenue.

The distribution $\tilde{F}$ can be approximated by atomless distributions with unbounded densities around zero. These distributions violate Assumption 1. However, we can drop Assumption 1 and generalize the result of Theorem 2 as follows: under Assumption 2, an efficient auction is optimal if and only if $n > \overline{N}(F) \equiv 1 + \sqrt{2+\phi}/(1+\phi)$. Without Assumption 1, it is possible for ϕ to take any value above -1 and thus for $\overline{N}(F)$ to take any value above 1. For example, for power function distributions $F(v) = v^k$ on $[0, 1]$, $\overline{N}(F) = 1 + \sqrt{k+1}/k$. As $k \to 0$, the distribution $F(v) = v^k$ puts a lot of probability mass at points near zero as if there is an atom at zero. Note that as $k \to 0$, $\overline{N}(F) \to \infty$, so an efficient auction is almost never optimal, which is consistent with our result for the case of an uncertain number of buyers.

An alternative way to link it to our main model is to consider the limit as the uncertainty about the number of serious buyers vanishes. This uncertainty is captured by $\lambda = \tilde{F}(0)$. Let the optimal solution to the auxiliary problem for $\lambda > 0$ be denoted by w_t^λ. We consider the limit as $\lambda \to 0$ so that the uncertainty about the number of buyers vanishes. If Assumptions 1 and 2 hold as in Theorem 2, and there are three or more buyers, we prove in Appendix A.6 that, as $\lambda \to 0$, the sequence of the cutoff paths $\{w_t^\lambda\}$ converges to a limiting path w_t^0 that satisfies $w_t^0 = 0$ for all $t > 0$. Moreover, the seller's profit converges to the profit of an efficient auction.

7 Concluding Remarks

In this paper we have studied the role of commitment power in auctions where the seller cannot commit to future reserve prices. Our analysis draws insights from the bargaining literature, and the auction and mechanism design literature. We conclude the paper by discussing our modeling assumptions and possible extensions.

Symmetry Restriction. Throughout the paper, we have restricted attention to buyer-symmetric equilibria. If we allow for asymmetric equilibria, we can formulate an asymmetric auxiliary problem in terms of a trading time function (or a sequence of cutoffs) for each buyer. Since the seller can only choose a single price in each period, however, the set of implementable cutoff sequences for a given buyer depends on the cutoff sequences chosen for the other buyers. Therefore, the asymmetric auxiliary problem requires additional constraints which are quite complex and not very tractable. A more fundamental problem for a tractable specification of the auxiliary problem arises because we do not know how to extend the proof of Lemma 2 (No Need for Randomization) to asymmetric equilibria.[36] Consequently, we cannot restrict attention to deterministic allocation rules. Finally, symmetry also helps to rule out that buyers play dominated strategies in second-price auctions, which is a standard assumption.[37] In light of these issues, it seems that the complications involved in studying asymmetric equilibria are on par with the complications that arise when analyzing general mechanisms. We believe that the analysis of general mechanisms is a fruitful direction for future research but is beyond the scope of this paper.

Modeling Limited Commitment. Our way of modeling limited commitment assumes that the seller can commit to the terms of trade within a single period: if $\Delta = \infty$, there is full commitment; as $\Delta \to 0$, the seller's commitment power vanishes. This approach is taken by Milgrom (1987) and McAfee and Vincent (1997).

An alternative modeling approach is to assume that the seller's opportunity of running an additional auction is uncertain. This can be cast into a continuous-time framework as follows. There is a Poisson arrival of auction opportunities, with constant arrival rate λ. An auction can only be held at time $t = 0$ or when there is an arrival. If $\lambda = 0$, there is full commitment; if $\lambda \to \infty$, the commitment power vanishes. This model is similar to ours except that the period length Δ is random, but $\Delta \to 0$ in distribution as $\lambda \to \infty$.

Yet another approach is to allow the seller to use long-term contracts. If the legal environment allows the seller to commit to the auction rules within a given period, why can she not write a contract that forces her to keep an object off the market and thereby gain commitment power? Intuitively, such contracts are not renegotiation proof, which may explain why we do not see them in practice.

A general formulation of the problem with long-term contracts with renegotiation exists for the case of bilateral contracts (see Hart and Tirole, 1988; Strulovici, 2013, and references

[36] In the proof for the symmetric case, for any (possibly mixed) equilibrium, we select the sequence of (symmetric) cutoffs implemented along one particular on-path history. Since every symmetric sequence of cutoffs is implementable by some sequence of reserve prices, we are able to construct a new equilibrium without on-path randomization and weakly higher profits. With asymmetric cutoffs, this is no longer possible because the cutoffs implemented along a particular history may not be implementable by a single deterministic price sequence.

[37] For $n > 2$, Blume and Heidhues (2004) show that the second-price auction has a unique equilibrium if the seller uses a non-trivial reserve price. Therefore, symmetry is not needed to rule out low-profit equilibria if $n > 2$. By posting a reserve price close to zero, the seller can end the game with probability arbitrarily close to one and guarantee herself a profit arbitrarily close to the profit of an efficient auction. This implies that the lower bound for the seller's equilibrium payoff that we obtain in Lemma 3 is independent of the symmetry assumption if there are at least three buyers.

therein). In our setup with multiple buyers, however, modeling renegotiation introduces new conceptual issues, such as the protocol of multiple-person bargaining and signaling in the renegotiation phase.

A Appendix

In this appendix, we sketch the key steps in characterizing the optimal solutions to the auxiliary problem, which will form the basis of our proofs of Theorems 1–3. The proof of Proposition 1 (existence of stationary equilibria and uniform Coase conjecture) is contained in the companion paper (Liu, Mierendorff, and Shi, 2018). All other proofs omitted from this appendix are collected in Section B of the Supplemental Material. Section C of the Supplemental Material constructs equilibria that approximate the solution to binding payoff floor constraint and proves Proposition 6 which is used in the proof of Theorem 3.

A.1 Analysis of the Auxiliary Problem

A.1.1 Optimal Value as Equilibrium Revenue Upper Bound

Based on Ausubel and Deneckere (1989) we start by showing existence of stationary equilibria, i.e., equilibria with stationary buyer-strategies that only depend on the valuation and the current reserve price. We also generalize the uniform Coase conjecture for stationary equilibria to the auction setting.

Proposition 1. *(i) (Existence) A stationary equilibrium exists for every $r > 0$ and $\Delta > 0$.*

(ii) (Uniform Coase Conjecture) Suppose Assumption 3 holds. For every $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta < \Delta_\varepsilon$, all $x \in [0, 1]$, and every symmetric stationary equilibrium (p, b) of the game with period length Δ and a truncated distribution $F(v|v \leq x)$ on $[0, x]$, the seller's profit associated with this equilibrium, $\Pi^\Delta(p, b|x)$, is bounded above by $(1 + \varepsilon)\Pi^E(x)$, where $\Pi^E(x)$ is the seller's profit from the efficient auction under this truncated distribution.

The second part of the proposition shows that the seller's profit in every symmetric stationary equilibrium converges to the profit of the efficient auction.[38] Uniform convergence,

[38] Notice that in contrast to the uniform Coase conjecture for one buyer (Ausubel and Deneckere, 1989), Proposition 1.(ii) does not show that the initial reserve price p_0 converges to zero. This is in fact not the case in the auction setting as was noted by McAfee and Vincent (1997). However, reserve prices for $t > 0$ converge to zero which is sufficient for the convergence of equilibrium profits to the profit of an efficient auction—the counterpart of the Coase conjecture in the auction setting.

in the sense that $\Pi^{\Delta}(p, b|x) / \Pi^{E}(x) \to 1$ uniformly for all $x \in (0, 1]$, will be used in the construction of trigger strategy equilibria for Theorem 3.

Clearly, the lower bound of the seller's profit for all equilibria is achievable by $T^{E}(v) \equiv 0$. This corresponds to a second-price auction with reserve price $p_t = 0$ at time $t = 0$, and $T^{E}(v) \equiv 0$ implies $v_t = 0$ for all $t > 0$. Therefore, the payoff floor constraint is trivially satisfied for both $t > 0$ and $t = 0$. The following result shows that the optimal value of the auxiliary problem is an upper bound for all equilibrium revenues in the original game.

Proposition 2 (Seller's Equilibrium Payoff Bounds)**.** *Let* (Δ_m) *be a decreasing sequence with* $\Delta_m \searrow 0$*, and let* $(p_m, b_m) \in \mathcal{E}(\Delta)$ *be a sequence of equilibria. Then* $\limsup_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) \in [\Pi^{E}, V]$*. In particular* $\Pi^* \leq V$*.*

A.1.2 Preliminary Observations

Before characterizing optimal solutions to the auxiliary problem, we note several Lemmas regarding the payoff floor constraint that will be used in the proofs.

First we consider solutions where a strictly positive measure of types trade at the same time t so that $v_t > v_t^+$. In other words, there is an "atom" of types that trade at t. The following lemma shows that if the payoff floor constraint is satisfied right after the atom, then the payoff floor constraint at t (right before the atom) is strictly slack. Moreover, if we reduce the size of the atom by lowering v_t to $v \in (v_t^+, v_t)$ so that some types in the atom trade earlier than t, the payoff floor constraint at t remains strictly slack for all choices $v \in (v_t^+, v_t)$.

Lemma 4 (Slack PF before Atom)**.** *Let* $T : [0,1] \to [0,1]$ *be non-increasing (not necessarily feasible) and denote the corresponding cutoff sequence by* v_t*. Suppose there is an "atom" at* $t \geq 0$*, that is,* $v_t > v_t^+$*. If the payoff floor constraint is satisfied at* t^+*, that is*

$$\int_0^{v_t^+} e^{-r(T(x)-t)} J(x|x \leq v_t^+) dF^{(n)}(x) \geq \int_0^{v_t^+} J(x|x \leq v_t^+) dF^{(n)}(x), \tag{A.1}$$

then we have, for all $v \in (v_t^+, v_t]$*,*

$$\int_0^{v} e^{-r(T(x)-t)} J(x|x \leq v) dF^{(n)}(x) \geq \int_0^{v} J(x|x \leq v) dF^{(n)}(x). \tag{A.2}$$

In particular, the payoff floor constraint is satisfied at t*. The inequality (A.2) is strict if* $v_t^+ > 0$*.*

Second, we show that a feasible solution to the auxiliary problem cannot end with a single atom where all remaining types trade.

Lemma 5 (No Final Atom)**.** *Let T be a feasible solution. Then for all $t > 0$ such that $v_t > 0$, there exists $w \in (0, v_t)$ such that $T(v) > t$ for all $v \leq w$.*

Finally, we observe that the payoff floor constraint must by strictly slack in quiet periods (a, b) where v_t is constant, i.e., where no trade takes place.

Lemma 6 (Slack PF in Quiet Period)**.** *Let T be a feasible solution and $a < b$ such that $v_t = v_b$ for all $t \in (a, b)$, then* (PF) *is a strict inequality for all $t \in (a, b]$.*

A.1.3 Characterizing Optimal Solutions

After introducing these observations about the payoff floor constraint, we now prove intermediate results which are used to characterize optimal solutions to the auxiliary problem and the set of feasible profits. The first observation is that an optimal solution exists which follows from standard arguments.

Proposition 3. *An optimal solution to the auxiliary problem exists.*

For $n = 1$, the case of a single buyer, the right-hand side of the payoff floor constraint is zero, and the optimal solution is T^M.[39] For $n \geq 2$ this is not the case, as shown in the following lemma.

Lemma 7 (Cutoffs Converge to Zero)**.** *For any T in the feasible set of the auxiliary problem, $T(v) < \infty$ for all $v > 0$ and $\lim_{t\to\infty} v_t = 0$.*

Next, we show that the efficient auction (T^E) is optimal if and only if it is the only feasible solution to the auxiliary problem. It is clear that any feasible solution yields a profit that is at least as high as the profit of the efficient auction. Otherwise, the payoff floor constraint would be violated at $t = 0$. The following proposition shows that if positive reserve prices are feasible, that is, if the feasible set includes a solution with delayed trade for low types, then the seller can achieve a strictly higher revenue than in the efficient auction.

Proposition 4. *An efficient auction (T^E) is an optimal solution to the auxiliary problem if and only if it is the only feasible solution.*

[39] This also implies the "folk theorem" obtained by Ausubel and Deneckere (1989).

To get an intuition for this result, compare the efficient auction in which all types trade at time zero, to an alternative feasible solution in which only the types in $(v_0^+, 1]$ trade at time zero, where $v_0^+ < 1$.[40] There are two effects that determine how the profits of these two solutions are ranked. First, in the alternative, the trade of low types is delayed, which creates an inefficiency. Second, the delay for the low types reduces information rents for higher types. We must argue that the total reduction in information rents exceeds the inefficiency, so that the ex-ante profit is higher under the alternative solution. We first consider the reduction in information rents only for the types in $[0, v_0^+]$. This is what matters for the continuation profit at time 0^+, right after the initial trade. Feasibility implies that the reduction in information rents for the types in $[0, v_0^+]$ must already (weakly) exceed the revenue loss from inefficiency. Otherwise, the continuation profit at 0^+ would be smaller than the profit from an efficient auction given the posterior v_0^+, and thus the payoff floor constraint would be violated. If we now include the types in $(v_0^+, 1]$ in the comparison, we must add the reduction in information rents for these types but there is no additional inefficiency because these types trade at time zero in both solutions. Therefore, the total reduction in information rents is strictly higher than the inefficiency, and the ex-ante profit under the alternative is strictly higher than under the efficient auction.

Proof of Proposition 4. The "if" part is trivial. For the "only if" part, suppose there is another feasible solution $\tilde{T}$ other than the efficient auction $T^E \equiv 0$. Let $\tilde{v}_t$ denote the cutoff path corresponding to $\tilde{T}$. Note first that the range of $\tilde{T}$ cannot be a singleton because this would imply that $\tilde{T}(v) = t$ for all $v \in [0, 1]$ for some $t > 0$. Then the expected revenue would be given by

$$e^{-rt} \int_0^1 J(v) dF^{(n)}(v),$$

which is strictly lower than the revenue from an efficient auction at time 0. Therefore, the payoff floor constraint would be violated at $t = 0$, contradicting the feasibility of $\tilde{T}$.

Hence, there exists some time s with $0 < \tilde{v}_s^+ = \tilde{v}_s < 1$ such that $\tilde{T}(v) < s$ for all $v > \tilde{v}_s$, and $\tilde{T}(v) > s$ for all $v < \tilde{v}_s$. Then we can define a new feasible solution

$$\hat{T}(v) := \begin{cases} 0 & \text{if } v > \tilde{v}_s, \\ \tilde{T}(v) - s & \text{if } v \leq \tilde{v}_s, \end{cases}$$

with corresponding cutoff path $\hat{v}_t$. Solution $\hat{T}$ is feasible because $\tilde{T}$ satisfies the payoff floor

[40] In the proof of Proposition 4, we show that we can always construct a feasible solution with $0 < v_0^+ < 1$, if there exists any feasible solution that differs from the efficient auction.

constraint for all $t \geq s$. Moreover, we have $0 < \hat{v}_0^+ < 1$ because $\hat{v}_0^+ = \tilde{v}_s$. We can invoke Lemma 4 (PF before atom) by setting $t = 0$ and $v = v_0 = 1$ to obtain

$$\int_0^1 e^{-r\hat{T}(x)} J(x) dF^{(n)}(x) > \int_0^1 J(x) dF^{(n)}(x).$$

The left hand side of the above inequality is the revenue from $\hat{T}$, while the right hand side is the revenue from $T^E \equiv 0$. This completes the proof. □

Proposition 4 implies that in order to decide whether the efficient auction is optimal or not, it suffices to determine whether it is the unique feasible solution. This will be particularly useful, if we are able to construct solutions with non-zero trading times. We approach such a construction by considering the binding payoff floor constraint.

Solutions to the binding payoff floor are also important for the characterization of optimal solutions of the auxiliary problem. The following proposition which shows that the payoff floor constraint must be locally binding for an optimal solution if the monopoly profit is locally concave.

Proposition 5. *If $v(1-F(v))$ is strictly concave on an interval $[0, \overline{v}]$, then for every optimal solution, the payoff floor constraint binds for all t such that $v_t \in (0, \overline{v})$.*

To clarify the role of concavity, we state the main lemma that is used in the proof. Remember than $Q(v) = e^{-rT(v)}(F(v))^{n-1}$ denotes the expected discounted winning probability of type v for any solution T. The following lemma shows that if for T and $\hat{T}$, discounted winning probabilities $\hat{Q}(v)$ are more spread out than discounted winning probabilities $Q(v)$,[41] then concavity implies that the ex-ante profit is higher for $\hat{T}$.

Lemma 8 (MPS)**.** *Let T be a feasible solution of the auxiliary problem with cutoffs v_t. Let $a < b$ be such that $v(1-F(v))$ is strictly concave on the interval $[v_b, v_a]$. Let $\hat{T} : [0, 1] \to [0, \infty]$ be non-increasing and satisfy $\hat{T}(v) = T(v)$ for all $v \notin (v_b, v_a)$, such that*

$$\int_{v_b}^{x} \hat{Q}(v) - Q(v) dv \leq 0, \quad \forall x \in [v_b, v_a], \tag{A.3}$$

with equality for $x = v_a$. Then $\hat{T}$ satisfies (PF) for all $t \notin (a, b)$. If (A.3) holds with strict inequality for a set with strictly positive measure, then the ex-ante profit is strictly higher for $\hat{T}$ then for T and (PF) is a strict inequality for all $t < a$.

[41]Formally, this means that $Q(v)$ is a mean-preserving spread of $\hat{Q}(v)$.

When the payoff floor constraint is slack for some interval (a, b), then we can construct an alternative trading time function $\hat{T}$ that differs from T only for types in (v_b, v_a).[42] We select a cutoff type $w \in (v_b, v_a)$, types above w are assigned an earlier trading time and types below w are assigned a later trading time. Clearly this implies that $\hat{Q}(v)$ is more spread out than $Q(v)$. Concavity of $v(1 - F(v))$ implies that this variation improves ex-ante expected profits. The additional results stated in Lemma 8 also allow us to show that the alternative solution $\hat{T}$ satisfies the payoff floor constraint.

In the proof of Theorem 2, we will use Proposition 5 on intervals of the form $(0, \varepsilon)$. In this case, the requirement of local concavity is satisfied for any distribution function without imposing Assumption 4, if ε is sufficiently close to zero (see the discussion at the end of Section 5.2.3). Since Lemma 5 shows that a feasible solution cannot end with a single atom, Proposition 5 has bite in this case: (PF) must be binding for all t such that $v_t \in (0, \varepsilon)$ in the optimal solution.

