Advertising Platforms and Privacy

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Abstract

We develop a general equilibrium model of informative advertising to examine the implications of privacy regulations on consumer welfare, focusing on its utilitarian aspects. In our model, firms reach consumers by placing ads on an advertising platform. Privacy regulations affect ad targetability by either facilitating or hindering the identification of consumers' preferences. The ad platform takes into account the level of targetability it can offer advertisers when choosing ad prices for different products. On the demand side, consumers are heterogeneous in their product preferences, and in how willing they are to consider a non-preferred product.

We show that it is possible for some consumers to exhibit a preference for privacy purely for instrumental reasons. Our analysis characterizes the conditions under which the privacy preferences of different types of consumers will be aligned or opposed. The platform's market power in the ad market and the possibility of cross-selling in the product market—products intended for choosier consumers selling to consumers with flexible preferences under privacy—are critical factors. If all consumers were picky enough to only consider offers of their preferred product, then equilibrium will feature within-product competition only and all consumers (as well as the platform) will be indifferent between privacy and no privacy. On the other hand, if ad prices are exogenous to the privacy regime—as might be the case if the ad market were competitive—then all consumers prefer no privacy.

In the popular discourse the issues around privacy are commonly posed as a tussle between the intrinsic privacy rights of consumers and the greater productivity of advertising under no privacy. This paper suggests that the terms of this debate are too narrow. Consumers can find value in privacy purely for instrumental reasons simply because the presence of consumers with flexible preferences introduces the possibility of greater competition in the product market leading to lower prices and greater consumption for some or all consumers.

1 Introduction

The past decade has seen a surge in online advertising through third-party platforms such as Google and Facebook. These platforms build profiles of users by tracking their online activities—browsing behavior, text and e-mail messages, photo and video posts, social media interactions—and sell them to advertisers. Advertisers value these profiles because they help them craft better ads and target them more efficiently. The multi-billion dollar valuations of companies like Alphabet and Meta is testimony to the value of targeted ads and the market power of these advertising platforms.

Concurrent with these developments, consumer privacy concerns have also risen, and are now front and center in the debate about how to regulate the online platforms (O'Neil, 2021; McKinnon, 2022). Some governments and firms have already taken measures to protect consumer privacy. For example, the European Union enacted the General Data Protection Regulation (GDPR) in 2016 to give users greater control over their personal information; in the U.S., the state of California passed the California Consumer Privacy Act (CCPA) in 2018 with a similar goal. Among companies, Apple, and to a lesser extent Alphabet, have attracted the most attention for their pro-privacy moves.[1] In 2021 Apple announced its App Tracking Transparency initiative which, from iOS 14.5 on, requires apps to seek users' explicit permission before tracking them across other apps and websites. In the message asking for permission, app developers are allowed to persuade consumers to opt in. For example, Roku's message says that opting in will "allow Roku to personalize ads in this app only. You will still see ads if you select 'Ask App Not to Track,' but ads may be less relevant to you."[2] Yet, early reports from Flurry Analytics suggested that only about 15% of U.S. users opted in; this number has since risen to about 40% in the most recent data reported by ad-analytics company AppsFlyer (Klosowski, 2022). Facebook's parent company, Meta Platforms, recently announced that it expected its sales in 2022 to go down by \$10 billion because "Apple's harmful policy is making it harder and more expensive for businesses of all sizes to reach their customers" (Haggin and Salvador, 2022).

Who gains and who loses from increased privacy protection? This is the question we examine in this paper. Our approach is strictly utilitarian. In other words, instead of thinking about privacy as an intrinsic preference, we view it as an instrumental preference—what it gives or does not give consumers in terms of consumer surplus (Becker, 1980; Lin, 2022).[3] Our analysis takes into account the possibility that some consumers are indeed served less relevant ads under privacy, which then leads them to make poorer product choices and/or pay higher prices. On the other hand, there may be other consumers who gain from hiding their preferences because

[1] Alphabet has announced, but not yet implemented, privacy restrictions to curtail tracking across apps on Android smartphones (Schechner and Patience, 2022).

[2] `https://www.attprompts.com/details/roku/r/recIkxx413ZkcfmJz`

[3] This is not to deny the existence of intrinsic privacy preferences; indeed, Lin's (2022) empirical work confirms that they exist. Rather, the point of this paper is to argue that a theory of privacy need not rest on intrinsic preference foundations. By grounding privacy preferences in instrumental motives, the theory acquires greater explanatory power.

doing so actually increases their consumer surplus. (Certainly the 40% opt-in rate quoted above suggests the possibility of heterogeneity on this issue.) We also examine the effect of privacy on ad prices, ad volume, and total consumption.

In order for consumers to differ in their privacy preferences they must be heterogeneous in some way. We craft the simplest such model possible: a model of horizontal product differentiation with two consumer types, each with its own distinct preferred product, but willing to accept the other type's preferred product provided it is offered at a discount. How much discount varies by type: one type is "pickier" than the other in that it demands a larger discount to buy a non-preferred product (Shilony, 1977; Narasimhan, 1988). (We call the former type "picky" and the latter type "flexible.") Firms making the two products send ads through a monopoly platform to inform consumers about their products and prices. Consumers who receive multiple ads will buy from the firm which offers them the highest consumer surplus; consumers who receive no ads will not buy. The product market is monopolistically competitive (Salop, 1979).

To give privacy its due, we examine its two most extreme manifestations: no privacy and full privacy. In the former, the advertising platform can identify consumers' types and make it possible for advertisers to target individual types. Thus advertisers can: (i) offer type-specific prices, and (ii) choose to reach only one of the types. In the latter, the platform can't identify consumers' types, so type-specific prices are not possible, nor is it possible to avoid reaching both consumer types—some consumers will necessarily end up receiving ads for the non-preferred product. Under each privacy regime, the platform choose ad prices to maximize its ad revenue. Advertisers, taking those ad prices and targeting constraints (if any) as given, choose product prices and how many ads to buy at each price. Our analysis is general-equilibrium in the sense that ad prices are endogenous, chosen by a platform which anticipates how ad prices affect the level and nature of the competition in the product market, and hence the total demand for ads, in different privacy regimes.

Our principal result is that privacy can have positive value for some or all consumers purely for instrumental reasons. However, it is not easy to find such value: giving up privacy has the attractive feature that it eliminates wasted advertising; an advertiser does not have to spend money trying to reach consumers it does not want to reach. When advertisers are more productive in reaching their target market, they will want to advertise more, potentially increasing consumption. Indeed, we show in Section 4 that this argument is powerful enough to ensure that consumers generally prefer no privacy if ad prices are exogenous, i.e., when they don't respond to changes in the privacy regime, as might be the case in a competitive ad market. In other words, in order for some consumers to prefer privacy for instrumental reasons, it is necessary that ad prices be endogenous to the privacy regime: the platform must be capable of changing ad prices in response to changes in the privacy regime (which is, of course, what we should expect given the market power of online ad platforms such as Facebook and Google).

With endogenous ad prices, a less-obvious effect of privacy is exposed: when consumers cannot be distinguished, it is difficult to prevent cross-selling. Ads for a given consumer type's preferred product will necessarily reach the other type of consumers. In our model, this translates to firms making the preferred product of picky consumers being tempted by the possibility of cross-selling to flexible consumers. This has the effect of lowering prices to those consumers. However, if one product's ads can cross-sell while the other product's ads can't, the former are automatically more productive than the latter. A higher per capita advertising cost induces firms making the flexible consumers' preferred product to price higher, while cross-selling pressure from the picky consumers' preferred product induces them to price lower. Under exogenous ad prices, these forces essentially cancel each other out so that the enhanced productivity of all ads under no privacy carries the day. However, when the platform has the ability to set ad prices, its own revenue-maximizing interests may sometimes align with tilting the playing field in favor of the firms making the flexible consumers' preferred product and sometimes in favor of the firms making the picky consumers' preferred product. As a result a variety of instrumental preferences for privacy may develop: consumers' preferences may be aligned—both types prefer privacy or both types prefer no privacy—or opposed—one type prefers privacy while the other prefers no privacy. However, in general, consumers with flexible preferences are more likely to benefit from privacy than consumers with rigid preferences.

It is worth noting that the product market in our model is monopolistically competitive: all product firms make zero profits in equilibrium, both under full privacy and under no privacy. The amount of advertising and the nature and level of price competition, however, depend crucially on ad prices, which is determined by a monopoly platform. Advertising levels and prices in the product market in turn determine consumer surplus, and hence their attitude toward privacy. Our model suggests that stricter privacy regulations may force ad platforms to favor some products over others. When advertising is an important tool for providing information to consumers, the market power of ad platforms may thus be more important than the competitiveness of the product market in determining consumers' attitudes toward privacy.

2 Literature review

The issues explored in this paper touch two streams of literature: the literature on targeting and the literature on privacy. Before we discuss those connections, however, it is useful to summarize here some properties of informative advertising with probabilistic reach, the advertising technology adopted in this paper (Butters, 1977; Grossman and Shapiro, 1984; Tirole, 1988, Section7.3.1; Stegeman, 1991; Bagwell, 2007). In this framework, consumers are unaware of the firms and uninformed about their products and prices unless reached by their advertising; reach itself is probabilistic in the sense that ads are not guaranteed to reach their target market: firms control ads sent, not ads received. Each consumer in the target market has an equal probability of receiving an ad; consumers who receive no ads can't buy; the rest choose, from all the ads received, the product maximizing consumer surplus. Butters (1977) shows that this advertising technology implies a distribution of prices in equilibrium even in a homogeneous market with competitive firms; furthermore, the amount of advertising is socially optimal. The latter result, however, turns out to be fragile. In Stegeman's (1991) model with heterogeneous reservation prices, the amount of advertising is less than socially optimal; in Grossman and Shapiro's (1984) model with horizontal product differentiation, the amount of advertising is more than socially optimal. Our model combines aspects of these two models: from Grossman and Shapiro (1984) we borrow horizontal product differentiation; from Stegeman (1991) we borrow a continuum of firms and consumers. But our model is also different from these papers: While these papers focus on untargeted advertising and prices, we compare targeted advertising and prices with untargeted advertising and prices; while ad prices are exogenous in these papers, we examine both exogenous and endogenous ad prices; finally, while there is no advertising platform in these papers, we have a monopoly ad platform setting ad prices for advertisers.