The next lemma shows that the binding payoff floor constraint implies that v_t must satisfy a second-order ordinary differential equation.

Lemma 9 (Binding PF yields ODE)**.** *Let v_t be a sequence of cutoffs for a feasible solution T, for which the payoff floor constraint is binding for all $t \in (a, b)$, where $0 \leq a < b \leq \infty$. Then v_t is twice continuously differentiable and strictly decreasing on (a, b) and satisfies the differential equation (5.6).*

Next we characterize precise conditions under which there exists a solution to this ODE that is non-increasing and thus is feasible in the auxiliary problem. It turns out that a feasible solution exists if $n < \overline{N}(F)$ and does not exist if $n > \overline{N}(F)$, where the distribution-specific cutoff $\overline{N}(F)$ for the number of buyers is defined as[43]

$$\overline{N}(F) := 1 + \frac{\sqrt{2 + \phi}}{1 + \phi}.$$

Depending on the type distribution, the cutoff $\overline{N}(F)$ can take any value above one. For example, if valuations are distributed according to $F(v) = v^k$ with support $[0, 1]$ and $k > 0$, we have $\phi = k - 1$ and $\overline{N}(F) = 1 + \sqrt{1 + k}/k$. If $k = 1$ we obtain the uniform distribution and $\overline{N}(F) = 1 + \sqrt{2}$. This verifies our claim in Section 2, that with three or more buyers, the seller cannot do better than running an efficient auction if the distribution is uniform.

[42] In the proof of Proposition 5, we also consider the case where $v_a = v_b$.

[43] Recall that $\phi = \lim_{v \to 0} \frac{f'(v)v}{f(v)}$, which exists and is greater than -1 by Assumption 2.

If $k < 1$, the density is unbounded at zero which violates Assumption 1. In this case $\overline{N}(F)$ can be large. For the proofs of our main results we obtain the following lemma:

Lemma 10 (Low Cutoff)**.** *If Assumptions 1 and 2 are satisfied, then $\overline{N}(F) < 3$. If $f(0) > 0$ then $\overline{N}(F) \in (2, 3)$, and if $f(0) = 0$, then $\overline{N}(F) < 2$.*

The following lemma shows that the cutoff determines if a feasible solution to the binding payoff floor constraint exists. For the statement of the lemma, let v_t^x be the unique solution to (4.1) with $v_0 = x$.

Lemma 11 (Solution to Binding PF)**.** *(i) If $n > \overline{N}(F)$, there exists no non-increasing solution to (5.6) that satisfies $v_0^+ > 0$ and $\lim_{t\to\infty} v_t = 0$.*

(ii) If $n < \overline{N}(F)$, v_t^x is a decreasing solution to (5.6) that satisfies $v_0 = x$ and $\lim_{t\to\infty} v_t = 0$.

(iii) Suppose $n < \overline{N}(F)$, and Assumption 4 is satisfied. Let $\hat{v}_t$ be a decreasing solution to (5.6) that does not coincide with v_t^x for any $x \in [0, 1]$. Then there exists $\hat{x} \in [0, 1]$ such that $v_t^{\hat{x}}$ yields a strictly higher profit than $\hat{v}_t$.

Note that, when feasible solutions exist, they are not necessarily unique for a given boundary value v_0^+, because (5.6) is a second-order differential equation. Using Lemma 8, part (iii) of Lemma 11 shows that any solution to (5.6) that does not satisfy (4.1) is strictly dominated by the solution to (4.1) for some initial cutoff x.

A.2 Equilibrium Approximation of the Solution to the Binding Payoff Floor Constraint

The final ingredient for the proofs of our main results is an approximation of solutions to the binding payoff floor constraint with cutoff paths that arise as equilibrium outcomes of the discrete time game.

Proposition 6 (Equilibrium Approximation of Binding PF)**.** *Suppose Assumptions 2 and 3 are satisfied, and $n < \overline{N}(F)$. Then for any v_0^+, there exists a decreasing sequence $\Delta_m \searrow 0$ and a sequence of equilibria $(p^m, b^m) \in \mathcal{E}(\Delta_m)$ such that the sequence of trading functions T^m implemented by (p^m, b^m) and the seller's ex-ante revenue $\Pi^\Delta(p^m, b^m)$ converge to the profit achieved by the solution given by (4.1) with boundary condition v_0^+.*

To obtain the approximation, we construct trigger strategies as outlined in Section 5.2.3. We use a discrete trading time $T^\Delta : [0, 1] \to \{0, \Delta, 2\Delta, \ldots\}$, where $\Delta > 0$ is an arbitrarily chosen period length. T^Δ is constructed such that the payoff floor constraint is slack for all $t \in \{0, \Delta, 2\Delta, \ldots\}$. This approximation, together with the price sequence given by (5.2), will be used to define the equilibrium price path for a game with given Δ. On the equilibrium path, buyers best respond to this price path. If the seller deviates from the equilibrium price path, the buyers use a continuation strategy given by a stationary equilibrium. Note that buyers can react to a deviation by the seller in the same period. Therefore, the response to a deviation is immediate and the seller cannot obtain profits in excess of the stationary equilibrium profit. The uniform Coase conjecture (Proposition 1.(ii)) thus implies that the profit after a deviation converges to the profit of the efficient auction. The equilibrium path, on the other hand is carefully constructed such that it yields a profit above the profit of stationary equilibria. As $\Delta \to \infty$, T^Δ is constructed such that it converges to the solution to the binding payoff floor constraint, but sufficiently slowly so that stationary equilibria can be used to provide incentives for the seller. The details of the construction are rather technical and are deferred to Appendix C in the Supplemental Material.

A.3 Proof of Theorem 1

Proof. From Proposition 3 we know that an optimal solution to the auxiliary problem exists and hence V is attained by an element in the feasible set. By Lemma 7, $T(v) < \infty$ for all $v > 0$ for any feasible solution T. Under Assumption 1, we have $J(v) < 0$ for v in a neighborhood of zero, which implies $p^M > 0$. Therefore $T^M(v) = \infty$ for some v and hence T^M is not in the feasible set of the auxiliary problem. Moreover, T^M is the only non-increasing trading time function that attains Π^M. Therefore $V < \Pi^M$. The payoff bounds in Proposition 2 then imply $\Pi^* \leq V < \Pi^M$. □

A.4 Proof of Theorem 2

Lemma 10 implies that under Assumption 1 the cutoff $\overline{N}(F)$ is less than three, and less than two if $f(0) = 0$. Hence the conditions in Theorem 2 imply $n > \overline{N}(F)$. We proceed just using $n > \overline{N}(F)$ since Assumption 1 is not used elsewhere in the proof. The proof has two parts. The first characterizes the solution to the auxiliary problem. The second part shows that the value of the auxiliary problem is Π^* and that its optimal solution can be approximated by discrete time equilibria.

Value of the Auxiliary Problem. We use an indirect argument to show that if $n > \overline{N}(F)$, the feasible set of the auxiliary problem only contains the efficient auction. Informally:

if there was an alternative optimal solution, then it would have to satisfy the payoff floor constraint with equality, which is impossible if $n > \overline{N}(F)$. This informal argument has several gaps which are filled in the following formal proof.

Suppose by contradiction, that there exists a element T in the feasible set of the auxiliary problem for which $T(v) > 0$ for a positive measure of types.[44] Proposition 4 implies that in this case, the efficient auction is not optimal. T itself need not be optimal, but Proposition 3 implies that an optimal solution to the auxiliary problem exists, which we call $\hat{T}$ with cutoffs denoted by $\hat{v}_t$. To derive a contradiction, we show that for $\hat{T}$ the payoff floor constraint must be binding for all t such that $v_t \in [0, \varepsilon]$. By Lemma 11.(i), there exists no feasible solution to the binding payoff floor constraint if $n > \overline{N}(F)$, which yields the contradiction.

To show that the payoff floor constraint must be binding, we use the observation that for any distribution function, there exists $\varepsilon > 0$ such that $v(1-F(v))$ is concave for all $v \in [0, \varepsilon]$. Since $\phi > -1$ by Assumption 2, there exists a valuation $\varepsilon > 0$ such that for all $v \in [0, \varepsilon]$, $(f'(v)v)/f(v) > -2$ which implies that $v(1 - F(v))$ is concave on this interval. This local concavity in a neighborhood of zero is only useful if $\hat{v}_t \in [0, \varepsilon]$ for some $t > 0$. This is implied by Lemma 5, which shows that the optimal solution to the auxiliary problem does not end with an atom. Therefore, there must be some time $\bar{t} > 0$ such that $\hat{v}_{\bar{t}} \in (0, \varepsilon)$. Therefore, Proposition 5 implies that $\hat{T}$ must satisfy the payoff floor with equality for all $t > \bar{t}$.

Equilibrium. So far we have shown by contradiction that the value of the auxiliary problem is $V = \Pi^E$. The bounds on the seller's equilibrium payoff from Proposition 2 then imply that $\Pi^* = V = \Pi^E = \Pi_*$. Hence, the seller's equilibrium payoff is unique and given by V. Finally, Proposition 1 shows the existence of stationary equilibria, and since $\Pi^* = \Pi^E$, there must exist a sequence of stationary equilibria for which the seller's profit converges to Π^E.

A.5 Proof of Theorem 3

Lemma 10 implies that under Assumption 1 the cutoff $\overline{N}(F)$ is between two and three if $f(0) > 0$. Hence the conditions in Theorem 2 imply $n < \overline{N}(F)$. We proceed just using $n < \overline{N}(F)$ since Assumption 1 and $f(v) > 0$ are not used elsewhere in the proof. The proof has two parts. The first characterizes the solution to the auxiliary problem. The second part shows that the value of the auxiliary problem is Π^* and that its optimal solution can be approximated by discrete time equilibria.

[44] We identify trading time functions that coincide almost everywhere.

Value of the Auxiliary Problem. For part (i) we show that there exists a feasible (not necessarily optimal) solution of the auxiliary problem that yields a profit greater than Π^E and hence $V > \Pi^E$. By Lemma 11.(ii), there exists a feasible solution to the auxiliary problem that differs from the efficient auction if $n < \overline{N}(F)$. Together with Proposition 4, this implies that the efficient auction is not the optimal solution of the auxiliary problem if $n < \overline{N}(F)$.

For part (ii) we first show how V can be achieved. By Proposition 5 and Assumption 4, the payoff floor constraint must be binding at the optimal solution to the auxiliary problem. By Lemma 11.(iii), the optimal solution must satisfy (4.1) and is unique up to the choice of v_0^+. If we choose v_0^+ optimally, we thus obtain the optimal solution to the auxiliary problem which achieves V.

Next we show that any value in $[\Pi^E, V]$ can be achieved by a solution to the ODE in (4.1) by varying v_0^+. Let v_t^x be the sequence of cutoffs obtained from the ODE in (4.1) with boundary condition $v_0^+ = x \in [0,1]$ and denote the value of the objective function of the auxiliary problem evaluated at v_t^x by $\Pi(x)$. We thus have to show that the range of $\Pi(x)$ is $[\Pi^E, V]$. It is clear that $x = 0$ leads to $\Pi(x) = \Pi^E$ and we have shown above that that there exists x^* such that $\Pi(x^*) = V$. To complete the proof we show that $\Pi(x)$ is continuous. To see this, denote the trading time function corresponding to v_t^x by T^x. $\Pi(x)$ is obtained by substituting $T(v) = T^x(v)$ in the objective function of the auxiliary problem. Note that

$$T^x(v) = \begin{cases} 0, & \text{if } v \geq x, \\ T^1(v) - T^1(x), & \text{if } v \leq x. \end{cases}$$

Hence $T^x(v)$ is continuous in x for all $v > 0$ and therefore $e^{-rT^x(v)}$ is continuous in x for all $v > 0$. Since $e^{-rT^x(v)}$ is bounded, $\Pi(x)$ is continuous in x, which completes the proof.

Equilibrium Approximation. For part (i) we show that there exists a solution that yields a profit above Π^E which can be approximated by a sequence of equilibria. This shows $\Pi^* > \Pi^E$. Again by Lemma 11.(ii), a profit $\tilde{\Pi} > \Pi^E$ can be achieved by the solution to the ODE in (4.1) for some $v_0^+ \in (0,1)$. Proposition 6 shows that this solution to (4.1) can be approximated by discrete time equilibrium outcomes. Hence, there exists a sequence of equilibria $(p_m, b_m) \in \mathcal{E}(\Delta_m)$, for $\Delta_m \to 0$ as $m \to \infty$, such that $\lim_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) = \tilde{\Pi}$. This implies $\Pi^* \geq \tilde{\Pi} > \Pi^E$.

For part (ii) Proposition 6 shows that there exists a sequence of equilibria $(p_m, b_m) \in \mathcal{E}(\Delta_m)$, for $\Delta_m \to 0$ as $m \to \infty$, such that $\lim_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) = \Pi(x)$. Since this holds for any $x \in (0,1)$ and the range of $\Pi(x)$ is $[\Pi^E, V]$ this completes the proof.

A.6 Uncertain Number of Buyers

Suppose that $F(v)$ has support $[\underline{v}, 1]$, with $\underline{v} < 0$. The derivation of the ODE from the binding payoff floor constraint in the proof of Lemma 9 is unchanged if $\underline{v} < 0$ instead of $\underline{v} = 0$. Therefore the binding payoff floor constraint implies that v_t is twice continuously differentiable and satisfies (5.6) with $g(v)$ given by (4.2). Lemma 11, which gives a condition for the existence of a decreasing solution to (5.6) for $\underline{v} = 0$, has to be modified for the case that $\underline{v} < 0$. Remember that we have defined v_t^x as the unique solution to (4.1) with the boundary condition $v_0 = x$. We have the following modified version of Lemma 11:

Lemma 12 (Lemma 11 for $\underline{v} < 0$)**.** *Suppose buyers' valuations are independently drawn from distribution F on $[\underline{v}, 1]$ with $\underline{v} < 0$. Then*

(i) v_t^x is a decreasing solution to (5.6) that satisfies $v_0 = x$ and $\lim_{t\to\infty} v_t = 0$.

(ii) Suppose in addition that Assumption 4 is satisfied. Let $\hat{v}_t$ be a decreasing solution to (5.6) that does not coincide with v_t^x for any $x \in [0,1]$. Then there exists $\hat{x} \in [0,1]$ such that $v_t^{\hat{x}}$ yields a strictly higher profit than $\hat{v}_t$.

With Lemma 12 in hand, we can apply Proposition 4 which shows that the efficient auction is not the optimal solution to the auxiliary problem if the feasible set contains another solution. Proposition 4 holds unchanged if $\underline{v} < 0$. Hence, an efficient auction is no longer an optimal solution to the auxiliary problem for any n.

Next we investigate what happens to the optimal cutoff path w_t as we truncate the distribution $F(v)$ at $\underline{v}' \in (\underline{v}, 0)$, and let $\underline{v}' \to 0$. Denote the truncated distribution function by $F_{\underline{v}'}(v)$. Note that the proof of Lemma 12 does not depend on the precise shape of $F_{\underline{v}'}(v)$ for $v < 0$. Therefore, we can replace $F_{\underline{v}'}(v)$ by the distribution function

$$\tilde{F}_\lambda(v) \quad = \quad \lambda + (1-\lambda)F_0(v)$$

with support $[0,1]$, where $\lambda = (F(0) - F(\underline{v}'))\,/\,(1 - F(\underline{v}'))$, and $F_0(v)$ is $F(v)$ truncated at $v = 0$. It is easy to verify that for $v \geq 0$, $\tilde{F}_\lambda(v) = F_{\underline{v}'}(v)$, and $\tilde{F}_\lambda$ has an atom of mass $\lambda = F_{\underline{v}'}(0)$ at $v = 0$. Hence, $\underline{v}' \to 0$ corresponds to taking the limit $\lambda \to 0$.

We denote the optimal solution for $\lambda > 0$ by w_t^λ as in Section 6. We want to show that for all $t > 0$, $w_t^\lambda \to 0$ as $\lambda \to 0$. Consider any sequence $\lambda_m > 0$, $m = 0, 1, \ldots$ such that

$\lambda_m \to 0$ as $m \to 0$. Suppose by contradiction that $w_t^{\lambda_m}$ does not converge to 0 pointwise for all $t > 0$. This implies that there exists $s > 0$ and $\underline{w} > 0$ such that for a subsequence m_k, $w_s^{\lambda_{m_k}} > \underline{w}$, $\forall k$. Since $w_t^{\lambda_m}$ is non-increasing for all m, this also implies that $w_s^{\lambda_{m_k}} > \underline{w}$ for all $t \leq s$. By Helly's theorem, we can select another subsequence $\hat{m}_k$ such that $w_t^{\lambda_{\hat{m}_k}} \to w_t^0$ almost everywhere, where the sequence w_t^0 is non-increasing and $w_t^0 \in [0, 1]$. Moreover, since $w_t^{\lambda_{\hat{m}_k}}$ satisfies the payoff floor constraint for all k, by dominated convergence w_t^0 also satisfies the payoff floor constraint. Hence w_t^0 it is a feasible solution to the auxiliary problem.

If Assumptions 1 and 2 are satisfied and there are three or more buyers, any feasible solution to the auxiliary problem satisfies $v_t = 0$ for all $t > 0$. This implies that $w_t^0 = 0$ for all $t > 0$ which is a contradiction since $w_t^0 \geq \underline{w} > 0$ for $t < s$. Therefore, $w_t^\lambda \to 0$ for all $t > 0$. Hence, as the uncertainty about the number of buyers vanishes, the cutoffs in the revenue-maximizing equilibrium converge to zero for all $t > 0$. Dominated converges also implies that the seller's profit converges to the profit of an efficient auction.

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Supplemental Material for Online Publication

B Omitted Proofs

B.1 Proof of Lemma 2

Proof. In the main paper we slightly abuse notation by using p_t both for the seller's (possibly mixed) strategy and the announced reserve price at a given history. This should not lead to confusion in the main part but for this proof we make a formal distinction. We denote the reserve price announced in period t by x_t. A history is therefore given by $h_t = (x_0, \ldots, x_{t-\Delta})$. Furthermore we denote by $h_{t^+} = (h_t, x_t) = (x_0, \ldots, x_{t-\Delta}, x_t)$ a history in which the reserve prices $x_0, \ldots, x_{t-\Delta}$ have been announced in periods $t = 0, \ldots, t-\Delta$ but no buyer has bid in these periods, and the seller has announced x_t in period t, but buyers have not yet decided whether they bid or not. For any two histories $h_t = (x_0, x_\Delta, \ldots, x_{t-\Delta})$ and $h'_s = (x'_0, x'_\Delta, \ldots, x'_{s-\Delta})$, with $s \leq t$, we define a new history

$$h_t \oplus h'_s = (x'_0, x'_\Delta, \ldots, x'_{s-\Delta}, x_s, \ldots, x_{t-\Delta}).$$

That is, $h_t \oplus h'_s$ is obtained by replacing the initial period s sub-history in h_t with h'_s. Finally, we can similarly define $h_{t^+} \oplus h'_s$ for $s < t$. With this notation we can state the proof of the lemma.

Consider any equilibrium $(p, b) \in \mathcal{E}(\Delta)$ in which the seller randomizes on the equilibrium path. The idea of the proof is that we can inductively replace randomization on the equilibrium path by a deterministic reserve price and at the same time weakly increase the seller's ex-ante revenue. We first construct an equilibrium $(p^0, b^0) \in \mathcal{E}(\Delta)$ in which the seller earns the same expected profit as in (p, b), but does not randomize at $t = 0$. If the seller uses a pure action at $t = 0$, we can set $(p^0, b^0) = (p, b)$. Otherwise, if the seller randomizes over several prices at $t = 0$, she must be indifferent between all prices in the support of $p_0(h_0)$. Therefore, we can define $p^0_0(h_0)$ as the distribution that puts probability one on a single price $x_0 \in \operatorname{supp} p_0(h_0)$. If we leave the seller's strategy unchanged for all other histories ($p^0_t(h_t) = p_t(h_t)$, for all $t > 0$ and all $h_t \in H_t$) and set $b^0 = b$, we have defined an equilibrium (p^0, b^0) that gives the seller the same payoff as (p, b) and specifies a pure action for the seller at $t = 0$.

Next we proceed inductively. Suppose we have already constructed an equilibrium (p^m, b^m) in which the seller does not randomize on the equilibrium path up to $t = m\Delta$, but uses a mixed action on the equilibrium path at $(m+1)\Delta$. We want to construct an equilibrium (p^{m+1}, b^{m+1}) with a pure action for the seller on the equilibrium path at $(m+1)\Delta$. Suppose that in the equilibrium (p^m, b^m), the highest type in the posterior at $(m+1)\Delta$ is some type $\beta^0_{(m+1)\Delta} > 0$. We select a price in the support of the seller's mixed action at $(m+1)\Delta$, which we denote by $x^0_{(m+1)\Delta}$, such that the expected payoff of $\beta^0_{(m+1)\Delta}$ at $h_{t^+} = (h_t, x^0_{(m+1)\Delta})$ is weakly smaller than the expected payoff at h_t. In other words, we pick a price that is (weakly) bad news for the buyer with type $\beta^0_{(m+1)\Delta}$. This will be the equilibrium price announced in period $t = (m+1)\Delta$ in the equilibrium (p^{m+1}, b^{m+1}). The formal construction of the equilibrium is rather complicated. The rough idea is that, first we posit that after $x^0_{(m+1)\Delta}$ was announced in period $(m+1)\Delta$, (p^{m+1}, b^{m+1}) prescribes the same continuation as (p^m, b^m). Second, on the equilibrium path up to period $m\Delta$, we change the reserve prices such that the same marginal types as before are indifferent between buying immediately and waiting in all periods $t = 0, \ldots, m\Delta$. Since we have chosen $x^0_{(m+1)\Delta}$ to be bad news, this leads to (weakly) higher prices for $t = 0, \ldots, m\Delta$, and therefore we can show that the seller's expected profit increases weakly. Finally, we have to specify what happens after a deviation from the equilibrium path by the seller in periods $t = 0, \ldots, (m+1)\Delta$. Consider the on-equilibrium history h_t in period t for (p^{m+1}, b^{m+1}). We identify a history $\hat{h}_t$ for which the posterior in the original equilibrium (p, b) is the same posterior as at h_t in the new equilibrium. If at h_t, the seller deviates from p^{m+1} by announcing the reserve price $\hat{x}_t$, then we define (p^{m+1}, b^{m+1}) after $h_{t^+} = (h_t, \hat{x}_t)$ using the strategy prescribed by (p, b) for the subgame starting at $\hat{h}_{t^+} = (\hat{h}_{t^+}, \hat{x}_t)$. We will show that with this definition, the seller does not have an incentive to deviate.

Next, we formally construct the sequence of equilibria (p^m, b^m), $m = 1, 2, \ldots$, and show that this sequence converges to an equilibrium (p^∞, b^∞) in which the seller never randomizes on the equilibrium path and achieves an expected revenue at least as high as the expected revenue in (p, b). We first identify a particular equilibrium path of (p^0, b^0) with a sequence of reserve prices $h^0_\infty = (x^0_0, x^0_\Delta, \ldots)$ and the corresponding buyer cutoffs $\beta^0 = (\beta^0_0, \beta^0_\Delta, \ldots)$

that specify the seller's posteriors along the path $h_\infty^0 = (x_0^0, x_\Delta^0, \ldots)$.[45] Then we construct an equilibrium (p^m, b^m) such that the following properties hold: for $t = 0, \ldots, m\Delta$, the equilibrium prices x_t^m chosen by the seller are weakly higher than x_t^0 and the equilibrium cutoffs β_t^m are exactly β_t^0; for $t > m\Delta$, or off the equilibrium path, the strategies coincide with what (p^0, b^0) prescribes at some properly identified histories, so that the two strategy profiles prescribe the same continuation payoffs at their respective histories.

In order to determine $h_\infty^0 = (x_0^0, x_\Delta^0, \ldots)$ and $\beta^0 = (\beta_0^0, \beta_\Delta^0, \ldots)$ we start at $t = 0$ and define x_0^0 as the seller's pure action in period zero in the equilibrium (p^0, b^0) and set $\beta_0^0 = 1$. Next we proceed inductively. Suppose we have fixed x_t^0 and β_t^0 for $t = 0, \Delta, \ldots$. To define $x_{t+\Delta}^0$, we select a price in the support of the seller's mixed action at history $h_{t+\Delta}^0 = (x_0^0, \ldots, x_t^0)$ in the equilibrium (p^0, b^0) such that the expected payoff of the cutoff buyer type β_t^0, conditional on $x_{t+\Delta}^0$ is announces, is no larger than this type's expected payoff at the beginning of period $t + \Delta$ before a reserve price is announces.[46] We then pick $\beta_{t+\Delta}^0$ as the cutoff buyer type following history $\left(x_0^0, \ldots, x_t^0, x_{t+\Delta}^0\right)$.

(p^0, b^0) was already defined. We proceed inductively and construct equilibrium (p^{m+1}, b^{m+1}) for $m = 0, 1, \ldots$ as follows.

(1) On the equilibrium path at $t = (m+1)\Delta$, the seller plays a pure action and announces the reserve price $x_{(m+1)\Delta}^{m+1} := x_{(m+1)\Delta}^0$.

(2) On the equilibrium path at $t = 0, \Delta, \ldots, m\Delta$, the seller's pure action x_t^{m+1} is chosen such that the buyers' on-path cutoff types in periods $t = \Delta, \ldots, (m+1)\Delta$ is $\beta_t^{m+1} = \beta_t^0$, where β_t^0 was defined above.

(3) On the equilibrium path at the history $h_{t^+} = (x_0, \ldots, x_t)$ for $t = 0, \Delta, (m+1)\Delta$, each buyer bids if and only if $v^i \geq \beta_t^{m+1} = \beta_t^0$.

(4) at $t > (m+1)\Delta$: for any history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which no deviation has occurred at or before $(m+1)\Delta$, the seller's (mixed) action is $p^{m+1}(h_t) := p^0\left(h_t \oplus \left(x_0^0, \ldots, x_{(m+1)\Delta}^0\right)\right)$. For any history $h_{t^+} = (x_0, \ldots, x_{t-\Delta}, x_t)$ in which no deviation has occurred at or before $(m+1)\Delta$, the buyer's strategy is defined by $b^{m+1}(h_{t^+}) := b^0\left(h_{t^+} \oplus \left(x_0^0, \ldots, x_{(m+1)\Delta}^0\right)\right)$.

(5) For any off-path history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which the seller's first deviation from the equilibrium path occurs at $s \leq (m+1)\Delta$, the seller's (mixed) action is prescribed by $p^{m+1}(h_t) := p^0\left(h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)\right)$. For any off-path history $h_{t^+} = (x_0, \ldots, x_{t-\Delta}, x_t)$ in which the seller's first deviation from the equilibrium path occurs in period $s \leq (m+1)\Delta$, the buyer's strategy is $b^{m+1}(h_{t^+}) := b^0\left(h_{t^+} \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)\right)$.

In this definition, (1) and (2) define the seller's pure actions on the equilibrium path up to $(m+1)\Delta$. The prices defined in (1) and (2) are chosen such that bidding according to

[45] Note that the cutoffs β_t^0 are the equilibrium cutoffs which may be different from the cutoffs that would arise if the seller used pure actions with prices $x_0^0, x_\Delta^0, \ldots$ on the equilibrium path.

[46] If the seller plays a pure action at $h_{t+\Delta}^0$, then $x_{t+\Delta}^0$ the price prescribed with probability one by the pure action. If the seller randomizes at $h_{t+\Delta}^0$, there must be one realization, which, together with the continuation following it, gives the buyer a payoff weakly smaller than the average.

the cutoffs β_t^{m+1} is optimal for the buyers. Part (4) defines the equilibrium strategies for all remaining on-path histories and after deviations that occur in periods after $(m+1)\Delta$, that is, in periods where the seller can still mix on the equilibrium path. The equilibrium proceeds as in (p^0, b^0) at the history where the seller used the prices $x_0^0, \ldots, x_{(m+1)\Delta}^0$ in the first $m+1$ periods. This ensures that the continuation strategy profile is taken from the continuation of an on-path history of the equilibrium (p^0, b^0), where the seller's posterior in period $(m+1)\Delta$ is the same as in the equilibrium (p^{m+1}, b^{m+1}). Finally, (5) defines the continuation after a deviation by the seller at a period in which we have already defined a pure action. If the seller deviates at a history $h_t = \left(x_0^m, \ldots, x_{s-\Delta}^m\right)$, then we use the continuation strategy of (p^0, b^0), at the history $\left(x_0^0, \ldots, x_{s-\Delta}^0\right)$.

We proceed by proving a series of claims showing that we have indeed constructed an equilibrium.

Claim 1. *The expected payoff of the cutoff buyer* $\beta_{(m+1)\Delta}^m = \beta_{(m+1)\Delta}^0$ *at the on-path history* $h_{(m+1)\Delta}^m = (x_0^m, \ldots, x_{m\Delta}^m)$ *in the candidate equilibrium* (p^m, b^m) *is the same as its payoff at the on-path history* $h_{(m+1)\Delta}^0 = (x_0^0, \ldots, x_{m\Delta}^0)$ *in the candidate equilibrium* (p^0, b^0).

Proof. This follows immediately from (1)–(3) above. ☐

Claim 2. *The expected payoff of the cutoff buyer* $\beta_{(m+1)\Delta}^{m+1} = \beta_{(m+1)\Delta}^0$ *at the on-path history* $h_{((m+1)\Delta)^+}^{m+1} = \left(x_0^{m+1}, \ldots, x_{m\Delta}^{m+1}, x_{(m+1)\Delta}^{m+1}\right)$ *in the candidate equilibrium* (x^{m+1}, x^{m+1}) *is the same as this cutoff type's expected payoff at the on-path history* $h_{((m+1)\Delta)^+}^0 = \left(x_0^0, \ldots, x_{m\Delta}^0, x_{(m+1)\Delta}^0\right)$ *in the candidate equilibrium* (p^0, b^0).

Proof. By construction, $x_{(m+1)\Delta}^{m+1} = x_{(m+1)\Delta}^0$. It follows from part (4) that (p^{m+1}, b^{m+1}) and (p^0, b^0) are identical on the equilibrium path from period $(m+2)\,\Delta$ onwards. The claim follows. ☐

Claim 3. *The expected payoff of the cutoff buyer* $\beta_{(m+1)\Delta}^{m+1} = \beta_{(m+1)\Delta}^0$ *at the on-path history* $h_{(m+1)\Delta}^{m+1} = \left(x_0^{m+1}, \ldots, x_{m\Delta}^{m+1}\right)$ *in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly lower than this cutoff type's expected payoff at the on-path history* $h_{(m+1)\Delta}^0 = (x_0^0, \ldots, x_{m\Delta}^0)$ *in the equilibrium* (p^0, b^0).

Proof. In the candidate equilibrium (p^{m+1}, b^{m+1}), the cutoff type's payoffs at histories $h_{(m+1)\Delta}^{m+1}$ and $h_{((m+1)\Delta)^+}^{m+1}$ are the same because the seller plays a pure action in period $(m+1)\,\Delta$. In the equilibrium (p^0, b^0), the cutoff type's payoff at history $h_{((m+1)\Delta)^+}^{m+1}$ is weakly lower than his payoff at history $h_{(m+1)\Delta}^0$ because of the definition of $x_{(m+1)\Delta}^0$ (which chosen to give the cutoff type a lower expected payoff than the expected payoff at $h_{(m+1)\Delta}^0$). The claim then follows from Claim 2. ☐

Claim 4. *The expected payoff of the cutoff buyer* $\beta_{(m+1)\Delta}^{m+1} = \beta_{(m+1)\Delta}^0$ *at the on-path history* $h_{(m+1)\Delta}^{m+1} = \left(x_0^{m+1}, \ldots, x_{m\Delta}^{m+1}\right)$ *in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly lower than this cutoff type's expected payoff at the on-path history* $h_{(m+1)\Delta}^m = (x_0^m, \ldots, x_{m\Delta}^m)$ *in the candidate equilibrium* (p^m, b^m).

Proof. By Claim 1, the cutoff type's expected payoff at the on-path history $h^m_{(m+1)\Delta} = (x^m_0, \dots, x^m_{m\Delta})$ in the candidate equilibrium (p^m, b^m) is the same as its payoff at the on-path history $h^0_{(m+1)\Delta} = (x^0_0, \dots, x^0_{m\Delta})$ in the candidate equilibrium (p^0, b^0). The claim then follows from Claim 3. □

Claim 5. *For each* $m = 0, 1, \dots$ *and* $t = 0, 1, \dots, m\Delta$, *we have* $x^{m+1}_t \geq x^m_t$.

Proof. By Claim 4, the cutoff type $\beta^{m+1}_{(m+1)\Delta} = \beta^m_{(m+1)\Delta} = \beta^0_{(m+1)\Delta}$ in period $(m+1)\Delta$ on the equilibrium path in the candidate equilibrium (p^{m+1}, b^{m+1}) has a weakly lower payoff than its expected payoff in the candidate equilibrium (p^m, b^m). To keep this cutoff indifferent in period $m\Delta$ in both candidate equilibria, we must have $x^{m+1}_{m\Delta} \geq x^m_{m\Delta}$. Then to keep the cutoff type $\beta^{m+1}_{m\Delta} = \beta^m_{m\Delta} = \beta^0_{m\Delta}$ indifferent in period $(m-1)\Delta$, we must have $x^{m+1}_{(m-1)\Delta} \geq x^m_{(m-1)\Delta}$. The proof is then completed by induction. □

Claim 6. *The seller's (time* 0*) expected payoff in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly higher than the seller's expected payoff in the equilibrium* (p^0, b^0).

Proof. By parts (1)–(3) of the construction, at $t = 0, \dots, m\Delta$, (p^{m+1}, b^{m+1}) and (p^m, b^m) have the same buyer cutoffs on the equilibrium path. At $t = (m+1)\Delta$, the seller in (p^{m+1}, b^{m+1}) chooses $x^{m+1}_{(m+1)\Delta}$ that is in the support of the seller's strategy in (p^m, b^m) in that period (note that even though we haven't show that (p^m, b^m) is an equilibrium, the seller is indeed indifferent in (p^m, b^m) at $(m+1)\Delta$ because the play switch to (p^0, b^0) with identical continuation payoffs by Part (4) of the construction). It then follows from Claim 5 that the seller's (time 0) expected payoff in (p^{m+1}, b^{m+1}) is weakly higher than the seller's (time 0) expected payoff in (p^m, b^m). The claim is proved by repeating this argument. □

Claim 7. *For* $t = \Delta, \dots, (m+1)\Delta$, *the seller's expected payoff at the on-path history* $\left(x^{m+1}_0, \dots, x^{m+1}_{t-\Delta}\right)$, *in the candidate equilibrium* (p^{m+1}, b^{m+1}) *is weakly higher than the seller's expected at the history* $\left(x^0_0, \dots, x^0_{t-\Delta}\right)$ *in equilibrium* (p^0, b^0).

Proof. Denote $m_t = t/\Delta$ so that $t = m_t\Delta$ and consider (p^{m_t}, b^{m_t}). By parts (1)–(3) of the construction, the buyer's cutoff type at $\left(x^{m_t}_0, \dots, x^{m_t}_{t-\Delta}\right)$ in this equilibrium is the same as the buyer's cutoff type at $\left(x^0_0, \dots, x^0_{t-\Delta}\right)$ in equilibrium (p^0, b^0). By part (4) of the construction, the seller's payoff at history $\left(x^{m_t}_0, \dots, x^{m_t}_{t-\Delta}\right)$ in (p^{m_t}, b^{m_t}) coincides with the seller's payoff at history $\left(x^0_0, \dots, x^0_{t-\Delta}\right)$ in equilibrium (p^0, b^0). Now consider the candidate equilibrium (p^{m_t+1}, b^{m_t+1}) and the history $\left(x^{m_t+1}_0, \dots, x^{m_t+1}_{t-\Delta}\right)$. By claim 5, $\left(x^{m_t+1}_0, \dots, x^{m_t+1}_{t-\Delta}\right) \geq \left(x^{m_t}_0, \dots, x^{m_t}_{t-\Delta}\right)$. Note that the candidate equilibrium (p^{m_t+1}, b^{m_t+1}) further differs from the equilibrium (p^{m_t}, b^{m_t}) on the equilibrium path in period $t + \Delta$. But $x^{m_t+1}_t$ is in the support of the seller's randomization in (p^{m_t}, b^{m_t}) (which makes the seller indifferent by part (4) of the equilibrium construction—see the proof in Claim 6). Therefore, the seller's payoff at $\left(x^{m_t+1}_0, \dots, x^{m_t+1}_{t-\Delta}\right)$ in the equilibrium (p^{m_t+1}, b^{m_t+1}) is weakly greater than at $\left(x^{m_t}_0, \dots, x^{m_t}_{t-\Delta}\right)$ in the equilibrium (p^{m_t+1}, b^{m_t+1}). This completes the proof of the claim. □

Claim 8. *For each* $m = 0, 1, \dots, (p^{m+1}, b^{m+1})$ *such constructed is indeed an equilibrium.*

Proof. The buyer's optimality condition follows immediately from the construction. Now consider the seller. By part (5) of the construction, for any off-path history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which the seller's first deviation from the equilibrium path occurs at $s \leq (m+1)\Delta$, the continuation strategy profile prescribed by (p^{m+1}, b^{m+1}) is exactly that prescribed by (p^0, b^0) at a corresponding history $h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)$ with exactly the same expected payoff (the payoff is the same due to the fact that the seller's strategies coincide and the fact that the buyer's cutoff at h_t in (p^{m+1}, b^{m+1}) is the same as that at $h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)$ in (p^0, b^0)). Hence there is no profitable deviation at h_t in (p^{m+1}, b^{m+1}) just as there is no profitable deviation at $h_t \oplus \left(x_0^0, \ldots, x_{s-\Delta}^0\right)$ in (p^0, b^0).