This paper belongs to the literature on targeting. The central question asked in this literature is how a change in targetability affects equilibrium outcomes such as advertising prices, product prices, producer surplus and consumer surplus. As observed by Bergemann and Bonatti (2011), "[T]he Internet has introduced at least two technological innovations in advertising, namely (i) the ability to relate payments and performance (e.g., pay per click), and (ii) an improved ability to target advertisement messages to users (p.435)."[4]. Most of the targeting literature, including Chen, Narasimhan, and Zhang (2001), Iyer, Soberman, and Villas-Boas (2005) and Bergemann and Bonatti (2011), focuses on effect (ii). For example, Chen, Narasimhan, and Zhang (2001) show that, in the model of Varian (1980) and Narasimhan (1988), for an initially low targetability, an improved ability to target price offers to their own loyal customers by either or both firms softens competition and benefits both firms. This is possible because firms may mistaken shoppers as loyal customers and charge them high prices. Iyer, Soberman, and Villas-Boas (2005) extends the analysis by allowing firms to use both targeted advertising and targeted prices. They find that each firm will advertise more to their own loyal consumers and less to shoppers, again leading to lower level of competition. In both papers, the cost of sending targeted price offers or targeted ads is either not explicitly modelled or assumed to be exogenously fixed and uniform across firms.[5] In comparison, ad prices in our model are endogenously determined by the ad platform and can differ across products. Bergemann and Bonatti (2011) develop a model with many firms and many advertising media to study how an increase in targetability affects equilibrium ad price and competition between online and offline media. They also use the advertising technology pioneered by Butters (1977) and allow ad price to be endogenously determined in equilibrium. They find that although targeting increases the social value of advertising, equilibrium ad price first increases and then decreases in the targeting ability. In their model, product prices are exogenous, ad prices

[4]See also Athey and Gans (2010) and Goldfarb (2014) for related discussion

[5]See also Shin and Yu (2021). In their model, ad prices are also fixed. Consumers do not know their own preferences and make inference based on targeted ads they receive.

are uniform across products, and in each media market there are an infinite number of publishers so the ad market is perfectly competitive. Our model shares the same advertising technology as theirs, but both ad prices and product prices are endogenously determined in equilibrium. More importantly, different from Bergemann and Bonatti (2011) and most of literature which assume that ad prices are uniform across products,[6] we allow ad prices to differ across products (and potentially also market segments), so we our model can capture not only effect (ii) but also effect (i) where ad platforms can relate ad prices to click-through rates and favour one type of product over another. Moreover, instead of assuming many competitive media markets, we assume a single monopoly ad platform, aiming to capture the market power of Facebook and Google as we mentioned earlier in the Introduction. Compared to the elaborate way of modelling targetability in Bergemann and Bonatti (2011), however, our modelling of targetability is rudimentary – with only two levels of targetability induced by privacy regulations.

Since the change in targetability in our model is motivated by a change in privacy regulation, our paper is naturally linked to the growing literature on privacy (see Acquisti, C. Taylor, and Wagman (2016) for a survey). One strand of this literature, e.g., Choi, Jeon, and Kim (2019), Acemoglu et al. (2019) and Bergemann, Bonatti, and Gan (2020), focuses on information externalities among users and shows how the social value of privacy may differ from individual value of privacy. See also Johnson (2013) and Garratt and Van Oordt (2021) for externalities in using electronic cash and ad avoiding technologies. Another strand of literature recognizes that firms may offer different prices and/or products to consumers based on their past purchasing behavior (Villas-Boas, 1999; Fudenberg and Tirole, 2000; Fudenberg and Villas-Boas, 2012) or on their voluntary disclosure (Ichihashi, 2020). The value of privacy arises because consumer privacy regulation may affect whether sellers can collect consumer data either through their past purchasing behavior or through their voluntary disclosure. One way to model the strength of privacy protection is the cost for consumers to stay anonymous after past purchase. Conitzer, C. R. Taylor, and Wagman (2012) show that consumer welfare has an inverted-U shaped relationship with the cost of remaining anonymous, and Baye and Sappington (2020) argue that privacy protection benefits myopic consumers at the cost of sophisticated ones. Similarly, in a duopoly setting where firms can use consumer information to price discriminate but consumers can pay some privacy cost to opt out, Montes, Sand-Zantman, and Valletti (2019) show that firms may be worse off while consumers may be better off from a higher privacy cost. Different from these papers, our static model does not assume informational externalities among users and privacy is exogenous. Different privacy settings affect the ability of the ad platform to target and we compare equilibrium outcomes under different privacy regimes.

Finally, there is a growing literature on the interaction between advertising platforms and con-

[6] A noticeable exception is Galeotti and Moraga-González (2008). In a related model with two consumer market segments where two homogeneous firms can each send targeted ads to one or both segments, they show that firms can earn strictly positive profit only if the per-consumer advertising costs differ significantly across segments. Their ad prices, however, are exogenous.

sumer privacy. In Casadesus-Masanell and Hervas-Drane (2015), consumers disclose information to firms to receive better service but disclosure entails a disutility. Firms compete for consumers in both privacy and price and can generate revenue from consumer purchases and/or from sales of consumer data to third parties. They show that it is optimal for firms to focus on one single revenue source and more intense competition does not necessarily lead to a higher level of privacy. In Campbell, Goldfarb, and Tucker (2015), firms differ in the scope of products they are offering, and a more specialized firm offers a narrower range of products but with higher product qualities. Consumer data can help firms optimize their product offering, but consumers suffer an intrinsic loss when their data are collected and used by these firms. They show that privacy regulation has a differential impact on firms selling different ranges of products. In comparison, consumers in our model do not have intrinsic preferences for privacy and their altitude towards privacy is completely determined by the consumer surplus they can receive under different privacy regimes. De Corniere and De Nijs (2016) are also interested in ad platforms and privacy and does not assume consumers have intrinsic preferences for privacy. In their model, firms compete in an auction for the monopoly right to serve a consumer. The ad platform (auctioneer) can choose whether to disclose consumer information to allow bidders to have better estimates about the value of serving this consumer. Their single-unit auction framework is very different from our general equilibrium framework. The privacy regime is determined by the platform who would prefer no privacy (i.e., disclosure) only when the number of firms is sufficient large. In contrast, in our model, the privacy regime is exogenous and the platform always prefers no privacy. Our paper focuses on the differential impact of privacy on different types of consumers, in contrast to their focus on the platform's incentive to disclose, and is complementary to their paper in studying ad platforms and privacy.

3 Model

The general framework is as follows. A large number of infinitesimal firms compete in a market with a continuum of infinitesimal horizontally differentiated consumers. They reach consumers by placing ads on a monopoly advertising platform. Exposure to a firm's ads is necessary for consumers to become informed about its product and price.

Consumers are of two types, $i \in \{1, 2\}$, depending on the product they prefer, $j \in \{1, 2\}$. They will buy up to one unit of one of the products. Both products are produced by a large number of infinitesimal firms; we call the firms producing product 1 type-1 firms and the firms producing product 2 type-2 firms. Horizontal product differentiation is captured in the following utility function:

$$u_{ij} = \begin{cases} u & \text{if} \quad i = j \\ \beta_i u & \text{if} \quad i \neq j \end{cases}$$

with $u > 0$ and $0 \leq \beta_i < 1$. In other words, a type-1 consumer is willing to pay u for a unit of product 1, but is only willing to pay $\beta_1 u < u$ for a unit of product 2; similarly, a type-2 consumer is willing to pay u for a unit of product 2, but is only willing to pay $\beta_2 u$ for a unit of product 1. The fraction of type i consumers in the population is $\gamma_i \in (0,1)$. There is no fixed cost of production, but there is a marginal cost of production $c \in (0,u)$, which is the same for both products.

To reduce the number of case distinctions and to simplify the analysis, we will assume that $\beta_1 \leq c/u < \beta_2$. That is, type-2 consumers are less picky than type-1 consumers, who are picky enough that product 2 firms would not find it cost-effective to serve them. As mentioned in the introduction, some heterogeneity among consumers is necessary for people to have different privacy preferences. In our model, the choosiness of consumers is that heterogeneity. While type-2 consumers view product 1 as a good substitute for product 2, type-1 consumers view product 2 as a poor substitute for product 1. Define $\rho \in (0,1)$ as the *ratio of total surpluses* when selling type-1 products to type-2 consumers versus selling type-2 products to type-2 consumers:

$$\rho \equiv \frac{\beta_2 u - c}{u - c}. \tag{1}$$

Given our assumption that $\beta_2 > c/u$, $\rho \in (0,1)$ and increasing in β_2.

Ads in our model are informative and have probabilistic reach, a framework pioneered by Butters (1977) and followed by virtually the entire informative advertising literature, including: Grossman and Shapiro (1984), Tirole (1988), Stegeman (1991), and Bergemann and Bonatti (2011, Section 7.3.2). Under this advertising technology, if a block A of ads is sent to *a unit measure* of consumers, the fraction of consumers which receives an ad is $1 - e^{-A}$.[7] The cost of advertising is linear, based on a price per unit, b, set by the platform, which may vary by product and by consumer type targeted if such targeting is possible.

The targetability of ads depends on privacy mode. We consider two privacy modes: no privacy (NP) and full privacy (P). Under NP the platform observes consumers' types, and can target ads to particular consumer types; advertisers can choose to target either or both types, and if the latter, they can customize their offers for different types. Under P the platform cannot identify consumers' types, making it impossible to target ads to specific types; both consumer types will necessarily see the same offer from a given firm. Consumers do not incur any disutility from targeted or non-targeted ads *per se*, nor do they have any intrinsic preference for privacy (Becker, 1980; Lin, 2022). To the extent they prefer privacy, then, it is because they expect a higher equilibrium consumer surplus—the difference between their reservation utility and the equilibrium price they pay—under privacy.

We assume that the ad platform sets ad prices to maximize its ad revenues. Under no privacy,

[7] To see this, consider a finite economy with n consumers. An ad falls randomly on these n consumers, so each consumer has $1/n$ chance of being hit by that ad. If firms send A units of ads per consumer, i.e., An total number of ads, then any given consumer observes none of the An ads with probability $(1-1/n)^{An}$. This probability converges to e^{-A} as n converges to infinity because $(1-1/n)^{An} = (1+1/(n-1))^{-An}$ and $\lim_{n\to\infty}(1+1/n)^n = e$.

ad prices may differ across product types as well as the consumer types they target. Under full privacy, ads are randomly sent to all consumers and hence ad prices can only differ across product types. We assume that the platform moves first setting ad prices; the firms then decide how many ads to buy and what prices to advertise. If a consumer receives only one ad she will buy the product featured in the ad if and only if it yields a positive consumer surplus; if she receives multiple ads, she compares their respective offers and chooses the firm offering the highest positive consumer surplus.

Our methodology going forward will be to solve for the equilibrium in each privacy mode, and then compare the resulting equilibria. Solving for each equilibrium will itself take two steps. In step one, we take the platform's ad prices as given and solve for the competitive equilibrium in the product market. This equilibrium will be defined by a set of advertising functions describing how many units of ads to send at every price; these advertising functions will be best responses to each other, i.e., form a Nash equilibrium. In step two, the platform, anticipating the equilibrium responses of firms, will choose ad prices to maximizes its ad revenue. After deriving the equilibrium under each privacy mode, we will compare equilibrium product-market outcomes and consumer welfare in the two privacy modes.

4 When ad prices are exogenous

To highlight the role of the platform's market power in affecting consumers' attitude towards privacy, we first consider a benchmark setting where ad prices are exogenous to the privacy regime—as would be the case, for example, if the ad market were perfectly competitive (which we know it is not), or if ad prices were determined solely by supply-side considerations (which, again, is unlikely). As noted above, much of the literature assumes exogenous ad prices. For that reason alone this analysis is of independent interest—we would like to know how exogenous ad prices plays out in our setting. However, the more important consideration is that it serves to demonstrate how hard it is to find a demand for privacy based solely on instrumental considerations. Consumers, it turns out, are generally better off giving up their privacy when ad prices are exogenous. In the process of establishing this result we will illustrate our analytical methodology: how we derive the equilibrium advertising-price distributions under no privacy and full privacy. This methodology will carry over substantially unchanged to the case of endogenous ad prices in Section 5.