By part (4) of the construction, at $t > (m+1)\Delta$, for any history $h_t = (x_0, \ldots, x_{t-\Delta})$ in which no deviation has occurred at or before $(m+1)\Delta$, the seller's strategy at h_t in (p^{m+1}, b^{m+1}) coincides with the seller's strategy at $h_t \oplus \left(x_0^m, \ldots, x_{(m+1)\Delta}^m\right)$, with exactly the same continuation payoffs (see the previous paragraph). Hence there is no profitable deviation at h_t in (p^{m+1}, b^{m+1}).

Now consider parts (1)–(3) of the construction, for $t = 0, \ldots, (m+1)\Delta$. By Claim 6 and 7, staying on the equilibrium path gives the seller a weakly higher payoff than that from the equilibrium (p^0, b^0) at the corresponding history. But deviation from the equilibrium path triggers a switch to (p^0, b^0) at a corresponding history. Since there is no deviation in (p^0, b^0), deviation becomes even less desirable in (p^{m+1}, b^{m+1}). This completes the proof of the claim. □

So far, we have obtained a sequence of equilibria $\{(p^m, b^m)\}_{m=0}^{\infty}$. Denote the limit of this sequence by (p^∞, b^∞). It is easy to check that the limit is well-defined. It remains to show that (p^∞, b^∞) is an equilibrium. It is clear that buyers do not have an incentive to deviate. For the seller, suppose the seller has a profitable deviation at some history $h_{m\Delta}$. By the definition of (p^∞, b^∞) and the construction of the sequence $\{(p^m, b^m)\}_{m=0}^{\infty}$, the continuation play at h_t in the candidate equilibrium (p^∞, b^∞), where h_t is a history with $h_{m\Delta}$ as its sub-history, will coincide with continuation play at h_t prescribed by equilibrium $\left(p^{m'}, b^{m'}\right)$ for any $m' \geq m$, which is in turn described by $p^0\left(h_t \oplus \left(x_0^0, \ldots, x_{(m-1)\Delta}^0\right)\right)$ and $b^0\left(h_{t^+} \oplus \left(x_0^0, \ldots, x_{(m-1)\Delta}^0\right)\right)$ by part (5) of the equilibrium construction. Since $\left(p^{m'}, b^{m'}\right)$ is an equilibrium, this particular deviation is not profitable in the equilibrium $\left(p^{m'}, b^{m'}\right)$ for any $m' \geq m$. But the on-path payoff of $\left(p^{m'}, b^{m'}\right)$ converges to that of (p^∞, b^∞), and we have just argued that the payoff after this particular deviation is the same for both $\left(p^{m'}, b^{m'}\right)$ and (p^∞, b^∞). This contradicts the assumption of profitable deviation. □

B.2 Proof of Lemma 3

Proof. Fix a history h_t. Note that if all buyers bid, then by the standard argument, it is optimal for each buyer to bid their true values. Therefore, it is sufficient to show that each buyer will submit a bid. By the skimming property (Lemma 1), we only need to show $\beta_t(h_t, p_t) = 0$. Suppose by contradiction that $\beta_t(h_t, p_t) > 0$. Consider a positive type $\beta_t(h_t, p_t) - \varepsilon$, where $\varepsilon > 0$. By Lemma 1, if this type follows the equilibrium strategy and waits, he wins only if his opponents all have types lower than $\beta_t(h_t, p_t) - \varepsilon$, and he can only win in period $t+\Delta$ or later at a price no smaller than 0. If he deviates and bids his true value in period t, it follows from Lemma 1 that he wins in period t at a price 0 if all of his opponents have types lower than $\beta_t(h_t, p_t)$. Therefore, the deviation is strictly profitable for type $\beta_t(h_t, p_t) - \varepsilon$, contradicting the definition of $\beta_t(h_t, p_t)$. □

B.3 Proof of Proposition 2

Proof. The lower bound follows directly from Lemma 3. For the upper bound, by Lemma 2, we can restrict attention to equilibria (p_m, b_m) in which the seller does not randomize on the equilibrium path.

We first define an ε-relaxed continuous-time auxiliary problem. We replace the payoff floor constraint by

$$\int_0^{v_t} e^{-r\,(T(x)-t)} J_t(v) dF_t^{(n)}(v) \geq (1-\varepsilon)\Pi^E(v_t).$$

By the maximum theorem, the value of this problem, which we denote by V_ε, is continuous in ε—that is,

$$\lim_{\varepsilon\to 0} V_\varepsilon = V. \tag{B.1}$$

Next, we formulate a discrete version of the auxiliary problem. For given Δ, the feasible set of this problem is given by

$$T : [0,1] \to \{0, \Delta, 2\Delta, \ldots\} \text{ non-increasing,}$$

$$\text{and} \qquad \int_0^{v_{k\Delta}} e^{-r\,(T(x)-k\Delta)} J_{k\Delta}(v) dF_{k\Delta}^{(n)}(v) \geq \Pi^E(v_{k\Delta}) \qquad \forall k \in \mathbb{N}.$$

We denote the value of this problem by $V(\Delta)$. Let $\mathcal{E}^d(\Delta) \subset \mathcal{E}(\Delta)$ denote the set of equilibria in which the seller does not randomize on the equilibrium path. The first constraint is clearly satisfied for outcomes of any equilibrium in $\mathcal{E}^d(\Delta)$. The second constraint requires that in each period, the seller's continuation profit on the equilibrium path exceeds the revenue from an efficient auction given the current posterior. Lemma 3 shows that this lower bound is a necessary condition for an equilibrium. Therefore, the seller's expected revenue in any equilibrium $(p, b) \in \mathcal{E}^d(\Delta)$ cannot exceed $V(\Delta)$. Moreover, for given ε, the feasible set of the discrete auxiliary problem is contained in the feasible set of the ε-relaxed continuous-time auxiliary problem if Δ is sufficiently small. Formally, we have:

Claim: Let $\varepsilon > 0$ and $\Delta_\varepsilon = -\ln(1-\varepsilon)/r$. For all $\Delta < \Delta_\varepsilon$ we have

$$\sup_{(p,b)\in\mathcal{E}^d(\Delta)} \Pi^\Delta(p, b) \leq V(\Delta) \leq V_\varepsilon.$$

Proof of the claim: The first inequality has been shown in the text above. For the second, let T^Δ be an element of the feasible set of the discrete auxiliary problem for $\Delta \leq \Delta_\varepsilon$. Let v_t^Δ be the corresponding cutoff path. Note that for $t \in (k\Delta, (k+1)\Delta]$ we have $v_t^\Delta = v_{(k+1)\Delta}^\Delta$ and hence

$$\int_0^{v_t^\Delta} e^{-r\,(T^\Delta(v)-t)} J_t(v) n(F(v))^{n-1} f(v) dv$$

$$
\begin{aligned}
=& e^{-r((k+1)\Delta - t)} \int_0^{v^\Delta_{(k+1)\Delta}} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v) dv \\
\geq& e^{-r\Delta} \int_0^{v^\Delta_{(k+1)\Delta}} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v) dv \\
\geq& e^{-r\Delta}\Pi^E(v^\Delta_{(k+1)\Delta}) = e^{-r\Delta}\Pi^E(v^\Delta_t) \geq (1-\varepsilon)\Pi^E(v^\Delta_t).
\end{aligned}
$$

The first inequality holds because $t \geq k\Delta$, the second inequality follows from the payoff floor constraint of the discretized auxiliary problem, and the last inequality holds because $\Delta \leq \Delta_\varepsilon$. Therefore, T^Δ is a feasible solution for the ε-relaxed continuous time auxiliary problem, and hence $V(\Delta) \leq V_\varepsilon$ if $\Delta < \Delta_\varepsilon$. Thus the claim is proved.

To complete the proof for Proposition 2, it suffices to show $\Pi^* \leq V$. We have:

$$
\Pi^* = \limsup_{\Delta\to 0} \sup_{(p,b)\in\mathcal{E}^d(\Delta)} \Pi^\Delta(p,b) \leq \lim_{\varepsilon\to 0} V_\varepsilon = V.
$$

The first equality follows from Lemma 2 which shows that the maximal revenue can be achieved without randomization on the equilibrium path by the seller. The previous claim implies that inequality. The second equality was shown above (see (B.1)). □

B.4 Proof of Proposition 3

Proof. Let $\delta(v) := e^{-rT(v)}$ denote the discount factor for type v who trades at time $T(v)$. We can rewrite the auxiliary problem as a maximization problem with $\delta(v)$ as the choice variable:

$$
\begin{aligned}
&\sup_\delta \int_0^1 \delta(v)\, J(v)\, f^{(n)}(v)\, dv \\
&\text{s.t. } \delta(v) \in [0,1], \text{ and non-decreasing,} \\
\forall v \in [0,1] : &\int_0^v \delta(s)\, J(s|s\leq v)\, f^{(n)}(s)\, ds \geq \delta(v) \int_0^v J(s|s\leq v)\, f^{(n)}(s)\, ds. \qquad \text{(B.2)}
\end{aligned}
$$

We show that (PF) is equivalent to (B.2). First suppose that (PF) holds. If v' is not part of an atom, i.e., $T^{-1}(T(v')) = \{v'\}$, then (PF) at $t' = T(v')$ is equivalent to (B.2) at v'. If v' is part of an atom, Lemma 4 (slack PF before atom), implies that if (PF) holds for all $t > T(v')$ in a neighborhood of $T(v')$, then (B.2) must hold for v'.

Conversely, suppose that (B.2) holds for all $v \in [0,1]$. If $t \in T([0,1])$ then the (B.2) for v_t implies that (PF) holds at t. Next, suppose that t is in a "quiet period," i.e., $t \notin T([0,1])$. Let t' be the end of the quiet periods, i.e., $t' = \max\{s \geq t | v_s = v_t\}$. This implies $t' \in T([0,1])$ and therefore (PF) holds at t' as we just argued. Since (PF) holds at t', Lemma 6 (slack PF in quiet period) shows that (PF) also holds at t'.

To summarize, we have shown that the constraint set of the above problem is isomorphic to the auxiliary problem (with $\delta(v) = e^{-rT(v)}$). This shows that existence of an optimal function δ in the above problem implies existence of an optimal solution to the payoff floor constraint which proves the Proposition.

Let $\bar{\pi}$ be the supremum of this maximization problem and let (δ_k) be a sequence of feasible solutions of this problem such that

$$\lim_{k\to\infty} \int_0^1 \delta_k(v)\, J(v)\, f^{(n)}(v)\, dv = \bar{\pi}.$$

By Helly's selection theorem, there exists a subsequence (δ_{k_ℓ}), and a non-decreasing function $\bar{\delta} : [0,1] \to [0,1]$ such that $\delta_{k_\ell}(v) \to \bar{\delta}(v)$ for all points of continuity of $\bar{\delta}$. Hence (after selecting a subsequence), we can take (δ_k) to be almost everywhere convergent with a.e.-limit $\bar{\delta}$. By Lebesgue's dominated convergence theorem, we also have convergence w.r.t. the L^2-norm and hence weak convergence in L^2. Therefore

$$\int_0^1 \bar{\delta}(v)\, J(v)\, f^{(n)}(v)\, dv = \lim_{k\to\infty} \int_0^1 \delta_k(v)\, J(v)\, f^{(n)}(v)\, dv = \bar{\pi}.$$

It remains to show that $\bar{\delta}$ satisfies the payoff floor constraint. Suppose not. Then there exists $\hat{v} \in [0,1)$ such that

$$\int_0^{\hat{v}} \bar{\delta}(s)\, J(s|s \le \hat{v})\, f^{(n)}(s)\, ds < \bar{\delta}(\hat{v}) \int_0^{\hat{v}} J(s|s \le \hat{v})\, f^{(n)}(s)\, ds.$$

Then there also exists $v \ge \hat{v}$ such that $\bar{\delta}$ is continuous at v, and

$$\int_0^{v} \bar{\delta}(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds < \bar{\delta}(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds.$$

Define

$$S := \bar{\delta}(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds - \int_0^{v} \bar{\delta}(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds.$$

Since v is a point of continuity we have $\bar{\delta}(v) = \lim_{k\to\infty} \delta_k(v)$. Therefore, there exists k_v such that for all $k > k_v$,

$$\left| \bar{\delta}(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds - \delta_k(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds \right| < \frac{S}{2},$$

and furthermore, since $\delta_k \to \bar{\delta}$ weakly in L^2, we can choose k_v such for all $k > k_v$ also

$$\left| \int_0^{v} \bar{\delta}(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds - \int_0^{v} \delta_k(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds \right| < \frac{S}{2}.$$

Together, this implies that for all $k > k_v$,

$$\int_0^{v} \delta_k(s)\, J(s|s \le v)\, f^{(n)}(s)\, ds < \delta_k(v) \int_0^{v} J(s|s \le v)\, f^{(n)}(s)\, ds,$$

which contradicts the assumption that δ_k is a feasible solution of the reformulated auxiliary problem defined above. □

B.5 Proof of Lemma 4

Proof. Fix $v \in (v_t^+, v_t]$. We obtain a lower bound for the LHS of (A.2) as follows:

$$\begin{aligned}
&\int_0^v e^{-r(T(x)-t)} J(x|x \le v) dF^{(n)}(x) \\
&= \int_{v_t^+}^v J(x|x \le v) dF^{(n)}(x) + \int_0^{v_t^+} e^{-r(T(x)-t)} J(x|x \le v_t^+) dF^{(n)}(x) \\
&\quad - \int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) \\
&\ge \int_{v_t^+}^v J(x|x \le v) dF^{(n)}(x) + \int_0^{v_t^+} J(x|x \le v_t^+) dF^{(n)}(x) \\
&\quad - \int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x).
\end{aligned}$$

The equality follows because all types in $(v_t^+, v]$ trade at time t, and the inequality follows from (A.1). To prove (A.2), it is sufficient to show that the RHS of (A.2) is smaller than the above lower bound. The RHS can be written as

$$\int_{v_t^+}^v J(x|x \le v) dF^{(n)}(x) + \int_0^{v_t^+} J(x|x \le v_t^+) dF^{(n)}(x) - \int_0^{v_t^+} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x).$$

Comparing this with the above lower bound, we only need to show:

$$- \int_0^{v_t^+} e^{-r(T(x)-t)} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) > - \int_0^{v_t^+} \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x),$$

or equivalently

$$\int_0^{v_t^+} (1 - e^{-r(T(x)-t)}) \left(\frac{F(v) - F(v_t^+)}{f(x)} \right) dF^{(n)}(x) > 0.$$

Since $T(x) > t$ for $x < v_t^+$ and $F(v) - F(v_t^+) > 0$ for $v > v_t^+$, the last inequality holds and the proof is complete. □

B.6 Proof of Lemma 5

Proof. Suppose by contradiction that for some t with $v_t > 0$, we have $T(v) = t$ for all $v \in [0, v_t]$. Then for all $\varepsilon > 0$ the payoff floor constraint at $t - \varepsilon$ is

$$\int_0^{v_t} e^{-r\varepsilon} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) + \int_{v_t}^{v_{t-\varepsilon}} e^{-r(T(v)-(t-\varepsilon))} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v) \ge \int_0^{v_{t-\varepsilon}} J_{t-\varepsilon}(v) dF_{t-\varepsilon}^{(n)}(v).$$

Rearranging this we get

$$\int_{v_t}^{v_{t-\varepsilon}} \left(e^{-r(T(v)-(t-\varepsilon))}-1\right) J_{t-\varepsilon}(v)dF_{t-\varepsilon}^{(n)}(v) \geq \left(1-e^{-r\varepsilon}\right)\int_0^{v_t} J_{t-\varepsilon}(v)dF_{t-\varepsilon}^{(n)}(v).$$

The RHS is strictly positive for $\varepsilon > 0$ but sufficiently small because, by the left-continuity of v_t and continuity of $J_t(v)$ in t, we have

$$\lim_{\varepsilon\to 0}\int_0^{v_t} J_{t-\varepsilon}(v)dF_{t-\varepsilon}^{(n)}(v) = \int_0^{v_t} J_t(v)dF_t^{(n)}(v) > 0.$$

On the other hand, since $J_t(v_t) = v_t > 0$, we have $J_{t-\varepsilon}(v) > 0$ for $v \in (v_t, v_{t-\varepsilon})$ with $\varepsilon > 0$ but sufficiently small. Note that

$$T(v) \geq t-\varepsilon \text{ for all } v \in (v_t, v_{t-\varepsilon})$$

Therefore, $e^{-r(T(v)-(t-\varepsilon))} \leq 1$ for all $v \in (v_t, v_{t-\varepsilon})$, and thus the LHS is non-positive. A contradiction. □

B.7 Proof of Lemma 6

Proof. For $t \in (a, b]$, the right-hand side of (PF) is independent of t since v_t is constant. The left-hand side is increasing in t, since t enters the discount factor. Feasibility of T implies that (PF) is satisfied at a^+ and therefore it must be strictly slack for $t \in (a, b]$. □

B.8 Proof of Lemma 7

Proof. Suppose by contradiction that T is feasible but $T(v) = \infty$ for some $v > 0$. Since T is non-increasing, there exists $w \in (0, 1)$ such that $T(v) = \infty$ for all $v \in [0, w)$ and $T(v) < \infty$ for all $v \in (w, 1]$. The left-hand side of the payoff floor constraint can be rewritten as, for all $t < \infty$,

$$\int_0^{v_t} e^{-r(T(x)-t)}J_t(x)dF^{(n)}(x) = \int_w^{v_t} e^{-r(T(x)-t)}J_t(x)dF^{(n)}(x).$$

Since $T(v) < \infty$ for all $v \in (w, 1]$, we have $v_t \to w$ as $t \to \infty$. Hence, as $t \to \infty$, the limit of the left-hand side is zero:

$$\lim_{t\to\infty}\int_w^{v_t} e^{-r(T(x)-t)}J_t(x)dF^{(n)}(x) = 0.$$

The limit of right-hand side of the payoff floor constraint as $t \to \infty$, however, is strictly positive:

$$\lim_{t\to\infty}\int_0^{v_t} J_t(x)dF^{(n)}(x) = \int_0^{w} J(x|x \leq w)dF^{(n)}(x) > 0.$$

Therefore, the payoff floor constraint must be violated for sufficiently large t, which contradicts the feasibility of T. □

B.9 Proof of Lemma 8

Proof. For $t > b$, $\hat{T}$ satisfies the (PF) because $\hat{T}(v) = T(v)$ for all $v < v_b$. If $\hat{v}_b = v_b$, the same argument extends to $t = b$. If $\hat{v}_b > v_b$, $\hat{T}$ satisfies (PF) for all $t > b$. Therefore, Lemma 4 (slack PF before atom) implies that $\hat{T}$ satisfies (PF) at $t = b$.

To show that $\hat{T}$ satisfies the (PF) for $t \leq a$, we define $\psi_t(v) := J_t(v)f(v)$. For any $t > 0$, (PF) can be written as

$$ne^{rt} \int_0^{v_t} Q(v)\psi_t(v)dv \geq \Pi^E(v_t).$$

For $t \leq a$, $\hat{v}_t = v_t$. Therefore the right-hand side does not change if we replace Q by $\hat{Q}$. Therefore, it suffices to show that

$$ne^{rt} \int_0^{v_t} \hat{Q}(v)\psi_t(v)dv \geq ne^{rt} \int_0^{v_t} Q(v)\psi_t(v)dv$$

for all $t \leq a$. Defining $\Psi_t(v) := \int_0^v \psi_t(x)dx$, this inequality is equivalent to

$$\begin{aligned} &\int_0^{v_t} \hat{Q}(v)\psi_t(v)dv \geq \int_0^{v_t} Q(v)\psi_t(v)dv \\ \iff \ &\hat{Q}(b)\Psi_t(b) - \hat{Q}(a)\Psi_t(a) - \int_0^{v_t} \Psi_t(v)d\hat{Q}(v) \geq Q(b)\Psi_t(b) - Q(a)\Psi_t(a) - \int_0^{v_t} Q(v)\Psi_t(v)d\hat{Q}(v) \\ \iff \ &\int_0^{v_t} \Psi_t(v)d\hat{Q}(v) \leq \int_0^{v_t} \Psi_t(v)dQ(v). \end{aligned}$$

To obtain the first and third lines we have used that $\hat{Q}(v) = Q(v)$ for $v \notin (a, b)$.

To establish the last line note first that both Q and $\hat{Q}$ are increasing, and hence up to an affine transformation, they are distribution functions on $[v_b, v_a]$. It follows from (A.3) that Q is a mean-preserving spread of $\hat{Q}$. Second, note that $\psi_t(v) = \psi_0(v) + (1 - F(v_t))$. Since $\psi_0(v) = J(v)f(v)$ is strictly increasing by assumption, ϕ_s is strictly increasing and Ψ_t is strictly convex. Convexity of Ψ together with the mean-preserving spread implies that the last line holds.

If (A.3) is a strict inequality for a set with strictly positive measure, then all inequalities are strict which implies that (PF) becomes a strict inequality for $t \leq a$, and the ex-ante revenue is strictly increased by replacing T with $\hat{T}$. □

B.10 Proof of Proposition 5

Proof of Proposition 5. Let T be an optimal solution to the auxiliary problem with associated cutoffs v_t, and suppose by contradiction that there exists $s > 0$ such that $v_s \in (0, \overline{v})$ and the payoff floor constraint is slack at s. Define

$$\begin{aligned} s' &:= \inf \{\sigma \in (T(\overline{v}), s] \,|\, \text{(PF) is a strict inequality for all } t \in [\sigma, s]\} \\ s'' &:= \sup \{\sigma \geq s \,|\, \text{(PF) is a strict inequality for all } t \in [s, \sigma]\}. \end{aligned}$$

Since v_t is left-continuous everywhere, $s' < s$ and hence $s' < s''$. In the following, we consider two cases:

Case 1: $v_{s'}^{+} > v_{s''}$

In this case, there exists an interval $(a, b) \subset [s', s'']$ such that $v_a > v_b$, and for a positive measure of types $v \in (v_b, v_a)$, $T(v) \in (a, b)$. In other words, (a, b) is not a "quiet period."

We construct an alternative solution $\hat{T}$ that satisfies the conditions of the MPS-Lemma 8 as follows:

$$\hat{T}(v) := \begin{cases} T(v), & \text{if } v \notin (v_b, v_a), \\ a, & \text{if } v \in (w, v_a), \\ b, & \text{if } v \in (v_b, w]. \end{cases}$$

We choose w such that

$$\begin{aligned} &\int_{v_b}^{v_a} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right) (F(v))^{n-1} dv \\ &= \int_{w}^{v_a} \left(e^{-ra} - e^{-rT(v)}\right) (F(v))^{n-1} dv + \int_{v_b}^{w} \left(e^{-rb} - e^{-rT(v)}\right) (F(v))^{n-1} dv = 0. \end{aligned} \tag{B.3}$$

The existence of such w follows from the intermediate value theorem: The second line is continuous in w. For $w = v_a$ the first integral in the second line vanishes and the second is negative. Conversely, for $w = v_b$ the second integral in the second line vanishes and the first is positive. Hence there exists $w \in (v_b, v_a)$ for which the second line is equal to zero.

Next, note that

$$\int_{v_b}^{x} \left(e^{-r\hat{T}(v)} - e^{-rT(v)}\right) (F(v))^{n-1} dv$$

is decreasing in x for $x < w$ and increasing for $x > w$. This together with (B.3) implies that $\hat{T}$ satisfies the conditions of Lemma 8. There is a positive measure of types $v \in (v_b, v_a)$ for which $T(v) \neq \hat{T}(v)$. $\hat{T}$ therefore satisfies the payoff floor constraint for $t \notin (a, b)$, and yields strictly higher ex-ante profit than T.

For the contradiction, it remains to show that $\hat{T}$ satisfies the payoff floor constraint for $t \in (a, b)$. Since $(a, b) \in [s', s'']$, the payoff floor constraint with T is a strict inequality for all $t \in (a, b)$. By choosing the interval (a, b) sufficiently small, we can ensure that replacing T by $\hat{T}$ does not violate the payoff floor constraint on (a, b). This concludes the proof for Case 1.

Case 2: *$v_t = v_{s'}$ for all $t \in (s', s'']$.*

In this case, the interval where the payoff floor constraint is slack is a "quiet period" without trade. This implies that v_t is discontinuous at s''. Otherwise the payoff floor constraint would be continuous in t at s'' which would require that it is binding at s''. However, if the payoff floor is binding at the endpoint of the "quiet period," it must be violated for $t \in (s', s'')$.[47] Therefore v_t must be discontinuous at s''—i.e., $v_{s''} > v_{s''}^{+}$.

Similar to Case 1, we construct an alternative solution $\hat{T}$ that satisfies the conditions of the MPS-Lemma 8. The alternative solution is parametrized by two trading times $a < s'' < b$

[47] For $t \in (s', s'')$, the right-hand side of (PF) is independent of t whereas the left-hand side is increasing in t.

and a cutoff valuation w which we set to $w = (v_{s''} + v_{s''}^+)/2$.

$$\hat{T}(v) := \begin{cases} T(v), & \text{if } v \notin (v_b, v_{s''}), \\ a, & \text{if } v \in (w, v_{s''}), \\ b, & \text{if } v \in (v_b, w). \end{cases}$$

In words, we "split the atom" at w. For the higher types in the atom we set an earlier trading time a and for the low types we delay the trading time to b. To preserve monotonicity we also delay the trading times of all $v \in (v_b, v_{s''}^+)$ to b.

If we fix $b > s''$ we need to select a such that we preserve the mean preserving spread of Q:

$$\int_w^{v_a} \left(e^{-ra} - e^{-rT(v)}\right)(F(v))^{n-1} dv + \int_{v_b}^{w} \left(e^{-rb} - e^{-rT(v)}\right)(F(v))^{n-1} dv = 0. \tag{B.4}$$

The second integral is negative and decreasing in b and the first is positive and decreasing in a. Therefore, for b sufficiently close to s'' there is a unique $a \in (s', s'')$ so that the equation is satisfied. $a \in (s', s'')$ implies that $v_a = v_{s''}$ so that $\hat{T}$ is monotone. We have constructed $\hat{T}$ such that (A.3) holds with equality for $x = v_a$ and by a similar argument as in case 1 it is satisfied for all $x \in [v_a, v_b]$. Therefore, by the MPS-Lemma 8, $\hat{T}$ yields higher ex-ante revenue than T and satisfies (PF) for all $t \notin (a, b)$. It remains to show that we can choose b such that (PF) is satisfied for all $t \in (a, b)$.

T satisfies (PF) for all t, and v_t is discontinuous at s''. Therefore, Lemma 5 (no final atom) implies that $v_{s''}^+ > 0$ and we can apply Lemma 4 (slack PF before atom). This yields

$$\int_0^w e^{-r(T(v)-s'')} J(v|v \leq w) dF^{(n)}(v) > \int_0^w J(v|v \leq w) dF^{(n)}.$$

If we choose b sufficiently close to s'' this inequality also holds for $\hat{T}$. Moreover, a is decreasing in b and $a \to s''$ for $b \to s''$, therefore we have

$$\int_0^w e^{-r(T(v)-a)} J(v|v \leq w) dF^{(n)} > \int_0^w J(v|v \leq w) dF^{(n)}.$$

This shows that $\hat{T}$ satisfies (PF) for $t = a^+$. Since the cutoff $\hat{v}_t$ defined by $\hat{T}$ is constant on (a, b), this implies that the payoff floor constraint is satisfied for all $t \in (a, b)$ (see footnote 47). This completes the proof for Case 2. □

B.11 Proof of Lemma 9

Proof. We first show that v_t is continuously differentiable for all $t \in (a, b)$ where $v_t > 0$. To show establish several claims.

Claim 1. *$v_a^+ > v_b$ and T is continuous on $v \in (v_b, v_a^+)$.*

Proof. $v_a^+ = v_b$ would imply a that (a, b) is a quiet period. By Lemma 6 (slack PF in quiet period) this would required that (PF) is a strict inequality for $t \in (a, b)$. Similarly, if T has a discontinuity at $v \in (v_b, v_a^+)$, then there is a quite period (s, s') which contradicts that (PF) is binding for all $t \in (a, b)$ by Lemma 6. □

Claim 2. *T is strictly decreasing for $v \in (v_b, v_a^+)$.*

Proof. Suppose by contradiction, that there exists a trading time $s \in (a, b)$ such that $T^{-1}(s) = (v_s^+, v_s]$ where $v_s^+ < v_s$. Since (PF) is satisfied for $t \in (s, b)$, Lemma 5 (no final atom) implies that $v_s^+ > 0$. By Lemma 4 (slack PF before atom), this implies that (PF) is a strict inequality at s which is a contradiction. □

Claim 3. *T is continuously differentiable with $T'(v) < 0$ for all $v \in (v_b, v_a^+)$*

Proof. Since T is continuous and strictly decreasing for $v \in (v_b, v_a^+)$, a binding payoff floor constraint for all $t \in (a, b)$ is equivalent to the condition that, for all $v \in (v_b, v_a^+)$,

$$\int_0^v e^{-rT(x)}\, J(x|x \leq v)\, dF^{(n)}(x) = e^{-rT(v)} \int_0^v J(x|x \leq v) dF^{(n)}(x),$$

which can be rearranged into

$$e^{-rT(v)} = \frac{\int_0^v e^{-rT(x)}\, J(x|x \leq v) dF^{(n)}(x)}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)}.$$

Continuity of T and continuous differentiability of F imply that the right-hand side of this expression is continuously differentiable, and thus T is also continuously differentiable.

Differentiating with respect to v and solving for $T'(v)$ yields

$$\begin{aligned}
T'(v) = &\frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right] \int_0^v e^{-r(T(x)-T(v))}\, J(x|x \leq v)\, dF^{(n)}(x)}{\left(\int_0^v J(x|x \leq v)\, dF^{(n)}(x)\right)^2} \\
&- \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v e^{-r(T(x)-T(v))} \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right]}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)} \\
= &\frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right] \int_0^v J(x|x \leq v)\, dF^{(n)}(x)}{\left(\int_0^v J(x|x \leq v)\, dF^{(n)}(x)\right)^2} \\
&- \frac{1}{r} \frac{\left[f^{(n)}(v)v - \int_0^v e^{-r(T(x)-T(v))} \frac{f(v)}{f(x)}\, dF^{(n)}(x)\right]}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)} \\
= &\frac{f(v)}{r} \frac{\int_0^v \left(e^{-r(T(x)-T(v))} - 1\right) \frac{1}{f(x)}\, dF^{(n)}(x)}{\int_0^v J(x|x \leq v)\, dF^{(n)}(x)}.
\end{aligned}$$

where the second equality follows from the binding payoff floor constraint. In the last line, the numerator is strictly negative and the denominator is positive. Therefore $T'(v) < 0$. □

Together Claims 1–3 imply that v_t is continuously differentiable for $t \in (a, b)$ where $v_t > 0$.

Next we derive a differential equation for v_t from the binding payoff floor constraint. In the process we also show that v_t is twice continuously differentiable.

Since v_t is continuously differentiable on (a, b), we can differentiate (PF) on both sides to obtain

$$\begin{aligned}
& e^{-rt} v_t f^{(n)}(v_t)\dot{v}_t - \int_0^{v_t} e^{-rT(x)} \frac{f(v_t)\dot{v}_t}{f(x)} dF^{(n)}(x) \\
= & -re^{-rt} \int_0^{v_t} J_t(x) dF^{(n)}(x) + e^{-rt} v_t f^{(n)}(v_t)\dot{v}_t - e^{-rt} \int_0^{v_t} \frac{f(v_t)\dot{v}_t}{f(x)} dF^{(n)}(x),
\end{aligned}$$

where we have used $\frac{\partial J_t(x)}{\partial t} = -\frac{f(v_t)\dot{v}_t}{f(x)}$, and $T(v_t) = t$ which follows from continuity of $T(v)$. This equation can be further simplified

$$-\int_0^{v_t} e^{-rT(x)} \frac{f(v_t)\dot{v}_t}{f(x)} dF^{(n)}(x) = -re^{-rt} \int_0^{v_t} J_t(x) dF^{(n)}(x) - e^{-rt} \int_0^{v_t} \frac{f(v_t)\dot{v}_t}{f(x)} dF^{(n)}(x).$$

Since T is continuous and has a bounded derivative, $\dot{v}_t < 0$. By assumption, $f(v_t) > 0$, so we can divide the previous equation by $-f(v_t)\dot{v}_t$ to obtain

$$\int_0^{v_t} e^{-rT(x)} \frac{1}{f(x)} dF^{(n)}(x) = \frac{re^{-rt}}{f(v_t)\dot{v}_t} \int_0^{v_t} J_t(x) dF^{(n)}(x) + e^{-rt} \int_0^{v_t} \frac{1}{f(x)} dF^{(n)}(x). \tag{B.5}$$

This equation, together with our assumption that $f(v)$ is continuously differentiable, implies that v_t is twice continuously differentiable. Differentiating (B.5) on both sides yields

$$\begin{aligned}
& e^{-rt} n F^{n-1}(v_t)\dot{v}_t \\
= & re^{-rt} \left(\frac{v_t f^{(n)}(v_t)\dot{v}_t - f(v_t) \int_0^{v_t} \frac{f^{(n)}(x)}{f(x)} dx \, \dot{v}_t}{f(v_t)\dot{v}_t} - \frac{\left(\dot{v}_t \frac{f'(v_t)}{f(v_t)} + \frac{\ddot{v}_t}{\dot{v}_t} + r\right) \int_0^{v_t} J_t(x) f^{(n)}(x) dx}{f(v_t)\dot{v}_t} \right) \\
& - re^{-rt} \int_0^{v_t} \frac{1}{f(x)} dF^{(n)}(x) + e^{-rt} n F^{n-1}(v_t)\dot{v}_t.
\end{aligned}$$

Multiplying both side by $f(v_t)\dot{v}_t$, and rearranging we get

$$\frac{\ddot{v}_t}{\dot{v}_t} + \underbrace{\left(\frac{f'(v_t)}{f(v_t)} - \frac{f^{(n)}(v_t) v_t - 2 f(v_t) n \int_0^{v_t} F^{n-1}(x) dx}{\int_0^{v_t} J_t(x) f^{(n)}(x) dx} \right)}_{=: g(v_t)} \dot{v}_t + r = 0.$$

Some further algebra yields

$$\int_0^{v_t} J_t(x) f^{(n)}(x) dx = (n-1)n \int_0^{v_t} (F(v_t) - F(x)) F^{n-2}(x) f(x) x dx,$$

which implies

$$g(v_t) = \frac{f'(v_t)}{f(v_t)} - \frac{\left\{v_t F^{n-1}(v_t) - 2\int_0^{v_t} F^{n-1}(v)\,dv\right\} f(v_t)}{(n-1)\int_0^{v_t}[F(v_t) - F(v)]\,F^{n-2}(v)\,f(v)\,vdv}.$$

□

B.12 Proof of Lemma 10

Proof. Since $f(v)$ is continuously differentiable $\lim_{v\to 0} vf'(v)$ exists. We first show that $\lim_{v\to 0} vf'(v) = 0$. Suppose by contradiction that $\lim_{v\to 0} vf'(v) = z \neq 0$. If $z > 0$, we must have $f'(v) \geq z/(2v)$ for a neighborhood $(0,\varepsilon)$, which implies that $f(\varepsilon) = f(0) + \int_0^{\varepsilon} f'(v)dv \geq f(0) + \int_0^{\varepsilon}(z/(2v))dv = \infty$ which contradicts the assumption of a finite density. If $z < 0$, we have $f'(v) \leq z/(2v)$ for a neighborhood $(0,\varepsilon)$, which implies that $f(\varepsilon) = f(0) + \int_0^{\varepsilon} f'(v)dv \leq f(0) + \int_0^{\varepsilon}(z/(2v))dv = -\infty$ which contradicts $f(v) > 0$. Since $f(0) > 0$ and $\lim_{v\to 0} vf'(v) = 0$ together imply $\phi = \lim_{v\to 0}\frac{vf'(v)}{f(v)} = 0$, we have $\overline{N}(F) := 1 + \frac{\sqrt{2+\phi}}{1+\phi} = 1 + \sqrt{2} \in (2,3)$.