Before we get into the derivations, we should note that the concept of equilibrium is the same in the two privacy regimes. In each case, following Stegeman (1991), it is a pair of non-decreasing continuous advertising functions, $(A_1(p), A_2(p))$, where $A_i(p)$ $(i = 1, 2)$ represents the equilibrium units of ads with prices less than or equal to p sent by type-i firms *per unit measure of the target market*. What changes is the target market. Under full privacy, for each type of firm, the target market is the entire market; under no privacy, for type-1 firms the target market is the sub-market of type-1 consumers and for type-2 firms the target market is the sub-market of type-2 consumers.[8] With that normalization in place, equilibrium advertising functions must satisfy three properties as specified in the following definition.

Definition 1 (Equilibrium under exogenous ad prices b_1 and b_2). *A pair of non-decreasing continuous functions $(A_1(p), A_2(p))$, $A_i(p) : [c + b_i, u] \to \mathbb{R}_+$, $i = 1, 2$, is an equilibrium under exogenous ad prices b_1 and b_2 if (i) $A_i(c + b_i) = 0$; (ii) $\pi_i(p; A_1, A_2) \leq 0$ for all prices $p \in [c + b_i, u]$; and (iii) $A_i(p') = A_i(p'')$ if $\pi_i(p; A_1, A_2) < 0$ for all $p \in [p', p'']$. Here $A_i(p)$ $(i = 1, 2)$ is the units of ads with prices less than or equal to p sent by type-i firms per unit measure of the target market, which is the sub-market of type-i consumers under no privacy and the entire market under full privacy.*

Equilibrium advertising functions have a domain $[c + b_i, u]$ because a price below $c + b_i$ will produce a sure loss for a type-i firm while consumers will not accept a price above u. Condition (i) requires that firms do not advertise at price $c + b_i$. If a firm did, while it breaks even on the ads accepted, it incurs a loss on the ads not accepted (which always occurs with positive probability). Condition (ii) says that the return to a marginal ad—the marginal profit—is non-positive at all feasible prices $p \in [c + b_i, u]$. Condition (iii) says that firms will not advertise prices that generate a negative marginal profit. We will rely on property (ii) to construct the equilibrium in each privacy regime.

Let the exogenous ad price for product i be $b_i > 0$. Then, if a product-i advertiser buys n units of ads it will pay nb_i to the ad platform.

4.1 No privacy

Under no privacy, the ad platform can observe consumers' preferences and classify them by type, creating two sub-markets: a type-1 sub-market and a type-2 sub-market. Advertisers can then use this classification to target consumers with sub-market-specific advertising and pricing strategies. As noted in footnote 8, even though it is possible for type-i firms to target type-$-i$ consumers, in equilibrium this cannot happen. Hence, to save notation, we will simply assume without loss of generality that under no privacy firms only send ads to consumers with matched preferences. Each sub-market then is a self-contained Butters (1977) economy.

An equilibrium under no privacy is a pair of non-decreasing continuous functions, $\left(A_1^{NP}(p), A_2^{NP}(p)\right)$, where $A_i^{NP}(p)$ $(i = 1, 2)$ represents the equilibrium units of ads with prices less than or equal to p sent by type-i firms to sub-market i per unit measure of type-i consumers, satisfying the three properties specified in Definition 1. To construct the equilibrium we use property (ii). That is, for $\left(A_1^{NP}(\cdot), A_2^{NP}(\cdot)\right)$ to be an equilibrium pair of advertising functions under no

[8] While it is possible under no privacy for type-1 firms to target type-2 consumers and type-2 firms to target type-1 consumers, this can't happen in equilibrium. To see this, suppose to the contrary that type $-i$ firms sell to sub-market i in equilibrium. Then, by definition of equilibrium (see below), those firms must be breaking even. However, this means that type-i firms must be earning positive profits—because they have a competitive advantage selling to type-i consumers—which contradicts the definition of equilibrium.

privacy, the marginal profit of sending an additional unit of ad of product i in sub-market i at any price must be non-positive. Suppose a type-i firm tries sending Z additional ad units (per unit measure of type-i consumers) with price p into sub-market i. Its additional profit will be

$$\Pi_i\left(Z; p, A_1^{NP}, A_2^{NP}\right) = \left(1 - e^{-Z}\right) e^{-A_i^{NP}(p)} (p - c) - b_i Z.$$

Here $e^{-A_i^{NP}(p)}$ is the *fraction* of type-i consumers who do not receive any matched ads with prices less than or equal to p in the putative equilibrium, and $1 - e^{-Z}$ is the *fraction* of type-i consumers who receive one of the Z additional ads. Hence, $\left(1 - e^{-Z}\right) e^{-A_i^{NP}(p)}$ is the fraction of type-i consumers who are receiving the new ads with price p but did not receive any ads with price lower than p in the putative equilibrium. The firm makes a sale at price p to these consumers, pays a marginal cost of c, and incurs an extra ad expenditure of $b_i Z$. The marginal profit generated by an additional unit of advertising at price p is then

$$\pi_i^{NP}\left(p; A_1^{NP}, A_2^{NP}\right) \equiv \frac{\partial \Pi_i^{NP}\left(Z; p, A_1^{NP}, A_2^{NP}\right)}{\partial Z}|_{Z=0} = e^{-A_i^{NP}(p)} (p - c) - b_i.$$

In equilibrium this marginal profit must be zero. The unique equilibrium $\left(A_1^{NP}(p), A_2^{NP}(p)\right)$ is given by

$$A_i^{NP}(p) = \ln(p - c) - \ln b_i \tag{2}$$

for all $p \in [c+b_i, u]$ and $i = 1, 2$.[9] Note that the density of the advertising function, $\partial A_i^{NP}(p)/\partial p = 1/(p - c)$, is decreasing in product price but independent of ad price. An increase in ad price b_i, by raising the lower bound of the advertising function's support, would therefore increase the average advertised price for product-i.

[9] To see that this is indeed the unique equilibrium, note that by condition (ii), for all $p \in [c + b_i, u]$, $\pi_i^{NP}\left(p; A_1^{NP}, A_2^{NP}\right) = e^{-A_i^{NP}(p)} (p - c) - b_i \leq 0$. Hence, $A_i^{NP}(p) \geq \ln(p - c) - \ln b_i$ for all i. It suffices to argue that this inequality must hold with equality in equilibrium. Suppose by contradiction that $A_i^{NP}(\hat{p}) > \ln(\hat{p} - c) - \ln b_i$ for some $\hat{p} \in [c + b_i, u]$. Since $A_i^{NP}(p)$ is continuous, there must exist an interval containing $\hat{p}$ such that $A_i^{NP}(p) > \ln(p - c) - \ln b_i$ for all p in that interval. Let (p', p'') denote the maximal such interval. It follows that, for all $p \in (p', p'')$, $\pi_i^{NP}\left(p; A_1^{NP}, A_2^{NP}\right) < 0$; hence by equilibrium condition (iii) and continuity, $A_i^{NP}(p') = A_i^{NP}(p'')$. Since $A_i^{NP}(c + b_i) = 0 = \ln(c + b_i - c) - \ln b_i$ and interval (p', p'') is maximal, we must have $A_i^{NP}(p') = A_i^{NP}(p'') = \ln(p' - c) - \ln b_i$. It follows that

$$\begin{aligned}
\pi_i^{NP}\left(p''; A_1^{NP}, A_2^{NP}\right) &= e^{-A_i^{NP}(p'')} \left(p'' - c\right) - b_i \\
&= e^{-A_i^{NP}(p')} \left(p' - c\right) - b_i + e^{-A_i^{NP}(p')} \left(p'' - p'\right) \\
&= e^{-A_i^{NP}(p')} \left(p'' - p'\right) \\
&> 0,
\end{aligned}$$

which contradicts condition (ii).

4.2 Full privacy

Under full privacy platforms cannot identify consumers' types, so advertisers cannot implement sub-market-specific marketing strategies. Instead, their strategies will have to treat the entire market as a single entity (while recognizing the heterogeneity underneath). Since ads for each product will descend randomly on the market, each consumer will have a positive probability of three types of ad exposure: (i) seeing ads only for her preferred product; (ii) seeing ads only for a non-preferred product, and (iii) seeing ads for both preferred and non-preferred products. A type-1 consumer receiving ads from both types of firms will simply ignore the ads for type-2 products given her picky preferences. However, a type-2 consumer in the same situation will consider both products; she will buy a type-2 product if and only if $u - p_2 \geq \beta_2 u - p_1$ where p_1 and p_2 are the lowest type-1 and type-2 prices received by the consumer.

Let $\left(A_1^P(p), A_2^P(p)\right)$ be an equilibrium under full privacy. Since each firm's target market is the entire market now, $A_i^P(p)$ denotes the units of ads with prices less than or equal to p sent by type-i firms *per unit measure of all consumers*. Note that the total amount of type-i ads with prices no higher than p received by type-j consumers is $\gamma_j A_i^P(p)$, and hence the amount of type-i ads with prices no higher than p received by a unit measure of type-j consumers is also $A_i^P(p)$. The fraction of consumers of either type who do not receive a type-i ad with price less than or equal to p is $e^{-A_i^P(p)}$, and the fraction of consumers of either type who receive at least one unit of a type-i ad with price less than or equal to p is $1 - e^{-A_i^P(p)}$.

Denote the lowest and highest prices charged by type-i firms in equilibrium by $\underline{p}_i$ and $\bar{p}_i$, respectively. It is easy to see that $\underline{p}_i \geq c + b_i$. By definition of $\underline{p}_1$, if a type-2 firm charges a price $p \leq \underline{p}_1 + (1-\beta_2)u$, it faces competition from type-2 firms only; but if it charges $p \geq \underline{p}_1 + (1-\beta_2)u$, it faces competition from both types of firms.

$$-\underset{\underline{p}_2}{|}\overbrace{-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-}^{\text{competition from type-2 firms only}}\underset{\underline{p}_1+(1-\beta_2)u}{|}\overbrace{-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-}^{\text{competition from all firms}}\underset{u}{|}-\,- \quad > \quad \text{price of a type-2 firm}$$

Therefore:

- If a type-2 firm sends Z additional units of ads at price $p \in [\underline{p}_1 + (1-\beta_2)u, u]$, those additional ads will make a sale only with those type-2 consumers who receive neither ads of product 2 with prices lower than p nor ads of product 1 with prices lower than $p - (1-\beta_2)u$, and the fraction of these consumers is $\gamma_2 e^{-A_1^P(p-(1-\beta_2)u)} e^{-A_2^P(p)}$. Since the fraction of all consumers receiving one of those additional ads is $\left(1 - e^{-Z}\right)$, its net profit will be

$$\Pi_2^P\left(Z; p, A_1^P, A_2^P\right) = \gamma_2 e^{-A_1^P(p-(1-\beta_2)u)} e^{-A_2^P(p)} \left(1 - e^{-Z}\right)(p - c) - b_2 Z.$$