If $f(0) = 0$, we use a Taylor expansion of $f(v)$ at zero to obtain

$$\phi = \lim_{v\to 0}\frac{f'(v)v}{f(v)} = \lim_{v\to 0}\frac{f'(v)v}{f'(0)v} = 1.$$

This implies $\overline{N}(F) = 1 + \sqrt{3}/2 < 2$. □

B.13 Proof of Lemma 11

Proof. If Assumption 2 is satisfied, we can repeatedly use l'Hospital's rule, and

$$\lim_{v\to 0}\frac{vf(v)}{F(v)} = \lim_{v\to 0}\frac{f'(v)v + f(v)}{f(v)} = 1 + \phi \quad \text{and} \quad \lim_{v\to 0}\frac{F(v)}{vf(v)} = \frac{1}{1+\phi},$$

to get

$$\kappa := \lim_{v\to 0} g(v)v = \phi - \frac{((n-1)\phi + n - 2)(n\phi + n + 1)}{(n-1)(1+\phi)}.$$

Simple algebra shows that if $\phi > -1$,

$$\kappa > -1 \quad \iff \quad n < \overline{N}(F).$$

Next, we transform the ODE (5.6) using the change of variables $y = \dot{v}_t$. This yields

$$y'(v) + g(v)y(v) + r = 0.$$

The general solution is given by

$$y(v) = e^{-\int_m^v g(x)dx} \left(C - \int_m^v r e^{\int_m^w g(x)dx} dw \right), \tag{B.6}$$

where $m > 0$.[48] Feasibility requires that $y(v) \leq 0$ for all $v \in (0, v_0^+)$. This implies that

$$\forall v \in (0, v_0^+): \quad C \leq \int_m^v r e^{\int_m^w g(x)dx} dw,$$

Since the right-hand side is increasing in v this implies

$$C \leq \overline{C} := -\int_0^m r e^{\int_m^w g(x)dx} dw$$

and

$$\overline{C} = \lim_{v \to 0} \int_m^v r e^{\int_m^w g(x)dx} dw > -\infty. \tag{B.7}$$

(i) Suppose $\kappa < -1$. Since $\kappa = \lim_{v \to 0} g(v)v$, there must exist $\gamma > 0$ such that $g(v) \leq -\frac{1}{v}$ for all $v \in (0, \gamma]$. We may assume that $0 < m < \gamma$. In this case, the limit in (B.7) can be computed as follows:

$$\begin{aligned} \lim_{v \to 0} \int_m^v r e^{\int_m^w g(x)dx} dw &= \lim_{v \to 0} - \int_v^m r e^{-\int_w^m g(x)dx} dw \\ &\leq \lim_{v \to 0} - \int_v^m r e^{\int_w^m \frac{1}{x} dx} dw = \lim_{v \to 0} - \int_v^m r \frac{m}{w} dw = -\infty. \end{aligned}$$

Given that $C \leq \overline{C}$ there exits no finite C such that the general solution in (B.6) satisfies $y(v) \leq 0$ for all $v \in (0, v_0^+)$. This shows part (i).

To prove part (ii), we set $C = \overline{C}$. We show that the resulting solution

$$y(v) = -e^{-\int_m^v g(x)dx} \int_0^v r e^{\int_m^w g(x)dx} dw = -\int_0^v r e^{-\int_w^v g(x)dx} dw, \tag{B.8}$$

is negative and finite for all v. It is clear that $y(v) < 0$, so it suffices to rule out $y(v) = -\infty$. Since $\kappa = \lim_{v \to 0} g(v)v > -1$, there exist $\hat{\kappa} > -1$ and $\gamma > 0$ such that $g(v) \geq \frac{\hat{\kappa}}{v}$ for all $v \in (0, \gamma]$. Hence the limit in (B.7) can be computed as (where we may again assume that $0 < m < \gamma$):

$$\begin{aligned} \lim_{v_t \to 0} \int_m^{v_t} r e^{\int_m^v g(x)dx} dv &= \lim_{v_t \to 0} - \int_{v_t}^m r e^{-\int_v^m g(x)dx} dv \\ &\geq \lim_{v_t \to 0} - \int_{v_t}^m r e^{-\hat{\kappa} \ln \frac{m}{v}} dv = \lim_{v_t \to 0} - \int_{v_t}^m r \left(\frac{v}{m} \right)^{\hat{\kappa}} dv \\ &= -r m^{-\hat{\kappa}} \frac{1}{\hat{\kappa}+1} \lim_{v_t \to 0} \left(m^{\hat{\kappa}+1} - v_t^{\hat{\kappa}+1} \right) > -\infty. \end{aligned}$$

[48] For $m = 0$, the solution candidate is not well defined for all κ because $e^{-\int_m^v g(x)dx} = \infty$.

Therefore, $y(v)$ is finite and $y(v) < 0$ for all v. Next we have to show that (B.8) can be integrated to obtain a feasible solution of the auxiliary problem. It suffices to verify that the following boundary condition from Lemma 7 (cutoffs converge to zero):

$$\lim_{t\to\infty} v_t = 0, \tag{B.9}$$

is satisfied. Recall that $\dot{v}_t = y(v_t)$. Therefore, we have

$$\dot{v}_t = -e^{-\int_m^{v_t} g(v)dv} \left(\int_0^{v_t} re^{\int_m^v g(x)dx} dv \right).$$

We first show that, for any $v_0^+ \in [0,1]$, the solution to this differential equation satisfies (B.9). Since the term in the parentheses is strictly positive we have

$$\frac{e^{\int_m^{v_t} g(v)dv}\, \dot{v}_t}{\int_0^{v_t} e^{\int_m^v g(x)dx} dv} = -r.$$

Integrating both sides with respect to t, we get

$$\ln \int_0^{v_t} e^{\int_m^v g(x)dx} dv - \ln \int_0^{v_0^+} e^{\int_m^v g(x)dx} dv = -rt.$$

Now take $t \to \infty$. The RHS diverges to $-\infty$ and the second term on the LHS is constant, so we must have

$$\lim_{t\to\infty} \ln \int_0^{v_t} e^{\int_m^v g(x)dx} dv = -\infty$$

which holds if and only if $\lim_{t\to\infty} v_t = 0$. Therefore, we have found a solution that satisfies the boundary condition and is decreasing for all starting values v_0^+. This completes the proof of part (ii).

(iii) Let $\hat{v}_t$ be a decreasing solution to the binding payoff floor constraint that does not satisfy (4.1). Then $z(v) = \dot{\hat{v}}_t$ must be given by (B.6) for some $C \leq \overline{C}$. The solution v_t^x satisfies (4.1). If we define $y^x(v) = \dot{v}_t^x$, $y^x(v)$ satisfies (B.8). Therefore we have

$$z(v) = y(v) - \left(\overline{C} - C\right) e^{-\int_m^v g(x)dx} < y(v).$$

This implies that $\hat{v}_t = v_t^x$ implies $\dot{\hat{v}}_t < \dot{v}_t^x$. We have established a single crossing property: For any $x \in [0,1]$, $\hat{v}_t$ crosses v_t^x at most once, and from above.

Now we pick x so that we can apply the MPS-Lemma 8. Let $\hat{Q}(v)$ be the expected discounted winning probability times associated with the cutoff path $\hat{v}_t$ and $Q^x(v)$ the one associated with v_t^x. Define

$$D(x) = \int_0^1 Q^x(v) - \hat{Q}(v)(v)dv.$$

Clearly $D(x)$ is continuous in x. $x = 0$ implies that $Q^x(v) = (F(v))^{(n-1)} > (F(v))^{(n-1)} e^{-r\widehat{T}(v)} =$

$\hat{Q}(v)$ for all $v < \hat{v}_t^+$ Therefore $D(0) > 0$. If we set $x = \hat{v}_t^+$, then v_t^x and $\hat{v}_t$ intersect at $t = 0$ and the crossing property implies that $v_t^x > \hat{v}_t$ for all $t > 0$. This implies $Q^x(v) < \hat{Q}(v)$ for all $v < \hat{v}_t^+$ and thus $D(v_0^+) < 0$. Hence, the intermediate value theorem implies that there exists $x^* \in (0, v_0^x)$ such that $D(x^*) = 0$. Moreover, $\hat{v}_t$ crosses $v_t^{x^*}$ exactly once and from above. This implies that $\hat{Q}(v)$ crosses $Q^x(v)$ once and from below. Therefore we must have

$$\int_0^z Q^{x^*}(v) - \hat{Q}(v)dv \leq 0, \forall z \in [0,1].$$

Hence Lemma 8 implies that v_t^x yields strictly higher profit than $\hat{v}_t$. □

B.14 Proof of Lemma 12

Proof. The proof follows the same steps as in the proof of Lemma 11 but when taking the limit $\kappa = \lim_{v \to 0} g(v)\, v$, we have to take into account that $F(0) > 0$. Applying l'Hospital's rule, we can compute κ as

$$\kappa = \phi - \lim_{v \to 0} \frac{(n-1)v^2 f^2(v) F^{n-2}(v) + v^2 f'(v) F^{n-1}(v) - 2\left(f'(v)v + f(v)\right) \int_0^v F^{n-1}(s)\, ds}{(n-1) f(v) \int_0^v s\, F^{n-2}(s) f(s)\, ds}$$

Noting that $F(0) > 0$, we can again apply l'Hospital's rule to obtain

$$\lim_{v \to 0} \frac{(n-1)v^2 f^2(v) F^{n-2}(v)}{(n-1) f(v) \int_0^v s\, F^{n-2}(s) f(s)\, ds} = 2$$

and

$$\lim_{v \to 0} \frac{v^2 f'(v) F^{n-1}(v) - 2\left(f'(v)v + f(v)\right) \int_0^v F^{n-1}(s)\, ds}{(n-1) f(v) \int_0^v s\, F^{n-2}(s) f(s)\, ds} = -\infty$$

It follows that

$$\kappa = \lim_{v \to 0} v g(v) = +\infty.$$

The rest of the proof of part (i) follows from the proof of Part (ii) of Lemma 11. Part (ii) is proven by the same steps as in the proof of Part (iii) of Lemma 11. □

C Equilibrium Approximation of the Solution to the Binding Payoff Floor Constraint

C.1 Equilibrium Approximation (Proof of Proposition 6)

In this section we construct equilibria that approximate the solution to the binding payoff floor constraint. We proceed in three steps. First, we show that if the binding payoff floor constraint has a decreasing solution, then there exists a nearby solution for which the payoff floor constraint is strictly slack. In particular, we show that for each $K > 1$ sufficiently small, there exists a solution with a decreasing cutoff path to the following *generalized* payoff floor constraint:

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF_t^{(n)}(x) = K\,\Pi^E(v_t). \tag{C.1}$$

For $K = 1$, (C.1) reduces to the original payoff floor constraint in (PF) (divided by $F_t(v_t)$). Therefore, a decreasing solution that satisfies (C.1) for $K > 1$ is a feasible solution to the auxiliary problem. Moreover, the slack in the original payoff floor constraint is proportional to $\Pi^E(v_t)$.

Lemma 13. *Suppose Assumption 2 holds and $n < \overline{N}(F)$. Then there exists $\Gamma > 1$ such that for all $K \in [1, \Gamma]$, there exists a feasible solution T^K to the auxiliary problem that satisfies (C.1). For $K \searrow 1$, $T^K(v)$ converges to $T(v)$ for all $v \in [0, 1]$, and the seller's expected revenue converges to the value of the auxiliary problem.*

In the second step, we discretize the solution obtained in the first step so that all trades take place at times $t = 0, \Delta, 2\Delta, \ldots$. For given K and Δ, we define the discrete approximation $T^{K,\Delta}$ of T^K by delaying all trades in the time interval $(k\Delta, (k+1)\Delta]$ to $(k+1)\Delta$:

$$T^{K,\Delta}(v) := \Delta \min\left\{k \in \mathbb{N} \,\middle|\, k\Delta \geq T^K(v)\right\}. \tag{C.2}$$

In other words, we round up all trading times to the next integer multiple of Δ. Clearly, for all $v \in [0, 1]$ we have,

$$\lim_{K\to 1}\lim_{\Delta\to 0} T^{K,\Delta}(v) = \lim_{\Delta\to 0}\lim_{K\to 1} T^{K,\Delta}(v) = T(v),$$

and the seller's expected revenue also converges. Therefore, if we show that the functions T^{K_m,Δ_m} for some sequence (K_m, Δ_m) describe equilibrium outcomes for a sequence of equilibria $(p^m, b^m) \in \mathcal{E}(\Delta_m)$, we have obtained the desired approximation result.

The discretization changes the continuation revenue, but we can show that the approximation loss vanishes as Δ becomes small. In particular, if Δ is sufficiently small, then the approximation loss is less than half of the slack in the payoff floor constraint at the solution T^K. More precisely, we have the following lemma.

Lemma 14. *Suppose Assumption 3 is satisfied and let $n < \overline{N}(F)$. For each $K \in [1, \Gamma]$, where Γ satisfies the condition of Lemma 13, there exists $\bar{\Delta}_K^1 > 0$ such that for all $\Delta < \bar{\Delta}_K^1$,* *and all* $t = 0, \Delta, 2\Delta, \ldots$,

$$\int_0^{v_t^{K,\Delta}} e^{-r\left(T^{K,\Delta}(x)-t\right)} J_t(x) dF_t^{(n)}(x) \geq \frac{K+1}{2} \Pi^E(v_t^{K,\Delta}).$$

This lemma shows that if Δ is sufficiently small, at each point in time $t = 0, \Delta, 2\Delta, \ldots$, the continuation payoff of the discretized solution is at least as high as $1 + (K-1)/2$ times the profit of the efficient auction.

In the final step, we show that the discretized solution $T^{K,\Delta}$ can be implemented in an equilibrium of the discrete time game. To do this, we use weak-Markov equilibria as a threat to deter any deviation from the equilibrium path by the seller. The threat is effective because the uniform Coase conjecture (Proposition 1.(ii)) implies that the profit of a weak-Markov equilibrium is close to the profit of an efficient auction for any posterior along the equilibrium path. More precisely, let $\Pi^\Delta(p, b|v)$ be the continuation profit at posterior v for a given equilibrium $(p, b) \in \mathcal{E}(\Delta)$ as before.[49] Then Proposition 1.(ii) implies that for all $K \in [1, \Gamma]$, where Γ satisfies the condition of Lemma 13, there exists $\bar{\Delta}_K^2 > 0$ such that, for all $\Delta < \bar{\Delta}_K^2$, there exists an equilibrium $(p, b) \in \mathcal{E}(\Delta)$ such that, for all $v \in [0, 1]$,

$$\Pi^\Delta(p, b|v) \leq \frac{K+1}{2} \Pi^E(v). \tag{C.3}$$

Now suppose we have a sequence $K_m \searrow 1$, where $K_m \in [1, \Gamma]$ as in Lemma 13. Define $\bar{\Delta}_K := \min\left\{\bar{\Delta}_K^1, \bar{\Delta}_K^2\right\}$. We can construct a decreasing sequence $\Delta_m \searrow 0$ such that for all m, $\Delta_m < \bar{\Delta}_{K_m}$. By Lemma 14 and (C.3), there exists a sequence of (punishment) equilibria $(\hat{p}^m, \hat{b}^m) \in \mathcal{E}(\Delta_m)$ such that for all m and all $t = 0, \Delta_m, 2\Delta_m, \ldots$

$$\int_0^{v_t^{K_m,\Delta_m}} e^{-r\left(T^{K_m,\Delta_m}(x)-t\right)} J_t(x) dF^{(n)}(x) \geq \frac{K_m+1}{2} \Pi^E(v_t^{K_m,\Delta_m}) \geq \Pi(\hat{p}^m, \hat{b}^m | v_t^{K_m,\Delta_m}). \tag{C.4}$$

The left term is the continuation profit at time t on the candidate equilibrium path given by T^{K_m,Δ_m}. This is greater or equal than the second expression by Lemma 14. The term on the right is the continuation profit at time t if we switch to the punishment equilibrium. This continuation profit is smaller than the middle term by Proposition 1.(ii). Therefore, for each m, $(\hat{p}^m, \hat{b}^m)$ can be used to support T^{K_m,Δ_m} as an equilibrium outcome of the game indexed by Δ_m. Denote the equilibrium that supports T^{K_m,Δ_m} by $(p^m, b^m) \in \mathcal{E}(\Delta_m)$. It is defined as follows: On the equilibrium path, the seller posts reserve prices given by T^{K_m,Δ_m} and (5.2). A buyer with type v bids at time $T^{K_m,\Delta_m}(v)$ as long as the seller does not deviate. As argued in Section 5.2.3, this is a best response to the seller's on-path behavior because the prices given by (5.2) implement the reading time function T^{K_m,Δ_m}. After a deviation by the seller, she is punished by switching to the equilibrium $(\hat{p}^m, \hat{b}^m)$. Since the seller anticipates the switch to $(\hat{p}^m, \hat{b}^m)$ after a deviation, her deviation profit is bounded above by $\Pi(\hat{p}^m, \hat{b}^m | v_t^{K_m,\Delta_m})$. Therefore, (C.4) implies that the seller does not have a profitable deviation. To summarize, we have an approximation of the solution to the binding payoff floor constraint by discrete

[49] If the profit differs for different histories that lead to the same posterior, we could take the supremum, but this complication does not arise with weak-Markov equilibria.

time equilibrium outcomes. This concludes the proof of Proposition 6.

C.2 Proof of Lemma 13

The key step of the approximation is to discretize the solution to the binding payoff floor constraint. In order to do that, we first need to find a feasible solution such that the payoff floor constraint is strictly slack. We have the following generalization of Lemma 9.

Lemma 15. *Suppose $T(x)$ satisfies (C.1) for all $t \in (a,b)$ and suppose T is continuously differentiable with $-\infty < T'(v) < 0$ for all $v \in (v_b, v_a)$ and v_t is continuously differentiable for all $t \in (a,b)$. Then v_t is twice continuously differentiable on (a,b) and is characterized by*

$$\frac{\ddot{v}_t}{\dot{v}_t} + g(v_t, K)\dot{v}_t + h(v_t, K)\left(\dot{v}_t\right)^2 + r = 0,$$

where

$$g(v_t, K) = \frac{f'(v_t)}{f(v_t)} - \frac{\left\{\left(2 - \frac{1}{K}\right) v_t F^{n-1}(v_t) - 2\int_0^{v_t} F^{n-1}(v)\, dv\right\} f(v_t)}{(n-1)\int_0^{v_t}\left[F(v_t) - F(v)\right] F^{n-2}(v) f(v) v dv},$$

and

$$h(v_t, K) = \frac{K-1}{rK} \frac{F^{n-2}(v_t) f^2(v_t) v_t}{\int_0^{v_t}\left[F(v_t) - F(v)\right] F^{n-2}(v) f(v) v dv}.$$

Proof. The proof follows similar steps as the proof of Lemma 9. □

Repeatedly applying l'Hospital's rule yields

Lemma 16. *If Assumption 2 is satisfied, we have*

$$\kappa := \lim_{v\to 0} g(v)v = \phi - \frac{((n-1)\phi + n - 2)(n\phi + n + 1)}{(n-1)(1+\phi)}, \tag{C.5}$$

$$\lim_{v\to 0} g(v,K)v = \kappa - \frac{K-1}{K}\left(n\phi + n + 2 + \frac{\phi+2}{(n-1)(1+\phi)}\right), \tag{C.6}$$

and

$$\lim_{v\to 0} h(v,K)v^2 = \frac{1}{r}\frac{K-1}{K}(n + \phi n + 1)(n + \phi n - \phi). \tag{C.7}$$

We use the change of variables $y = \dot{v}_t$ to rewrite the ODE obtained in Lemma 15 as

$$y'(v) = -r - g(v,K)y(v) - h(v,K)\left(y(v)\right)^2. \tag{C.8}$$

Any solution to the above ODE with $K > 1$ would lead to a strictly slack payoff floor constraint. Our goal is to show that the solution to the ODE exists for any K sufficiently close to zero and converges to the solution given by (4.1) as $K \searrow 1$. We will verify below that (4.1) satisfies the boundary condition $\lim_{v\to 0} y(v) = 0$. Given this observation, we want to show the existence of a solution $y_K(v) < 0$ of (C.8) that satisfies the same boundary condition. If the RHS is locally Lipschitz continuous in y for all $v \geq 0$ the Picard-Lindelof Theorem would imply existence and uniqueness and moreover, Lipschitz continuity would imply that the $y_K(v)$ is continuous in K. Unfortunately, although the RHS is locally Lipschitz continuous for all $v > 0$, its Lipschitz continuity may fail at $v = 0$. Therefore, for v strictly away from 0, the standard argument applies given Lipschitz continuity, but for neighborhood around 0, we need a different argument. In what follows, we will center our analysis on the neighborhood of $v = 0$.

We start by rewriting (C.8) by changing variables again, $z(v) = y(v)v^m$:

$$z'(v) = -rv^m - (g(v,K)v - m)\frac{z(v)}{v} - h(v,K)\frac{z(v)^2}{v^m}. \tag{C.9}$$

First, we show that the operator

$$L_K(z)(v) = \int_0^v -rs^m - (g(s,K)s - m)\frac{z(s)}{s} - h(s,K)\frac{z(s)^2}{s^m}ds. \tag{C.10}$$

is a contraction mapping on a Banach space of solutions that includes (4.1). This extends the Picard-Lindelof Theorem to our setting and thus implies existence and uniqueness. Next, we show that the fixed point of L_K converges uniformly to the fixed point of L_1 as $K \searrow 1$. Finally, we show that we can obtain a sequence of solutions T^K that converge (pointwise) to the solution of the binding payoff floor constraint (with $K = 1$) and show that the revenue of these solutions also converges to the value of the auxiliary problem.

Before we introduce the Banach space on which the contraction mapping is defined, we first derive bounds for the RHS of (C.9).

Lemma 17. *Suppose Assumption 2 is satisfied. For any $\kappa > -1$, there exist $\overline{K} > 1$, an integer $m \geq 0$, and strictly positive real numbers α, η, ξ such that the following holds.*

(a) $m < |\kappa| + \eta$,

(b) $\frac{(|\kappa|+\eta-m)\alpha+\eta\alpha^2+r}{m+1} \in [0,\alpha]$,

(c) $\frac{|\kappa|+\eta(1+2\alpha)-m}{m+1} \in (0,1)$,

(d) $\frac{\kappa+\eta(1+\alpha)-m}{m+1}, \frac{\kappa-\eta(1+\alpha)-m}{m+1} \begin{cases} \in (0,1) & \text{if } \kappa > m \\ \in (-\frac{1}{2},\frac{1}{2}) & \text{if } \kappa = m \\ \in (-1,0) & \text{if } \kappa < m \end{cases}$.

(e) $|h(v,K)v^2| < \eta$ *for any* $v < \xi$ *and* $K \in [1,\overline{K}]$,

(f) $|g(v,K)v - \kappa| < \eta$ *for any* $v < \xi$ *and* $K \in [1,\overline{K}]$,

Proof. First we choose m. If $\kappa \geq 1$, let $m = \lfloor \kappa \rfloor$; if $\kappa \in (-1,1)$, let $m = 0$. Thus $0 \leq m \leq |\kappa|$ and (a) is satisfied for any $\eta > 0$. In addition, $0 \leq \frac{|\kappa|-m}{m+1} < 1$ and $0 \leq |\kappa| < m+1$. Note that by the choice of m, $\kappa < m$ if and only if $\kappa < 0$; $\kappa = m$ if and only if $\kappa = 0, 1, \ldots$; $\kappa > m$ if and only if $\kappa > 0$ and κ is not an integer.

Next we choose α. Consider (b). By the choice of m, the expression in (b) is non-negative for any $\eta, \alpha > 0$. Given this, Part (b) is equivalent to

$$\eta\alpha^2 - (2m + 1 - |\kappa| - \eta)\alpha + r \leq 0.$$

Hence, $\frac{(2m+1-|\kappa|-\eta)-\left[(2m+1-|\kappa|-\eta)^2-4r\eta\right]^{\frac{1}{2}}}{2\eta} \leq \alpha \leq \frac{(2m+1-|\kappa|-\eta)+\left[(2m+1-|\kappa|-\eta)^2-4r\eta\right]^{\frac{1}{2}}}{2\eta}$. Since $2m + 1 - |\kappa| > 0$, as $\eta \to 0$, the upper bound of α goes to $+\infty$ while the lower bound converge to

$\frac{r}{2m+1-|\kappa|}$ by L'Hospital's rule. We choose $\alpha = \frac{2r}{2m+1-|\kappa|}$. Then there exists $\eta_0 > 0$ such that Part (b) holds for any $\eta \in (0, \eta_0)$.

For m, α,and η_0 chosen above, since $0 \leq \frac{|\kappa|-m}{m+1} < 1$, there exists $\eta_1 \in (0, \eta_0)$ such that Part (c) holds for any $\eta \in (0, \eta_1)$.

For Part (d), consider the limit

$$\lim_{\eta \to 0} \frac{\kappa \pm \eta(1+\alpha) - m}{m+1} = \frac{\kappa - m}{m+1} \begin{cases} \in (0,1) & \text{if } \kappa > m \\ = 0 & \text{if } \kappa = m \\ \in (-1,0) & \text{if } \kappa < m \end{cases}$$

By continuity in both cases there exists $\eta \in (0, \eta_1)$ such that Part (f) holds.

Finally, given η chosen for Part (f), it follows from Lemma 16 that we can choose ξ and $\overline{K}$ jointly such that (e) and (f) hold. The proof of Lemma 16 shows that ξ can be chosen independently of K if $K < \overline{K}$. $\square$

Note that $(\overline{K}, m, \alpha, \eta, \xi)$ in Lemma 17 only depend on the number of buyers n and the distribution function F. Since Lemma 13 is a statement for a fixed distribution and fixed n, we treat $(\overline{K}, m, \alpha, \eta, \xi)$ as fixed constants for the rest of this section. In the following, we slightly abuse notation by using n as an index for sequences. The number of buyers does not show up in the notation in the remainder of this section except in the final proof of Lemma 13.

We define a space of real-valued functions

$$\mathcal{Z}_0 = \left\{ z : [0, \xi] \to \mathbb{R} \,\middle|\, \sup_v \left|\frac{z(v)}{v^{m+1}}\right| \in \mathbb{R} \right\},$$

and equip it with the norm

$$||z||_m = \sup_v \left| \frac{z(v)}{v^{m+1}} \right|.$$

Define a subset of $\mathcal{Z}_0$ by

$$\mathcal{Z} = \{ z : [0, \xi] \to \mathbb{R} \,|\, ||z||_m \leq \alpha \}.$$

Note that these definitions are independent of $K < \overline{K}$.

Lemma 18. *Suppose Assumption 2 is satisfied. $\mathcal{Z}_0$ is a Banach space with norm $||\cdot||_m$ and $\mathcal{Z}$ is a complete subset of $\mathcal{Z}_0$.*

Proof. For any $\gamma_1, \gamma_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathcal{Z}_0$ and $v \in [0, \xi]$, we have

$$\begin{aligned} \left| \frac{\gamma_1 z_1(v) + \gamma_2 z_2(v)}{v^{m+1}} \right| &\leq |\gamma_1| \left| \frac{z_1(v)}{v^{m+1}} \right| + |\gamma_2| \left| \frac{z_2(v)}{v^{m+1}} \right| \\ &\leq |\gamma_1| ||z_1||_m + |\gamma_2| ||z_2||_m \\ &< \infty. \end{aligned}$$

Therefore $\mathcal{Z}_0$ is a linear space. It's straight forward to see that $||\cdot||_m$ is a norm on $\mathcal{Z}_0$. We now show $\mathcal{Z}_0$ is complete. Consider a Cauchy sequence $\{z_n\} \subset \mathcal{Z}_0$: for any $\varepsilon > 0$, there exists N_ε such that $||z_{n'} - z_n||_m < \varepsilon$ for any $n', n \geq N_\varepsilon$.

First, notice that for any $n > 0$, $||z_n||_m \leq \beta := \max_{n' \leq N_\varepsilon} \{||z_{n'}||_m\} + \varepsilon < \infty$. Next we claim that z_n converges pointwise. To see this, note that $\sup_v |\frac{z_{n'}(v)-z_n(v)}{v^{m+1}}| < \varepsilon$ implies that $|\frac{z_{n'}(v)-z_n(v)}{v^{m+1}}| = |\frac{z_{n'}(v)}{v^{m+1}} - \frac{z_n(v)}{v^{m+1}}| < \varepsilon$ for any v. Since $|\frac{z_n(v)}{v^{m+1}}| \leq \beta$, completeness of real interval with the regular norm implies that there exists $x(\cdot)$ such that $\frac{z_n(v)}{v^{m+1}} \to x(v)$ pointwise and $|x(v)| \leq \beta$. Now define $z(v) = x(v)v^{m+1}$. It's straightforward that $z_n(v) \to z(v)$ pointwise.

Finally, we show that z_n converges under $|| \cdot ||_m$. To see this notice that $||z_n - z|| = \sup_v |\frac{z_n(v)}{v^{m+1}} - x(v)| \leq \varepsilon$ for any $n > N_\varepsilon$. In addition, since $|x(v)| \leq \beta$, $||z||_m \leq \beta$. This proves that $\mathcal{Z}$ is complete. The same argument shows that $\mathcal{Z}$ is complete, by replacing the bound β by α. □

To study the ODE (C.9) for each $K \in [1, \overline{K}]$,we define an operator L_K on $\mathcal{Z}$ as in (C.10).

Lemma 19. *Suppose Assumption 2 is satisfied. The operator L_K is a contraction mapping on $\mathcal{Z}$ with a common contraction parameter $\rho < 1$ for all $K \in [1, \overline{K}]$.*

Proof. First we show that $L_K \mathcal{Z} \in \mathcal{Z}$. For any $z \in \mathcal{Z}$ and $v \in [0, \xi]$,

$$\begin{aligned}
|L_K(z)(v)| =& \left| \int_0^v -rs^m - (g(s,K)s - m)\frac{z(s)}{s} - h(s,K)s^2 \frac{z(s)^2}{s^{m+2}} ds \right| \\
\leq& \frac{rv^{m+1}}{m+1} + \left| \int_0^v (g(s,K)s - m)\frac{z(s)}{s} ds \right| + \eta \left(||z||_m\right)^2 \int_0^v s^{2m+2-m-2} ds \\
\leq& \frac{rv^{m+1}}{m+1} + \sup_{s \in [0,\xi]} |g(s,K)s - m| \, ||z||_m \int_0^v \frac{s^{m+1}}{s} ds + \eta\alpha^2 \frac{v^{m+1}}{m+1} \\
\leq& \frac{rv^{m+1}}{m+1} + (|\kappa| + \eta - m)\alpha \frac{v^{m+1}}{m+1} + \eta\alpha^2 \frac{v^{m+1}}{m+1} \\
=& \frac{(|\kappa| + \eta - m)\alpha + \eta\alpha^2 + r}{m+1} v^{m+1} \\
\leq& \alpha v^{m+1}.
\end{aligned}$$

The first inequality follows from the triangle inequality of real numbers, Part (e) of Lemma 17 and $|z(s)| \leq ||z|| s^{m+1}$. The second inequality follows from $|z(s)| \leq ||z|| s^{m+1}$ and $||z|| \leq \alpha$. The third inequality follows from Lemma 17: for any $s \in [0, \xi]$ and $K \in [1, \overline{K}]$:

$$\begin{aligned}
|g(s,K)s - m| &\leq |g(s,K)s - \kappa| + |\kappa - m| \\
&\leq \begin{cases} \eta + \kappa - m \text{ if } \kappa \geq 1 \\ \eta + |\kappa| \text{ if } \kappa \in (-1, 1) \end{cases} \\
&= |\kappa| + \eta - m.
\end{aligned}$$

We now show $L_K : \mathcal{Z} \to \mathcal{Z}$ is a contraction mapping. For any $z_1, z_2 \in \mathcal{Z}$ and $v \in [0, \xi]$,

$$\begin{aligned}
|L_K(z_1)(v) - L_K(z_2)(v)| &= \left| \int_0^v -(g(s,K)s - m)\frac{z_1(s) - z_2(s)}{s} - h(s,K)s^2 \frac{z_1(s)^2 - z_2(s)^2}{s^{m+2}} ds \right| \\
&\leq \int_0^v \sup_{s \in [0,\xi]} |g(s,K)s - m| \frac{|z_1(s) - z_2(s)|}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{s\in[0,\xi]}|h(s,K)s^2|\frac{|z_1(s)+z_2(s)||z_1(s)-z_2(s)|}{s^{m+2}}ds\\
\le&(|\kappa|+\eta-m)\int_0^v ||z_1-z_2||_m\frac{s^{m+1}}{s}ds\\
&+\int_0^v \eta(||z_1||_m+||z_2||_m)||z_1-z_2||_m\frac{s^{2m+2}}{s^{m+2}}ds\\
\le&(|\kappa|+\eta-m)\frac{v^{m+1}}{m+1}||z_1-z_2||_m+\eta 2\alpha\frac{v^{m+1}}{m+1}||z_1-z_2||_m\\
=&v^{m+1}\frac{|\kappa|+\eta-m+\eta 2\alpha}{m+1}||z_1-z_2||_m
\end{aligned}
$$

The first inequality follows from the triangle inequality for real numbers. The second inequality follows from $\sup|g(s,K)s-m|<|\kappa|+\eta-m$ which was shown above, $|z_1(s)-z_2(s)|\le||z_1-z_2||_m s^{m+1}$, $\sup|h(s,K)s^2|<\eta$, and $|z_1(s)+z_2(s)|\le|z_1(s)|+|z_2(s)|\le(||z_1||_m+||z_2||_m)s^{m+1}$. The third inequality follows from $||z||_m\le\alpha$.

It follows immediately that $||L_K(z_1)-L_K(z_2)||_m\le\frac{|\kappa|+\eta-m+\eta 2\alpha}{m+1}||z_1-z_2||_m$. By Part (c) of Lemma 17, $\rho:=\frac{|\kappa|+\eta-m+\eta 2\alpha}{m+1}\in(0,1)$, which is independent of $K\in\overline{K}$. Hence L_K is contraction mapping on $\mathcal{Z}$, with a common contraction parameter for all $K\in[1,\overline{K}]$. □

Since $L_K:\mathcal{Z}\to\mathcal{Z}$ is a contraction mapping, the Banach fixed point theorem implies that there exists a unique fixed point of L_K in $\mathcal{Z}$. For any $K\in\left[1,\overline{K}\right]$, we denote the fixed point by z_K, i.e., $z_K=L_K(z_K)\in\mathcal{Z}$. By the Banach fixed point theorem we have $z_K=\lim_{n\to\infty}L_K^n(0)$.

Lemma 20. *Suppose Assumption 2 is satisfied.e The fixed point of L_K on $\mathcal{Z}$, and hence the solution to the ODE (C.9) must be strictly negative for $v>0$.*

Proof. Let $\rho_1=\frac{\kappa+\eta-m+\eta\alpha}{m+1}$, $\rho_2=\frac{\kappa-\eta-m-\eta\alpha}{m+1}$. We claim that there exists M_1,M_2 such that

$$M_1\le\frac{L_K^n(0)\,(v)}{v^{m+1}}\le M_2<0 \tag{C.11}$$

for any $n\ge1$.

For any $n>1$,

$$
\begin{aligned}
L_K^n(0)(v)&=-\frac{r}{m+1}v^{m+1}-\int_0^v(g(s,K)s-m)\,\frac{L_K^{n-1}(0)\,(s)}{s}+h(s,K)s^2\frac{\left(L_K^{n-1}(0)\,(s)\right)^2}{s^{m+2}}ds\\
&=-\frac{r}{m+1}v^{m+1}+\int_0^v\left((g(s,K)s-m)\frac{1}{s}+h(s,K)s^2\frac{L_K^{n-1}(0)\,(s)}{s^{m+2}}\right)\left(-L_K^{n-1}(0)\,(s)\right)ds
\end{aligned}
\tag{C.12}
$$

We prove separate the three cases $\kappa>m$, $\kappa=m$, $\kappa<m$ (which is equivalent to $\kappa<0$) separately.