- If a type-2 firm sends Z additional units of ads at price $p \in \left[\underline{p}_2, \underline{p}_1 + (1-\beta_2)u\right]$, no prices advertised by type-1 firms would be competitive against p. Hence, its net profit will be

$$\Pi_2^P\left(Z; p, A_1^P, A_2^P\right) = \gamma_2 e^{-A_2^P(p)}\left(1-e^{-Z}\right)(p-c) - b_2 Z.$$

In short, a type-2 firm's marginal profit from sending an additional unit of advertising at price p is

$$\pi_2^P\left(p; A_1^P, A_2^P\right) = \begin{cases} \gamma_2 e^{-A_2^P(p)}(p-c) - b_2 & \text{if} \quad p \in \left[\underline{p}_2, \underline{p}_1 + (1-\beta_2)u\right] \\ \gamma_2 e^{-A_1^P(p-(1-\beta_2)u)} e^{-A_2^P(p)}(p-c) - b_2 & \text{if} \quad p \in \left(\underline{p}_1 + (1-\beta_2)u, u\right] \end{cases} \tag{3}$$

For type-1 firms, a price $p \leq \beta_2 u$ can appeal to both types of consumers, but a price $p > \beta_2 u$ can attract type-1 consumers only.

$$-\overbrace{|_{\underline{p}_1} - - - - - - - - - - - -}^{\text{appeal to all consumers}}\overbrace{|_{\beta_2 u} - - - - - - - - - - - - - |_{u}}^{\text{appeal to type-1 consumers only}} - - \;>\quad \text{price of a type-1 firm}$$

Hence the marginal profit of a type-1 firm sending an additional ad at price p is

$$\pi_1^P\left(p; A_1^P, A_2^P\right) = \begin{cases} \left(\gamma_1 + \gamma_2 e^{-A_2^P(p+(1-\beta_2)u)}\right) e^{-A_1^P(p)}(p-c) - b_1 & \text{if} \quad p \in [\underline{p}_1, \beta_2 u] \\ \gamma_1 e^{-A_1^P(p)}(p-c) - b_1 & \text{if} \quad p \in (\beta_2 u, u] \end{cases} \tag{4}$$

Further progress depends on recognizing that the supports of the equilibrium advertising functions can take different forms depending on how the two types of firms compete with each other in equilibrium. There are three possibilities: (i) equilibrium with *within-product* competition only, (ii) equilibrium with *cross-product* competition, and (iii) equilibrium with *exclusion* (only one product is sold). We label these "W equilibrium," "C equilibrium," and "E equilibrium," respectively [10].

W equilibrium (within-product competition only). In this type of equilibrium type-2 firms compete against each other only (and, of course, type-1 firms compete against each other only). This can occur in two scenarios: when type-2 consumers are also picky (small β_2), or when β_2 is not small but the ad price b_1 is high relative to the ad price b_2. In the latter scenario, type-1 firms are essentially being forced to charge high prices by the high price of advertising making their product uncompetitive for type-2 consumers.[11] In the former scenario, the discount required for type-1 firms to sell to type-2 consumers is too large. Regardless, a W-equilibrium is

[10] The formal derivations of these three types of equilibrium are given in Section 7.1

[11] As we will see in Section 5, this scenario will play a major role when ad prices are chosen by the platform. The platform will raise b_1 strategically to avoid cross-product competition.

characterized by: the support of type-1 firms' advertised prices is $[\underline{p}_1, u]$ with $\underline{p}_1 \geq \beta_2 u$ and the support of type-2 firms' advertised prices is $[\underline{p}_2, u]$.

In a W-equilibrium, then, the zero-marginal profit condition $\pi_i^P\left(p; A_1^P, A_2^P\right) = 0$ for any price in $[\underline{p}_i, u]$ implies, for $i = 1, 2$, $\underline{p}_i = c + b_i/\gamma_i$ and

$$A_i^P(p) = \ln\gamma_i + \ln(p - c) - \ln b_i \quad \text{with} \quad p \in [c + b_i/\gamma_i, u]\,.$$

Once again, an increase in b_i reduces advertising quantity for low prices of product i and hence increases its average price.

C equilibrium (cross-product competition). In this type of equilibrium, some type-2 consumers buy product 1 in equilibrium. For this to occur β_2 has to be large so that the price discount needed for product 1 to attract type-2 consumers is small. Nevertheless, the two types of firms do not compete head-to-head with each other: the price offers on product 2 will always be more attractive to type-2 consumers than the price offers on product 1. The only way type-1 firms sell to type-2 consumers is when they receive no offers of product 2. Hence, when cross-selling to type-2 consumers, type-1 firms will advertise prices between $\underline{p}_1$ and $\beta_2 u$, while the maximal price advertised by type-2 firms is $\bar{p}_2 = \underline{p}_1 + (1 - \beta_2)u$.

Since type-1 firms with $p < \beta_2 u$ can attract both types of consumers while type-1 firms with $p > \beta_2 u$ can attract only type-1 consumers, there is a discrete drop in the demand for product 1 at $p = \beta_2 u$. Due to this discontinuity, type-1 firms will not advertise any prices immediately above $\beta_2 u$. However, they may start advertising at some price $\tilde{u} > \beta_2 u$. That is, the set of equilibrium prices advertised by type-1 firms takes the form $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$. For large β_2 price $\tilde{u}$ may surpass u; in this case the upper interval collapses, and the set of equilibrium prices advertised by type-1 firms becomes $[\underline{p}_1, \beta_2 u]$. In the former case (which we label as "C1"), type-1 firms advertise both high prices ($p > \beta_2 u$) to attract type-1 consumers only and low prices ($p \leq \beta_2 u$) to attract both types of consumers. In the latter case (which we label as "C2"), type-1 firms only advertise prices that are low enough to attract both types of consumers.

In both C1- and C2-equilibrium, the zero-marginal profit condition $\pi_2^P\left(p; A_1^P, A_2^P\right) = 0$ for any price $p \in \left[\underline{p}_2, \underline{p}_1 + (1 - \beta_2)u\right]$ implies that $\underline{p}_2 = c + b_2/\gamma_2$ and

$$A_2^P(p) = \ln\gamma_2 + \ln(p - c) - \ln b_2 \quad \text{with} \quad p \in \left[c + b_2/\gamma_2, \underline{p}_1 + (1 - \beta_2)u\right].$$

The zero-marginal profit condition $\pi_1^P\left(p; A_1^P, A_2^P\right) = 0$ for any equilibrium price $p \in [\underline{p}_1, \beta_2 u]$ implies that, for both C1-equilibrium and C2-equilibrium,

$$A_1^P(p) = \ln\left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right) + \ln(p - c) - \ln b_1 \quad \text{if} \quad p \in [\underline{p}_1, \beta_2 u]$$

where $\underline{p}_1 \geq c + b_1$ is implicitly determined by $A_1^P(\underline{p}_1) = 0$. In a C1-equilibrium, the zero-profit condition $\pi_1^P\left(p; A_1^P, A_2^P\right) = 0$ for any equilibrium price $p \in (\beta_2 u, \tilde{u}) \cup [\tilde{u}, u]$ further implies that

$$A_1^P(p) = \begin{cases} \ln\left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right) + \ln(\beta_2 u - c) - \ln b_1 & \text{if} \quad p \in (\beta_2 u, \tilde{u}) \\ \ln \gamma_1 + \ln(p - c) - \ln b_1 & \text{if} \quad p \in [\tilde{u}, u] \end{cases}$$

where $\tilde{u}$ is determined by $A_1^P(\beta_2 u) = A_1^P(\tilde{u})$.

When there is cross-selling, an increase in ad price for one product will affect the advertising quantity for both products. As we show in Section 7.4, when b_1 increases, the average product price for type-1 consumers increases, while the number of "effective" ads received by type-2 consumers decreases. When b_2 increases, the average "effective" product price for type-2 consumers increases, while the average product price for type-1 consumers decreases.

E equilibrium (exclusion). In this type of equilibrium only one product is advertised in the market. When that product is product 1 (E1), the support of $A_2^P(p)$ is empty and the support of $A_1^P(p)$ is either $[\underline{p}_1, u]$ (mimicking the W-type equilibrium) or $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ (mimicking the C1-type equilibrium) or $[\underline{p}_1, \beta_2 u]$ (mimicking the C2-type equilibrium). When that product is product 2 (E2), the support of $A_1^P(p)$ is empty and the support of $A_2^P(p)$ is $[\underline{p}_2, u]$. In an E equilibrium where only type i product is advertised, once again an increase in b_i reduces advertising quantity for low prices of product i and hence increases its average price.

Before moving on to the analysis of consumer welfare, we first list the equilibrium properties of market size on both advertising and product market. Specifically, we focus on the total number of "matched" ads between type-i product and type-i consumers for the advertising market, and the total number of sales for each type product for the product market.

4.3 Comparing no privacy and full privacy

We compare product-market outcomes under no privacy and full privacy on three dimensions: (i) advertising and sales volume, (ii) consumer welfare, and (iii) comparative statics with respect to ad prices. The last of these will play a crucial rule in Section 5 when we examine the advertising platform's choice of ad prices.

Advertising and sales volume. Under no privacy, total quantity of advertising is the same as the total quantity of matched advertising—type-i ads reaching type-i consumers ($i = 1, 2$). However, under full privacy, in general, total advertising quantity will exceed total matched advertising quantity: some product ads will invariably land on the "wrong" type—consumers who don't prefer that product. The interesting question, then, is whether the quantity of matched ads is higher or lower under full privacy. Proposition 1 states that even on this narrower metric, full privacy leads to lower advertising volume. This translates directly to lower sales volume in the W and E2 equilibria. However, in the C and E1 equilibria, this is not necessarily the case.

Proposition 1. *Compared to no privacy, under full privacy:*

- *Total units of matched ads is lower.*
- *Total sales is lower in the W and E2 equilibria; however, in the C and E1 equilibria, total sales could be higher for low b_1 and high b_2.*
- *The average advertised product price is higher in the W and E2 equilibria, and lower in E1 and C1 equilibria; in the C2 equilibrium, average price could be higher or lower.*

The mechanical effect of full privacy is that some ads will necessarily be mistargeted. This makes advertising less productive, inducing firms to advertise less. Type-1 firms are less affected because they can cross-sell to type-2 consumers; however, since the mismatched type-2 consumers always have a lower total surplus than matched type-1 consumers, in equilibrium the number of matched ads will necessarily go down. The cross-selling opportunity of type-1 firms to type-2 consumers also explains why total sales may actually go up under privacy. Under no privacy, type-1 firms cannot compete with type-2 firms for type-2 consumers as they offer an "inferior" product. Under full privacy, however, type-1 firms may develop a competitive advantage over type-2 firms due to their greater advertising productivity. The resulting increase in demand for type-1 products may increase total sales when b_1 is low and b_2 is high.

Consumer welfare. The proposition below summarizes the main welfare result.

Proposition 2 (Consumer welfare under exogenous ad prices). *When ad prices are exogenous and $b_1/b_2 \geq \rho$, both types of consumers strictly prefer no privacy to full privacy.*

Proposition 2 says that in general consumers will be better off without privacy when ad price are exogenous. Why might this be so? First, under no privacy advertising is more productive. Each type of firm is targeting the consumers most receptive to its product, and, even though there is always a positive probability of ads not reaching anyone—that type of wastage can never be ruled out—the ads that do reach someone are reaching the "right someone." For example, type-2 firms are not wasting money sending ads to type-1 consumers who will never consider their product. Second, by making ads more productive, more ads can be sent, expanding the market.