Case 1: $\kappa>m$. In this case, $\rho_1,\rho_2>0$ by Lemma 17. Let $M_1=-\frac{r}{m+1}$ and $M_2=-\frac{r}{m+1}(1-\rho_1)$. By part (d) of Lemma 17: $M_1\le\frac{-r}{m+1}\le M_2<0$. Therefore we have

$L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (C.11). We prove the desired result by induction. For $n > 1$, consider (C.12):

$$\begin{aligned}
L_K^n(0)(v) &\le -\frac{r}{m+1}v^{m+1} + \int_0^v \left(\frac{\kappa - m + \eta}{s} + \frac{\eta\alpha s^{m+1}}{s^{m+2}}\right)\left(-L_K^{n-1}(0)(s)\right) ds \\
&\le -\frac{r}{m+1}v^{m+1} + (\kappa - m + \eta(1+\alpha)) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_1 M_1\right) v^{m+1} \\
&= M_2 v^{m+1}
\end{aligned}$$

The first inequality follows from $-L_K^{n-1}(0) > 0$, $L_K \in \mathcal{Z}$, and replacing the coefficient of $-L_K^{n-1}(0)$ by its upper bound. The second inequality follows from $\kappa - m + \eta(1+\alpha) > 0$ and replacing $-L_K^{n-1}(0)$ with its upper bound $-M_1 s^{m+1}$ (by the induction hypothesis). In addition,

$$\begin{aligned}
L_K^n(0)(v) &\ge -\frac{r}{m+1}v^{m+1} + \int_0^v \left(\frac{\kappa - m - \eta}{s} - \frac{\eta\alpha s^{m+1}}{s^{m+2}}\right)\left(-L_K^{n-1}(0)\,(s)\right) ds \\
&\ge -\frac{r}{m+1}v^{m+1} + (\kappa - m - \eta(1+\alpha)) \int_0^v \left(-M_2 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_2 M_2\right) v^{m+1} \\
&\ge M_1 v^{m+1}
\end{aligned}$$

The first inequality follows from $-L_K^{n-1}(0)\,(s) > 0$ and replacing the coefficient of $\left(-L_K^{n-1}(0)\,(s)\right)$ by its lower bound. The second inequality follows from $\kappa - m - \eta(1+\alpha) > 0$ and replacing $-L_K^{n-1}(0)$ with its upper bound $-M_2 s^{m+1}$ (by the induction hypothesis). The last inequality follows from $-\rho_2 M_2 > 0$ and the choice of M_1.

Case 2: $\kappa < m$. In this case, $\rho_1, \rho_2 \in (-1, 0)$ by part (d) of Lemma 17. Let $M_1 = -\frac{r}{m+1}\frac{1}{1+\rho_2}$ and $M_2 = -\frac{r}{m+1}$. $\rho_2 < 0$ implies $M_1 \le -\frac{r}{m+1} \le M_2 < 0$. Therefore we have $L_K^1(0)\,(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (C.11). For $n > 1$, consider $(C.12)$:

$$\begin{aligned}
L_K^n(0)(v) &\le -\frac{r}{m+1}v^{m+1} + (\kappa - m + \eta(1+\alpha)) \int_0^v \left(-M_2 \frac{s^{m+1}}{s}\right) ds \\
&= \left(-\frac{r}{m+1} - \rho_1 M_2\right) v^{m+1} \\
&\le -\frac{r}{m+1}v^{m+1} \\
&= M_2 v^{m+1}
\end{aligned}$$

The first inequality follows from a similar derivation as in the case $\kappa > m$. However here $\kappa - m + \eta(1+\alpha) < 0$, therefore $-L_K^{n-1}(0)$ is replaced by its lower bound $-M_2 s^{m+1}$. The second inequality follows because $\rho_1 M_2 > 0$. In addition,

$$L_K^n(0)(v) \geq -\frac{r}{m+1}v^{m+1} + (\kappa - m - \eta(1+\alpha)) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds$$
$$= \left(-\frac{r}{m+1} - \rho_2 M_1\right) v^{m+1}$$
$$= M_1 v^{m+1}.$$

Case 3: $\kappa = m$. Then $\rho_1 = -\rho_2 = \frac{\eta(1+\alpha)}{m+1} \in (-1/2, 1/2)$ by part (d) of Lemma 17. Let $M_1 = -\frac{r}{m+1}\frac{1}{1-\rho_1}$ and $M_2 = -\frac{r}{m+1}\frac{1-2\rho_1}{1-\rho_1}$. Since $m \geq 0$ we have $\rho_1 \in (0, 1/2)$. This implies $M_1 \leq -\frac{r}{m+1} \leq M_2 < 0$. Therefore we have $L_K^1(0)(v) = -\frac{r}{m+1}v^{m+1}$ satisfying (C.11). For $n > 1$, consider $(C.12)$:

$$L_K^n(0)(v) \leq -\frac{r}{m+1}v^{m+1} + \eta(1+\alpha) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds$$
$$= \left(-\frac{r}{m+1} - \frac{\eta(1+\alpha)}{m+1} M_1\right) v^{m+1}$$
$$= M_2 v^{m+1}$$

To obtain the first inequality, we replace $-L_K^{n-1}(0)$ by its upper bound $-M_1 s^{m+1}$ since $\eta(1+\alpha) > 0$. In addition,

$$L_K^n(0)(v) \geq -\frac{r}{m+1}v^{m+1} - \eta(1+\alpha) \int_0^v \left(-M_1 \frac{s^{m+1}}{s}\right) ds$$
$$= \left(-\frac{r}{m+1} + \frac{\eta(1+\alpha)}{m+1} M_1\right) v^{m+1}$$
$$= M_1 v^{m+1}$$

To obtain the first inequality, we replace $-L_K^{n-1}(0)(v)$ by its upper bound $-M_2 s^{m+1}$ since $-\eta(1+\alpha) < 0$. □

Lemma 21. *Suppose Assumption 2 is satisfied. Then* $\sup_{v\in[0,\xi]} \left|\frac{z_K(v)}{v^m} - \frac{z_1(v)}{v^m}\right| \to 0$ *as* $K \to 1$.

Proof. First note that for any $\varepsilon > 0$, it follows from Lemma 16 that $g(v, K)v$ and $h(v, K)v^2$ are bounded over $v \in [0, \xi]$ and $K \in [1, \overline{K}]$. Hence there exists $\Gamma \in \left(1, \overline{K}\right)$ such that

$$\sup_{v\in[0,\xi], K\in[1,\Gamma]} |g(v, K)v - g(v, 1)v| < \varepsilon,$$
$$\sup_{v\in[0,\xi], K\in[1,\Gamma]} |h(v, K)v^2 - h(v, 1)v^2| < \varepsilon.$$

Since $\sup_{v\in[0,\xi]} \left|\frac{z_K(v)}{v^m} - \frac{z_1(v)}{v^m}\right| \leq \sup_v ||z_K - z_1||_m \frac{v^{m+1}}{v^m} \leq \xi ||z_K - z_1||_m$, it's sufficient to show that $\lim_{K\to 1} ||z_K - z_1||_m = 0$. The proof follows from Lee and Liu (2013, Lemma 13(b)). Let $\rho = \frac{|\kappa|+\eta-m+\eta 2\alpha}{m+1} < 1$ be the contraction parameter, which is independent of K. For all $z \in \mathcal{Z}$

and $K \in [1, \Gamma]$,

$$\begin{aligned}
|L_K(z)(v) - L_1(z)(v)| =& \left| \int_0^v (g(s,K)s - g(s,1)s) \frac{z(s)}{s} + (h(s,K)s^2 - h(s,1)s^2) \frac{z(s)^2}{s^{m+2}} ds \right| \\
\leq& \varepsilon \int_0^v \frac{z(s)}{s} ds + \varepsilon \int_0^v \frac{z(s)^2}{s^{m+2}} ds \\
\leq& \varepsilon \left(||z||_m \frac{v^{m+1}}{m+1} + ||z||_m^2 \frac{v^{m+1}}{m+1} \right) \\
\leq& \varepsilon \frac{\alpha + \alpha^2}{m+1} v^{m+1}
\end{aligned}$$

Therefore, $||L_K(z) - L_1(z)||_m \leq \varepsilon \frac{\alpha+\alpha^2}{m+1}$.

For any $n > 1$,

$$\begin{aligned}
||L_K^n(z) - L_1^n(z)||_m =& ||L_K \left(L_K^{n-1}(z) \right) - L_1 \left(L_K^{n-1}(z) \right) + L_1 \left(L_K^{n-1}(z) \right) - L_1 \left(L_1^{n-1}(z) \right) ||_m \\
\leq& ||L_K \left(L_K^{n-1}(z) \right) - L_1 \left(L_K^{n-1}(z) \right) || + ||L_1 \left(L_K^{n-1}(z) \right) - L_1 \left(L_1^{n-1}(z) \right) ||_m \\
\leq& \varepsilon \frac{\alpha + \alpha^2}{m+1} + \rho ||L_K^{n-1}(z) - L_1^{n-1}(z)||_m \\
\leq& \varepsilon \frac{\alpha + \alpha^2}{m+1} \sum_{k=0}^{n-1} \rho^k \\
\leq& \varepsilon \frac{\alpha + \alpha^2}{m+1} \frac{1}{1-\rho}
\end{aligned}$$

Given $z_K = \lim_{n\to\infty} L_K^n(0)$, there exists N_ε s.t. $\forall n \geq N_\varepsilon$, $||z_K - L_K^n(0)|| \leq \varepsilon$:

$$\begin{aligned}
||z_K - z_1||_m \leq& ||z_K - L_K^n(0)||_m + ||z_1 - L_1^n(0)||_m + ||L_K^n(0) - L_1^n(0)||_m \\
\leq& 2\varepsilon + \varepsilon \frac{\alpha + \alpha^2}{m+1} \frac{1}{1-\rho} \\
=& \left(2 + \frac{\alpha + \alpha^2}{m+1} \frac{1}{1-\rho} \right) \varepsilon
\end{aligned}$$

Therefore $\lim_{K\to 1} ||z_K - z_1||_m = 0$. □

Given definition $z(v) = y(v)v^m$, let $y_K(v) = \frac{z_K(v)}{v^m}$, where z_K is the fixed point of L_K. It follows from the previous two lemmas that $y_K(v)$ is negative and $\lim_{K\to 1} ||y_K - y_1|| = 0$ under standard sup norm. Now we have all the ingredients necessary to prove Lemma 13.

Proof of Lemma 13. The uniform convergence of y_K implies that the cutoff sequence v_t^K given by $v(t) = v(0) + \int_0^t y_K(v(s))\, ds$ converges pointwise to the cutoff sequence $v_t = v_t^1$ associated with the trading time function $T(v) = T^1(v)$. Since v_t is continuous and strictly decreasing (by Lemma 9), this implies that the trading time function

$$T^K(v) = \sup \left\{ t : v_t^K \geq v \right\}$$

converges pointwise to $T(v)$. To see this, note that $\sup\{t : v_t \geq v\} = \sup\{t : v_t > v\}$, since v_t is continuous and strictly decreasing. Now, for all t such that $v_t > v$, there exists K^t such that $v_t^K > v$ for all $K < K^t$. Hence,

$$\lim_{K \searrow 1} \sup\left\{t : v_t^K \geq v\right\} \geq \sup\{t : v_t > v\}.$$

Similarly, for all t such that $v_t < v$, there exists K^t such that $v_t^K < v$ for $K < K^t$. Hence,

$$\lim_{K \searrow 1} \sup\left\{t : v_t^K \geq v\right\} \leq \sup\{t : v_t \geq v\}.$$

Therefore, for all v, we have

$$\lim_{K \searrow 1} \sup\left\{t : v_t^K \geq v\right\} = \sup\{t : v_t \geq v\},$$

or equivalently,

$$\lim_{K \searrow 1} T^K(v) = T(v).$$

It remains to show that the seller's ex ante revenue converges. Notice that the sequence $e^{-rT^K(v)}$ is uniformly bounded by 1. Therefore, the dominated convergence theorem implies that

$$\lim_{K \searrow 1} \int_0^1 e^{-rT^K(x)} J(x) dF^{(n)}(x) = \int_0^1 e^{-rT(x)} J(x) dF^{(n)}(x).$$

□

C.3 Proof of Lemma 14

Proof. For $t \in \{0, \Delta, 2\Delta, \ldots\}$, define

$$\begin{aligned} \mathcal{V}_t^{K,\Delta} &: = \left\{v \in \left[0, v_t^{K,\Delta}\right] \Big| J(v | v \leq v_t^{K,\Delta}) \geq 0\right\}, \\ \overline{\mathcal{V}}_t^{K,\Delta} &: = \left[0, v_t^{K,\Delta}\right] \setminus \mathcal{V}_t^{K,\Delta}. \end{aligned}$$

Consider the LHS of the payoff floor constraint at $t = k\Delta$, $k \in \mathbb{N}_0$. Notice that, for $k > 0$, the new posterior at this point in time is equal to the old posterior at $((k-1)\Delta)^+$. Therefore, we can approximate the LHS of the payoff floor at $t = k\Delta$ as:

$$\begin{aligned} &\int_{\left[0, v_{k\Delta}^{K,\Delta}\right]} e^{-r\left(T^{K,\Delta}(v) - k\Delta\right)} J(v | v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v | v \leq v_{k\Delta}^{K,\Delta}) dv \\ =& \int_{\left[0, v_{k\Delta}^{K,\Delta}\right]} e^{-r\left(T^K(v) - (k-1)\Delta\right)} e^{-r\left(T^{K,\Delta}(v) - T^K(v) - \Delta\right)} J(v | v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v | v \leq v_{k\Delta}^{K,\Delta}) dv \\ =& \int_{\mathcal{V}_{k\Delta}^{K,\Delta}} e^{-r\left(T^K(v) - (k-1)\Delta\right)} e^{-r\left(T^{K,\Delta}(v) - T^K(v) - \Delta\right)} J(v | v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v | v \leq v_{k\Delta}^{K,\Delta}) dv \\ &+ \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\left(T^K(v) - (k-1)\Delta\right)} e^{-r\left(T^{K,\Delta}(v) - T^K(v) - \Delta\right)} J(v | v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v | v \leq v_{k\Delta}^{K,\Delta}) dv \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\mathcal{V}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad + \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} e^{r\Delta} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\geq \int_{\mathcal{V}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad + \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad - \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} \left(1 - e^{r\Delta}\right) J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&= \int_{[0, v_{k\Delta}^{K,\Delta}]} e^{-r\,(T^K(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
&\quad - \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} \left(1 - e^{r\Delta}\right) J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv.
\end{aligned}
$$

The first term in the last expression is equal to the LHS of the payoff floor constraints at $((k-1)\Delta)^+$ for the original solution v^K. Hence it is equal to $K\Pi^E(v_{k\Delta}^{K,\Delta})$. Therefore, we have

$$
\begin{aligned}
&\int_0^{v_{k\Delta}^{K,\Delta}} e^{-r\,(T^{K,\Delta}(v)-k\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
=& K\Pi^E(v_{k\Delta}^{K,\Delta}) + \left(e^{r\Delta} - 1\right) \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} e^{-r\,(T^K(v)-(k-1)\Delta)} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
\geq& K\Pi^E(v_{k\Delta}^{K,\Delta}) + \left(e^{r\Delta} - 1\right) \int_{\overline{\mathcal{V}}_{k\Delta}^{K,\Delta}} J(v|v \leq v_{k\Delta}^{K,\Delta})\, f^{(n)}(v|v \leq v_{k\Delta}^{K,\Delta}) dv \\
=& K\Pi^E(v_{k\Delta}^{K,\Delta}) - \left(e^{r\Delta} - 1\right) \left[\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta}) - \Pi^E(v_{k\Delta}^{K,\Delta})\right] \\
=& K\Pi^E(v_{k\Delta}^{K,\Delta}) - \left(e^{r\Delta} - 1\right) \left[\frac{\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1\right] \Pi^E(v_{k\Delta}^{K,\Delta}),
\end{aligned}
$$

where

$$
\widetilde{\Pi}^M(w) := \int_{[0,w]} \max\{0, J(v|v \leq w)\}\, f^{(n)}(v|v \leq w) dv \ < w.
$$

Next we show that $\frac{\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1$ is uniformly bounded. Recall that by Assumption 3, there exist $0 < M \leq 1 \leq L < \infty$ and $\alpha > 0$ such that $Mv^\alpha \leq F(v) \leq Lv^\alpha$ for all $v \in [0,1]$. This implies that the rescaled truncated distribution

$$
\tilde{F}_x(v) := \frac{F(vx)}{F(x)},
$$

for all $v \in [0,1]$ is dominated by a function that is independent of x:

$$\tilde{F}_x(v) \leq \frac{Lv^\alpha x^\alpha}{Mx^\alpha} = \frac{L}{M}v^\alpha.$$

Next, we observe that the revenue of the efficient auction can be written in terms of the rescaled expected value of the second-highest order statistic of the rescaled distribution:

$$\Pi^E(v) = \int_0^1 vs\tilde{F}_v^{(n-1:n)}(s)ds.$$

If we define $\hat{F}(v) := \min\left\{1, \frac{L}{M}v^\alpha\right\}$ and $B := \int_0^1 s\hat{F}^{(n-1:n)}(s)ds$, then given $\tilde{F}_x(v) \leq \frac{L}{M}v^\alpha$ we can apply Theorem 4.4.1 in David and Nagaraja (2003) to obtain $\Pi^E(v) \geq Bv > 0$. Since $\widetilde{\Pi}^M(v) \leq v$, we have

$$\frac{\widetilde{\Pi}^M(v_{k\Delta}^{K,\Delta})}{\Pi^E(v_{k\Delta}^{K,\Delta})} - 1 \leq \frac{1}{B} - 1.$$

Therefore, LHS of the payoff floor at $t = k\Delta$ is bounded below by

$$\left[K - \left(e^{r\Delta} - 1\right)\left(\frac{1}{B} - 1\right)\right]\Pi^E(v_{k\Delta}^{K,\Delta}).$$

Clearly, for Δ sufficiently small, the term in the square bracket is no less than $(K+1)/2$. $\square$