By contrast, under full privacy, all firms' advertising becomes less productive, but type-2 firms' especially so. This is because while type-1 firms' ads can potentially appeal to type-2 consumers, type-2 firms ads can't appeal to type-1 consumers. So there are more wasted ads from type-2 firms than type-1 firms. Less productive ads means less advertising, which reduces consumption across the board. However, type-1 firms have another revenue source potentially: type-2 consumers not reached by type-2 firms' ads. This lowers the average price paid by type-2 consumers (as well as type-1 consumers). The import of the proposition is that if $b_1/b_2 \geq \rho$, then the second effect is insufficient to compensate for the first, making full privacy strictly worse for all consumers than no privacy.

The condition in question is a weak restriction on ad prices. It has two interpretations. First, it may be seen as ruling out the possibility of ad prices being so lopsided in favor of type-1 products that no type-2 consumer prefers a type-2 firm's offer to a type-1 firm's offer (when they receive both offers). In other words, $b_1/b_2 \geq \rho$ ensures that type-2 consumers prefer price offers from their favored product to the price offers from their unfavored product. The second interpretation comes from rewriting the condition as $(u-c)/b_2 \geq (\beta_2 u - c)/b_1$. Now the left-hand side is the *total* surplus from the type-2 sub-market when those consumers are buying type-2 products divided by the unit advertising cost for those products; the right-hand side is the *total* surplus from the type-2 sub-market when those consumers are buying type-1 products divided by the advertising cost for those products. So the condition amounts to saying that if the *net* total surplus from type-2 consumers buying their preferred product—net of advertising cost—is higher than the *net* total surplus from type-2 consumers buying their non-preferred product, then all consumers will prefer no privacy to full privacy under exogenous prices.

Comparative statics with respect to ad prices. Proposition 3 below summarizes the effect of ad prices on the average price paid by different types of consumers and the number of "relevant" ads they see in the various equilibria. By "relevant ads" we mean ads that a consumer type will consider buying from. For type-1 consumers, relevant ads are ads for type-1 products; for type-2 consumers, relevant ads are ads for type-2 products and ads for type-1 products with prices below $\beta_2 u$.

Proposition 3 (Comparative statics of ad prices). *When ad price b_i ($i = 1, 2$) increases, it has the following effects on the average price paid by each type of consumer and the amount of relevant ads they see in the different product-market equilibria:*

1. ***No privacy:*** *Average product price will increase and the amount of relevant ads will decrease for type-i consumers only.*

2. ***Full Privacy:***

   - ***W or E equilibrium:*** *Average product price will increase and the amount of relevant ads will decrease for type-i consumers only.*
   - ***C1 equilibrium:*** *Average product price will increase and the amount of relevant ads will decrease for type-i consumers. In addition, an increase in b_1 will decrease the amount of relevant ads for type-2 consumers, while an increase in b_2 will not affect the amount of relevant ads, but decrease the average product price for type-1 consumers.*
   - ***C2 equilibrium:*** *Average product price will increase and the amount of relevant ads will decrease for type-i consumers. In addition, an increase in b_1 will decrease the amount of relevant ads for type-2 consumers, while an increase in b_2 will increase the amount of relevant ads and decrease the average product price for type-1 consumers.*

In general, a marginal increase in ad price b_i will decrease ad demand from type-i firms. This leads to fewer relevant type-i ads and pushes up the average product price. This explains the comparative statics under no privacy, and in the W or E equilibrium under full privacy. In the C1 and C2 equilibria under full privacy, in addition to these "own" effects, cross-effects may emerge. First, type-2 consumers may consider type-1 ads with low price offers as "relevant." Second, the advertising behavior of type-2 firms will affect the probability of selling for type-1 firms as they both compete for flexible consumers. As a result, in general when cross-selling exists, a marginal increase in ad price b_2 will increase the ad demand from type-1 firms which leads to larger number of type-1 ads and lowers the average product price for type-1 consumers. A marginal increase in ad price b_1 will increase the ad demand from type-2 firms which leads to larger number of type-2 ads, but this increase cannot compensate the loss of relevant type-1 ads which leads to fewer total relevant ads for type-2 consumers.

Proposition 3 only speaks to marginal changes in ad prices, i.e., changes in ad prices that don't alter the nature of equilibrium itself. Non-marginal changes in ad prices may change the nature of product-market equilibrium itself. This consideration plays an important role when examining the monopoly platform's choice of ad prices. Effectively, the platform will be deciding what sort of competitive equilibrium it wants to see in the product market under privacy. It will do so taking into account γ_1, the proportion of picky consumers, and β_2, the flexibility of flexible consumers. In general, a large b_1, by pushing up the product prices advertised by type-1 firms, will induce these firms to focus on their own sub-market, i.e., induce the W equilibrium. A smaller b_1 or a larger b_2 will give type-1 firms opportunities to offer lower product prices and cross-sell to type-2 consumers, inducing a C equilibrium. Finally, when a particular ad price b_i is large enough, the corresponding product may be excluded from the market (E equilibrium).

5 When ad prices are chosen by the advertising platform

When ad prices are chosen by the monopoly platform to maximize its ad revenue, those prices can vary by product and privacy mode, as well as by consumer type under no privacy. However, it is easy to see that it will never be optimal for the platform to set ad prices such that type-1 firms sell to type-2 consumers under no privacy.[12] The reason is, under no privacy, both types of firms' ads are equally productive, and type-2 firms have a competitive advantage selling to type-2 consumers. The platform has no interest in upsetting a level playing field. Therefore, in what follows, we will simply assume that under no privacy the platform directs product-i ads to type-i consumers only.

The platform serves as a Stackelberg leader, setting ad prices first, to which the product-market firms react by choosing their advertising-price functions. As before, equilibrium in the product market will be a pair of non-decreasing continuous functions, $(A_1(p), A_2(p))$, with $A_i(p)$

[12]The other possibility of type-2 firms selling to type-1 consumers is ruled out by the pickiness of type-1 consumers.

($i = 1, 2$) representing the equilibrium units of ads with prices less than or equal to p sent by type-i firms per unit measure of the target market. The target market, of course, changes from no privacy to full privacy. In the former, type-i firms will target type-i consumers only; in the latter, type-i firms will target the entire market. The definition of equilibrium under endogenous ad prices below is the same as the definition of equilibrium under exogenous ad prices with the sole difference that there is now an added requirement that ad prices maximize the platform's ad revenue anticipating the product-market equilibrium to follow.

Definition 2 (Equilibrium under endogenous ad prices). *A pair of ad prices* (b_1, b_2) *and a pair of non-decreasing continuous advertising functions* $(A_1(p), A_2(p))$, $A_i(p) : [c+b_i, u] \to \mathbb{R}_+$, $i = 1, 2$, *form an equilibrium under endogenous ad prices if the following conditions hold: (i)* $A_i(c + b_i) = 0$*; (ii)* $\pi_i(p; A_1, A_2) \leq 0$ *for all prices* $p \in [c + b_i, u]$*; (iii)* $A_i(p') = A_i(p'')$ *if* $\pi_i(p; A_1, A_2) < 0$ *for all* $p \in [p', p'']$*; and (iv)* (b_1, b_2) *maximizes ad revenue for the monopoly platform anticipating the product-market equilibrium to follow. Here* $A_i(p)$ $(i = 1, 2)$ *is the units of ads with prices less than or equal to p sent by type-i firms per unit measure of the target market, which is the sub-market of type-i consumers under no privacy and the entire market under full privacy.*

In what follows, we will first characterize the equilibrium under no privacy. This is fairly straightforward; different types of firms compete in different sub-markets and we can follow the same procedure as in Section 4 to derive the product-market equilibrium advertising functions for given ad prices. Those advertising functions define the demand function for ads; the ad platform will use this demand function to compute its revenue-maximizing ad prices. We then turn to the analysis of equilibrium under full privacy, which is more complicated because it is now possible for type-1 firms to sell to type-2 consumers. Once we have the equilibrium under each privacy mode, we will compare ad prices, ad volume, customer acquisition cost, and consumer welfare in the two privacy modes.

5.1 No privacy

Under no privacy, the platform can target ads of product i only to type-i consumers. The platform's optimization problem is to choose b_1^{NP} and b_2^{NP} to maximize its total ad revenue

$$R^{NP} = \gamma_1 b_1^{NP} A_1^{NP}(u) + \gamma_2 b_2^{NP} A_2^{NP}(u).$$

To solve the optimal ad prices b_1^{NP} and b_2^{NP}, we first have to derive the total demand for ads of product i, $A_i^{NP}(u)$, for given ad prices. Following the same logic as in deriving equation (2) in Section 4, we can write the equilibrium advertising function $A_i^{NP}(p)$ as

$$A_i^{NP}(p) = \ln(p - c) - \ln b_i^{NP}.$$

The total demand for ads of product i is then $A_i^{NP}(u) = \ln(u-c) - \ln b_i^{NP}$. Therefore, the total ad revenue becomes

$$R^{NP} = \gamma_1 b_1^{NP}\left[\ln(u-c) - \ln b_1^{NP}\right] + \gamma_2 b_2^{NP}\left[\ln(u-c) - \ln b_2^{NP}\right].$$

The optimal ad prices are

$$b_1^{NP} = b_2^{NP} = \frac{u-c}{e},$$

and the equilibrium advertising function for type-i firms is

$$A_i^{NP}(p) = \ln(p-c) - \ln(u-c) + 1.$$

Note that the platform sets equal ad prices for the two products, independent of sub-market size. This is because each sub-market is effectively an independent market, and the platform's incentives are aligned with those of the product-market firms; it has no incentive to distort a level playing field. This will change under certain circumstances under full privacy.

The equilibrium sales functions are

$$S_i^{NP}(p) = \gamma_i - \frac{\gamma_i}{e}\frac{u-c}{p-c}.$$

with total sales of type i products being $\gamma_i(1-1/e)$. The consumer surplus for a type-i consumer is

$$\int_{c+(u-c)/e}^{u} (u-p)\, d\left(\frac{S_i^{NP}(p)}{\gamma_i}\right) = \left(1 - \frac{2}{e}\right)(u-c).$$

The profit for the platform is

$$R^{NP} = \frac{u-c}{e}.$$

The market size of type-i product in advertising market is

$$\gamma_i A_i^{NP}(u) = \gamma_i\left(\ln(u-c) - \ln b_i^{NP}\right) = \gamma_i,$$

and the market size of type-i product in product market is

$$S_{i\to i}^{NP}(u) = \gamma_i\left(1 - \frac{b_i^{NP}}{u-c}\right) = \gamma_i\frac{e-1}{e}.$$

The consumer acquisition cost per consumer for type-i firms is

$$\frac{b_i^{NP}\gamma_i A_i^{NP}(u)}{S_{i\to i}^{NP}(u)} = \frac{u-c}{e-1}.$$

5.2 Full privacy

Under full privacy, since ads are sent randomly to all consumers, the platform's optimization problem is to choose b_1 and b_2 to maximize its ad revenue

$$R^P = b_1 A_1^P(u) + b_2 A_2^P(u).$$

Here $A_1^P(u)$ and $A_2^P(u)$ represent the total ad demand for products 1 and 2, respectively, in the product-market equilibrium for ad prices (b_1, b_2). This analysis is the same as in Section 4. As seen there, there are three types of equilibrium under full privacy: equilibrium with only within-product competition (W), equilibrium with cross-product competition (C), and equilibrium with exclusion (E). To facilitate further analysis and discussion, we divide the type-W equilibrium into two cases: one with $\underline{p}_1 > \beta_2 u$ (W1) and one with $\underline{p}_1 = \beta_2 u$ (W2). Thus there are six possibilities for equilibrium in the product market:

- W1. Equilibrium with only within-product competition and $\underline{p}_1 > \beta_2 u$: the support of $A_2^P(p)$ is $[\underline{p}_2, u]$ and the support of $A_1^P(p)$ is $[\underline{p}_1, u]$ with $\underline{p}_1 > \beta_2 u$.
- W2. Equilibrium with only within-product competition and $\underline{p}_1 = \beta_2 u$: the support of $A_2^P(p)$ is $[\underline{p}_2, u]$ and the support of $A_1^P(p)$ is $[\underline{p}_1, u]$ with $\underline{p}_1 = \beta_2 u$.
- C1. Equilibrium with cross-product competition and $\bar{p}_1 = u$: the support of $A_2^P(p)$ is $[\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$ and the support of $A_1^P(p)$ is $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ with $\underline{p}_1 < \beta_2 u < \tilde{u} < u$.
- C2. Equilibrium with cross-product competition and $\bar{p}_1 = \beta_2 u$: the support of $A_2^P(p)$ is $[\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$ and the support of $A_1^P(p)$ is $[\underline{p}_1, \beta_2 u]$ with $\underline{p}_1 < \beta_2 u$.
- E1. Equilibrium with only product 1 is advertised: the support of $A_1^P(p)$ is either $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ or $[\underline{p}_1, \beta_2 u]$, and the support of $A_2^P(p)$ is $\emptyset$.
- E2. Equilibrium with only product 2 is advertised: the support of $A_1^P(p)$ is $\emptyset$, and the support of $A_2^P(p)$ is $[\underline{p}_2, u]$.

The platform's ad revenue can be rewritten as

$$R^P = \begin{cases} b_1 A_1^P(u) + b_2 A_2^P(u) & \text{for cases W1 and W2} \\ b_1 A_1^P(u) + b_2 A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right) & \text{for case C1} \\ b_1 A_1^P(\beta_2 u) + b_2 A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right) & \text{for case C2} \\ b_1 A_1^P(u) & \text{for case E1} \\ b_2 A_2^P(u) & \text{for case E2} \end{cases}$$

The key objects are $A_1^P(u)$, $A_1^P(\beta_2 u)$, $A_2^P(u)$, $A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)$, and $\underline{p}_1$. Depending on equilibrium type, these objects must satisfy a number of constraints (besides $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$).

W1 and W2 equilibria. A necessary constraint for a type-W equilibrium is

$$\underline{p}_1 \geq \beta_2 u \tag{5}$$

If constraint (5) is slack or equivalently $\pi_1^P(\beta u; A_1^P, A_2^P) < 0$, we are in a type-W1 equilibrium. If constraint (5) is binding or equivalently $\pi_1^P(\beta u; A_1^P, A_2^P) = 0$, we are in a type-W2 equilibrium.The zero-profit conditions needed to derive the total ad demands are

$$\pi_1^P(u; A_1^P, A_2^P) = \pi_2^P(u; A_1^P, A_2^P) = \pi_1^P(\underline{p}_1; A_1^P, A_2^P) = 0.$$

C1 and C2 equilibria. A necessary condition for a type-C equilibrium is $\underline{p}_1 < \beta_2 u$, or equivalently,

$$A_1^P(\beta_2 u) > 0. \tag{6}$$

Another necessary constraint is

$$A_1^P(u) \geq A_1^P(\beta_2 u). \tag{7}$$

If constraint (7) is slack, we are in a type-C1 equilibrium. If constraint (7) is binding, we are in a type-C2 equilibrium. For case C1, the zero-profit conditions needed to derive $A_1^P(u)$ and $A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)$ are

$$\pi_1^P(u; A_1^P, A_2^P) = \pi_2^P(\underline{p}_1 + (1-\beta_2)u; A_1^P, A_2^P) = \pi_1^P(\underline{p}_1; A_1^P, A_2^P) = 0.$$

For case C2, we use the zero-profit conditions to derive $A_1^P(\beta_2 u)$ and $A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)$:

$$\pi_1^P(\beta_2 u; A_1^P, A_2^P) = \pi_2^P(\underline{p}_1 + (1-\beta_2)u; A_1^P, A_2^P) = \pi_1^P(\underline{p}_1; A_1^P, A_2^P) = 0.$$

along with the additional constraint

$$\pi_1^P(u; A_1^P, A_2^P) \leq 0.$$

If β_2 is large (but γ_1 is not too large), it may be too costly for the platform to prevent cross-product competition for type-2 consumers.

E1 and E2 equilibria. For the equilibrium with only product 1 advertised (E1), $A_1(u) > 0$ and $A_2(u) = 0$. The zero-profit conditions needed to derive the ad revenue are either

$$\pi_1^P(u; A_1^P, A_2^P) = 0,$$

or

$$\pi_1^P(\beta_2 u; A_1^P, A_2^P) = 0 \text{ and } \pi_1^P(u; A_1^P, A_2^P) \leq 0.$$

For the equilibrium with only product 2 advertised (E2), $A_1(u) = 0$ and $A_2(u) > 0$. The zero-profit conditions needed to derive the ad revenue are

$$\pi_2^P(u; A_1^P, A_2^P) = 0.$$

To analyze the platform's choice of (b_1, b_2) the following definitions are useful. We define *welfare-neutral ad prices* as

$$b_i^* = \frac{\gamma_i\,(u - c)}{e}. \tag{8}$$

These ad prices are nothing but the optimal ad prices under no privacy scaled to the size of each sub-market, i.e., these are the ad prices the platform would use if it were compensating firms for the lower productivity of their ads under full privacy. We call them welfare-neutral because under these prices both platform and consumers will be indifferent between the W1- equilibrium under full privacy and no privacy.

It turns out that the platform's preference for different types of product-market equilibria can be completely characterized by (ρ, γ_1). If (ρ, γ_1) is such that only one type of equilibrium can arise, then the platform's preference is clear enough. This happens, for example, when $\rho < 1/e$; then only equilibrium W1 arises. But if (ρ, γ_1) are such that multiple types of product-market equilibria can coexist, then we need to compare the platform's ad revenues under these different types of equilibria and choose the one which yields the highest ad revenue to the platform. In the Appendix we show that there exist functions $h_1(\rho)$, $h_2(\rho)$, $h_3(\rho)$, $h_4(\rho)$, $h_5(\rho)$, and $h_6(\rho)$, which demarcate areas of the parameter space in which each type of product-market equilibrium is induced. See Figure 1 below.

What considerations govern the platform's decision to induce a particular type product-market equilibrium? Consider first why the W1 equilibrium is induced. When ρ is small, i.e., β_2 is small, type-2 consumers view product 1 as a poor substitute of product 2; the price discount needed on type-1 products to appeal to type-2 consumers—$(1 - \beta_2)u$—is large. Hence type-1 firms focus on their own natural market, and so do type-2 firms. In other words, for each type of firm, their pricing behavior is the same as under no privacy. What changes, however, is their advertising behavior. Advertising is less productive under full privacy—some advertising is invariably wasted as it reaches the "wrong" consumer type. Each type of firm will want to cut back on advertising, but the type with the smaller γ_i will want to cut back more. To compensate, the platform can lower ad prices, and set them differently for the two products. It turns out that the platform can compensate perfectly by setting $b_i^P = \gamma_i(u - c)/e$—the welfare-neutral ad prices. Under these prices, both consumers and the platform are indifferent between no privacy and full privacy. For consumers this is because they consume their preferred products at the same average price as

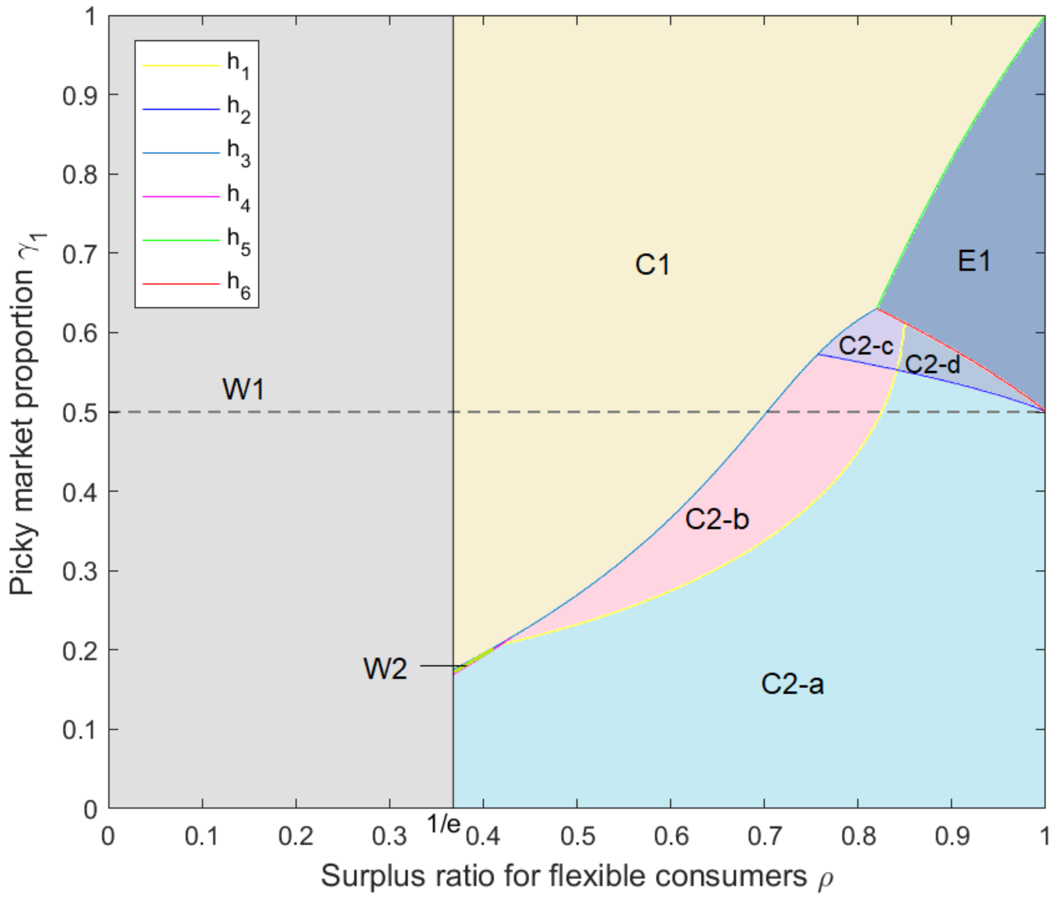

Figure 1: Platform's choice of equilibrium

under no privacy; for the platform, this is because it sells more ads at lower prices, but its revenue stays the same as under no privacy.

As ρ increases, the price discount needed for product 1 to attract type-2 consumers decreases, and type-1 firms are increasingly tempted to cross-sell to type-2 consumers. If they price below $\beta_2 u$ their product can appeal to both types of consumers, but if they price above $\beta_2 u$, they can only sell to type-1 consumers. If γ_1 is large enough—how large depends on β_2: the larger the β_2, the larger γ_1 has to be—we get equilibrium C1; if γ_1 is relatively small we get equilibrium C2. The difference between the two is that in the former type-1 firms are trying to balance two motivations—charging high prices to their own base versus charging lower prices to expand their market to include type-2 consumers—whereas in the latter, they have swung decisively in favor of market expansion.[13]

As ρ increases still further, equilibrium E1 emerges if γ_1 is large enough, Now the platform

[13]Even in C2, there is no head-to-head competition, however. Any type-2 consumer who receives ads from both type-1 and type-2 firms still prefers type-2 firms.

doesn't sell any type-2 product ads; both type-1 and type-2 consumers will be served type-1 products only. The loss in ad revenue from product 2 will be compensated for by gains in revenue on product 1 ads.

5.3 Comparing no privacy and full privacy

Having identified when each type of product-market equilibrium will arise, we are now ready to compare the equilibria under no privacy and full privacy. We will do so first on various firm-oriented criteria: advertising volume, sales volume, and customer acquisition cost. Later we will compare consumer welfare. The following proposition relies on functions g_1, g_2, and g_3, which are defined in the Appendix and illustrated in Figure 2.

Proposition 4 (Advertising volume, sales volume, and customer acquisition costs). *When ad prices are chosen by the platform, compared to no privacy, under full privacy:*

1. *In W1 equilibrium, advertising volume, sales volume, and consumer acquisition costs for both products are the same.*

2. *In W2 equilibrium, for type-2 products, advertising and sales volume are higher, and customer acquisition cost lower; for type-1 products it is exactly the opposite.*

3. *In C1 equilibrium, the total number of matched ads is lower for type-1 products and higher for type-2 products. For type-2 products, sales volume is higher, and customer acquisition cost is lower; for type-1 products, sales volume is higher when $\rho > g_1(\gamma_1)$, and customer acquisition cost is higher when $\rho > g_2(\gamma_1)$.*

4. *In C2 equilibrium, both advertising and sales volume are lower for both products, and customer acquisition cost is lower for type-2 products. For type-1 products, total number of matched ads is lower, sales volume is higher, and customer acquisition cost is lower when $\rho < g_3(\gamma_1)$.*

5. *In E1 equilibrium, type-1 products have the same number of matched ads, but sales volume is higher, and customer acquisition cost is lower. Type-2 products are excluded.*

The W1 equilibrium is induced when type-2 consumers are sufficiently picky that the platform can ignore the possibility of cross-selling; in this case the market functions the same under no privacy and full privacy. On the other hand, the W2 equilibrium is induced when the platform finds that while deterring cross-selling is still in its interest, it can only achieve that outcome by distorting ad prices. The lower ad price b_2 for type-2 firms bestows type-2 firms with lower consumer acquisition costs, and larger ad and sales volumes. The opposite holds for type-1 firms. In the C1 equilibrium, the same logic carries through for type-2 firms. However, for type-1 firms, sales volume may increase when type-2 consumers are sufficiently flexible. The consumer

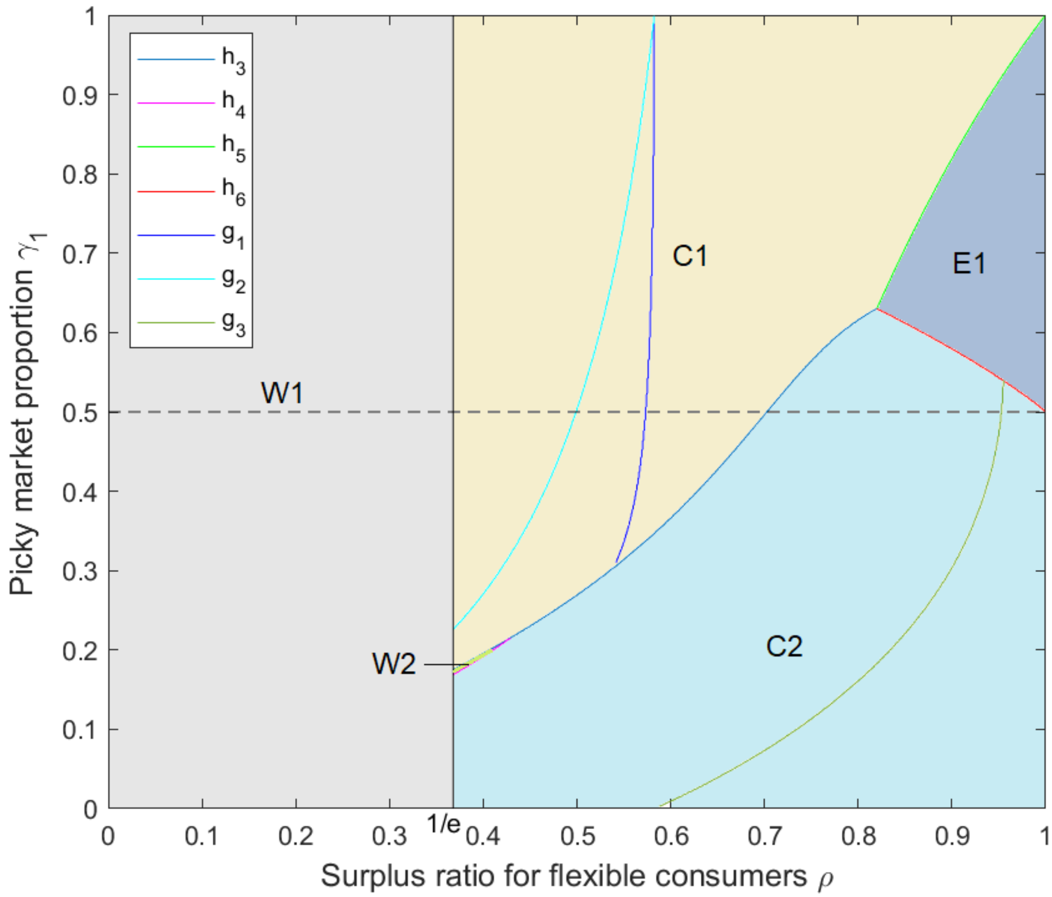

Figure 2: Illustrating Proposition 4

acquisition cost may also go down when type-2 consumers are not very flexible so that type-1 firms benefit from increased demand while the platform will not increase ad price b_1 that much in response. The C2 equilibrium is induced when type-2 consumers are quite flexible and make up a large fraction of the population. The platform finds it optimal to accommodate cross-selling by type-1 firms by letting them advertise low prices to attract type-2 consumers. This compresses the potential demand for type-2 firms. They will advertise less and sell less, but with a lower customer acquisition cost. Type-1 firms sell more even with fewer matched ads. Their customer acquisition cost could be higher under full privacy when type-2 consumers are so flexible that the platform can simply raise ad price b_1 while preserving cross-selling. In the E1 equilibrium, type-1 firms benefit from the increased demand from type-2 consumers. At the same time, since these consumers value type-1 products less than type-1 consumers, so the platform can't price b_1 too high. Now the customer acquisition cost goes down under full privacy.

Proposition 5 (Consumer welfare under endogenous ad prices). *When ad prices are cho-*

sen by the platform:

1. *If $\rho \leq \frac{1}{e}$, the platform will induce W1 equilibrium, and all consumers (as well as the platform) will be indifferent between no privacy and full privacy.*

2. *If $\rho > \frac{1}{e}$ and $\gamma_1 \in (h_4(\rho), h_5(\rho))$, the platform will induce W2 equilibrium. Type-2 consumers prefer full privacy while type-1 consumers prefer no privacy.*

3. *If $\rho > \frac{1}{e}$ and $\gamma_1 > \max\{h_3(\rho), h_5(\rho)\}$, the platform will induce C1 equilibrium. Type-2 consumers prefer full privacy while type-1 consumers prefer no privacy.*

4. *If $\rho > \frac{1}{e}$ and $\gamma_1 < \min\{h_3(\rho), h_4(\rho), h_6(\rho)\}$, the platform will induce C2 equilibrium. The welfare effect varies.*

   (a) *If $\gamma_1 < \min\{h_1(\rho), h_2(\rho)\}$, type-2 consumers prefer full privacy, while type-1 consumers prefer no privacy.*

   (b) *If $h_1(\rho) < \gamma_1 < h_2(\rho)$, both types of consumers prefer full privacy.*

   (c) *If $\gamma_1 > \max\{h_1(\rho), h_2(\rho)\}$, type-2 consumers prefer no privacy, while type-1 consumers prefer full privacy.*

   (d) *If $h_2(\rho) < \gamma_1 < h_1(\rho)$, both types of consumers prefer no privacy.*

5. *If $\gamma_1 \in (h_6(\rho), h_5(\rho))$, the platform will induce E1 equilibrium. Now type-2 consumers prefer no privacy, while type-1 consumers prefer full privacy.*

The intuition for the welfare results in Proposition 5 is as follows. If type-2 consumers are also picky (β_2 small or equivalently $\rho \leq 1/e$), then it is optimal for the platform to induce within-product competition only in the product market (i.e., induce the W1 equilibrium). In this case, the optimal ad prices under full privacy are the welfare-neutral prices b_i^*. By inspection, these welfare-neutral prices are related to the optimal ads prices under no privacy as follows:

$$b_i^* = \gamma_i b_i^{NP}.$$

Note that under full privacy, an ad of product i will reach type-i consumers with probability γ_i. Therefore, under welfare-neutral pricing, the expected advertising cost for an ad of product i to reach a type-i consumer under full privacy is b_i^*/γ_i, which is the same as the advertising cost for an ad of product i to reach a type-i consumer under no privacy (b_i^{NP}). Moreover, with only within-product competition, type-i firms face exactly the same trade-off as under no privacy. Both types of firms will buy the same amount of ads and choose the same equilibrium advertising functions for both privacy modes. From the perspective of type-i consumers, even though they receive more ads under full privacy, only a fraction γ_i of them match their preferences; the price distribution of those matched ads is exactly the same as the price distribution under no privacy.

Hence, both types of consumers are indifferent between the two privacy modes. The platform obtains the same revenue under both privacy modes. Since the platform always has the option of ignoring more granular consumer data, the platform's revenue under no privacy serves as an upper bound of the revenue it can obtain under full privacy; hence it is optimal for the platform to induce the W1 equilibrium whenever possible.

In the W2 equilibrium, which occurs when $\rho > 1/e$, the product market again features only within-product competition, but not without some platform-induced distortion: the platform has to strategically raise b_1^P above b_1^* and lower b_2^P below b_2^* to prevent type-1 firms from selling to type-2 consumers. Cross selling, in general, is bad for the ad revenue of the platform. It hurts type-2 firms' demand for ads of product 2 because the heightened competition with type-1 firms hurts their gross margins. The temptation of selling to type-2 consumers may also induce type-1 firms to lower their price on product 1. Since there is only within-product competition in the W2 equilibrium, the amount of product ads and the distribution of prices are then tied to the ad prices b_1^P and b_2^P. Since $b_1^P > b_1^*$ and $b_2^P < b_2^*$, type-1 consumers are worse off and type-2 consumers are better off under full privacy.

In the C1 equilibrium, the platform finds it too costly to prevent cross-selling. When this equilibrium occurs, γ_1 is not too small, so type-1 firms will charge prices both above $\beta_2 u$ (attractive to type-1 consumers only) and below $\beta_2 u$ (attractive to both types of consumers). Even though the platform cannot prevent cross-selling, the platform would still want to soften it by strategically raising b_1^P above b_1^* and lowering b_2^P below b_2^*. For type-1 consumers, a higher ad price b_1^P means fewer product 1 ads and on average higher advertised prices for product 1. Therefore, these picky consumers are worse off with full privacy. Type-2 firms have incentives to lower their advertised prices because of a lower ad price b_2^P and the competition from type-1 firms. Type 2 consumers benefit from full privacy in two ways. First, the reduction in ad price for type-2 products leads to more type-2 firms advertising average product prices. Second, type-2 consumers who receive type-1 ads only are now able to achieve positive surplus when they previously would have had none.

In E1 equilibrium, the set of equilibrium prices must be $\left[\underline{p}_1, \beta_2 u\right]$. That is, in equilibrium, type-1 firms only charge prices that appeal to all consumers. Type-1 consumers are better off because the prices on average are lower, while type-2 consumers are worse off because their preferred product is not offered. The intuition is that the demand for type-1 firms goes up when they appeal to type-2 consumers also, which prompts them to advertise more, increasing consumption among type-1 consumers also.

In the C2 equilibrium, the product market also features cross-selling. But different from the C1 equilibrium, here γ_1 is relatively small so type-1 firms only advertise prices below $\beta_2 u$ that are attractive for both types of consumers. In this equilibrium, it is not necessarily true that $b_1^P > b_1^*$ and $b_2^P < b_2^*$; the forces underlying the C1 equilibrium and the E1 equilibrium co-exist, and both types of consumers may be better off under privacy. Intuitively, type-2 consumers consider price offers from both types of firms and hence benefit from intensified competition. In contrast, type-1 consumers consider price offers only from type-1 firms, but they benefit from the increased advertising by type-1 firms. In fact, all four combinations of welfare effects may exist under C2 equilibrium.

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6 Conclusion

In this paper we asked the question, Is it possible to justify a preference for privacy on the part of consumers purely on instrumental grounds? And if so, is it possible for different consumers to have different privacy preferences? We have answered both in the affirmative in a general equilibrium model of manufacturers advertising to consumers through an ad platform.

The demand side of our model recognizes a natural heterogeneity that exists in most markets, namely, that, not only are consumers different in their product preferences, they are also different in how picky they are: some consumers have more rigid preferences than others. On the supply side, our model emphasizes the market power of the advertising platform while deemphasizing that of the manufacturers. (In fact, manufacturers have no market power in our model.) Our results suggest that the market power of the platform is the crucial consideration. If an ad platform lacks the market power to set ad prices—and by implication, is unable to vary ad prices in response to changes in privacy regulations—then it is impossible to justify a preference for privacy on instrumental grounds. This is because when consumers reveal their product preferences, advertising becomes more productive. This then has the knock-on effect of inducing product manufacturers to inform more consumers about their products and prices, increasing overall consumption. By contrast, if consumers' product preferences stay private, manufacturers can't avoid mistargeting ads, which lowers advertising productivity, lowering consumption ultimately. However, our analysis uncovers an interesting pro-competitive effect of privacy arising from an asymmetry between firms catering to picky consumers and firms catering to flexible consumers: while all firms' ads become less productive under privacy, the former suffer less because their ads can cross-sell to flexible consumers. Still, this potentially pro-competitive effect of privacy is not enough to carry the day for privacy without a little help from the platform. Our analysis shows that when the ad platform has market power, its own revenue-maximizing goal may sometimes align with tilting the privacy playing field in favor of firms catering to flexible consumers' product preferences and sometimes in favor of firms catering to picky consumers. Many varieties of instrumental privacy preferences can arise: all consumers may prefer full privacy, all consumers may prefer no privacy, flexible consumers may prefer full privacy while picky consumers prefer no privacy, and flexible consumers may prefer no privacy while picky consumers prefer full privacy.[14]

[14]The existing literature on privacy generally assumes away the ad platform, effectively making ad prices exogenous. In such models, it is hard to find a demand for privacy on instrumental grounds (Rhodes and Zhou, 2022). And when one does find one, as in Iyer, Soberman, and Villas-Boas (2005), it goes only in one direction:

However, the main message of Proposition 5 is that, in general, consumers with flexible preferences are more likely to prefer privacy than consumers with more rigid preferences.

In the popular discourse the issues around privacy are commonly posed as a tussle between the intrinsic privacy rights of consumers and the greater productivity of advertising under no privacy. This paper shows that the terms of this debate are too narrow. Consumers can find value in privacy purely for instrumental reasons simply because the presence of consumers with flexible preferences introduces the possibility of greater competition in the product market leading to lower prices and greater consumption for some or all consumers.

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consumers with flexible consumers prefer no privacy and picky consumers prefer privacy.

7 Appendix: Proofs

7.1 Characterization of Equilibrium Structure

The following six lemmas formally characterizes the structure of all possible equilibria.

Lemma 1. *For given ad prices, (b_1, b_2), in any equilibrium under full privacy where type-1 consumers are served, we must have $\bar{p}_1 = u$ or $\bar{p}_1 = \beta_2 u$.*

Proof. If $\bar{p}_1 \in (\beta_2 u, u)$, then free entry implies that

$$\pi_1^P(\bar{p}_1) = \gamma_1 e^{-A_1^P(\bar{p}_1)}(\bar{p}_1 - c) - b_1 = 0.$$

The marginal profit for a type-1 firm who sends an extra ad with price u is

$$\pi_1^P(u) = \gamma_1 e^{-A_1^P(\bar{p}_1)}(u - c) - b_1 = \gamma_1 e^{-A_1^P(\bar{p}_1)}(u - \bar{p}_1) > 0,$$

contradicting to free entry. Now suppose $\bar{p}_1 < \beta_2 u$. A similar reasoning implies that all prices $p \in (\bar{p}_1 + (1-\beta_2)u, u]$ must be advertised in equilibrium by type-2 firms, because otherwise a type-1 firm would earn a strictly positive profit by advertising at price $p \in (\bar{p}_1, \beta_2 u]$. By free entry, type-1 firms must earn zero profit by sending ads with price $\bar{p}_1$,

$$\pi_1^P(\bar{p}_1) = \left(\gamma_1 + \gamma_2 e^{-A_2^P(\bar{p}_1 + (1-\beta_2)u)}\right) e^{-A_1^P(\bar{p}_1)}(\bar{p}_1 - c) - b_1 = 0,$$

and type-2 firms must earn zero profit by sending ads with price $p' + (1-\beta_2)u \in (\bar{p}_1 + (1-\beta_2)u, u]$,

$$\pi_2^P(p' + (1-\beta_2)u) = \gamma_2 e^{-A_2^P(p' + (1-\beta_2)u)} e^{-A_1^P(\bar{p}_1)}(p' + (1-\beta_2)u - c) - b_2 = 0.$$

Consider a type-1 firm who sends an extra ad with price $p' \in (\bar{p}_1, \beta_2 u]$. Its marginal profit is

$$\begin{aligned}
\pi_1^P(p') &= \left(\gamma_1 + \gamma_2 e^{-A_2^P(p' + (1-\beta_2)u)}\right) e^{-A_1^P(\bar{p}_1)}(p' - c) - b_1 \\
&= \frac{\gamma_1 + \gamma_2 e^{-A_2^P(p' + (1-\beta_2)u)}}{\gamma_2 e^{-A_2^P(p' + (1-\beta_2)u)}} \frac{p' - c}{p' + (1-\beta_2)u - c} b_2 - b_1 \\
&> \frac{\gamma_1 + \gamma_2 e^{-A_2^P(\bar{p}_1 + (1-\beta_2)u)}}{\gamma_2 e^{-A_2^P(\bar{p}_1 + (1-\beta_2)u)}} \frac{\bar{p}_1 - c}{\bar{p}_1 + (1-\beta_2)u - c} b_2 - b_1 \\
&= \left(\gamma_1 + \gamma_2 e^{-A_2^P(\bar{p}_1 + (1-\beta_2)u)}\right) e^{-A_1^P(\bar{p}_1)}(\bar{p}_1 - c) - b_1 \\
&= 0,
\end{aligned}$$

where the inequality follows because each of the two ratios in the expression is increasing in p'. Again a contradiction to free entry. Therefore, we must have either $\bar{p}_1 = u$ or $\bar{p}_1 = \beta_2 u$. $\square$

Lemma 2. *For given ad prices, (b_1, b_2), in any W-type equilibrium under full privacy, the support of $A_1^P(p)$ is $[\underline{p}_1, u]$ and the support of $A_2^P(p)$ is $[\underline{p}_2, u]$ with $\underline{p}_1 \geq \beta_2 u$.*

Proof. Similar to the proof of Lemma 1, if $\bar{p}_i < u$, then free entry implies that

$$\pi_i^P(\bar{p}_i) = \gamma_i e^{-A_i^P(\bar{p}_i)}(\bar{p}_i - c) - b_i^P = 0.$$

The marginal profit for a type i firm who sends an extra ad with price u is

$$\pi_i^P(u) = \gamma_i e^{-A_1^P(\bar{p}_i)}(u - c) - b_i^P = \gamma_i e^{-A_i^P(\bar{p}_i)}(u - \bar{p}_i) > 0,$$

contradicting to free entry. $\underline{p}_1 \geq \beta_2 u$ simply ensures that indeed no type-2 consumers would buy type-1 products. $\square$

Lemma 3. *For given ad prices, (b_1, b_2), in any equilibrium, there does not exist an interval of prices $[a_1, a_2]$ such that all prices $p \in [a_1, a_2]$ are advertised by type-1 firms and all prices $p \in [a_1 + (1 - \beta_2)u, a_2 + (1 - \beta_2)u]$ are advertised by type-2 firms.*

Proof. Suppose by contradiction that such an interval $[a_1, a_2]$ exists. By free entry, for any $p \in [a_1, a_2]$, type-1 firms must earn zero profit by sending ads with price p,

$$\pi_1^P(p) = \left(\gamma_1 + \gamma_2 e^{-A_2^P(p+(1-\beta_2)u)}\right) e^{-A_1^P(p)}(p - c) - b_1 = 0,$$

and type-2 firms must earn zero profit by sending ads with price $p + (1 - \beta_2)u$,

$$\pi_2^P(p + (1 - \beta_2)u) = \gamma_2 e^{-A_2^P(p+(1-\beta_2)u)} e^{-A_1^P(p)}(p + (1 - \beta_2)u - c) - b_2 = 0.$$

It follows that, for all $p \in [a_1, a_2]$,

$$\frac{\gamma_1 + \gamma_2 e^{-A_2^P(p+(1-\beta_2)u)}}{\gamma_2 e^{-A_2^P(p+(1-\beta_2)u)}} \frac{p - c}{p + (1 - \beta_2)u - c} = \frac{b_1}{b_2}.$$

On the left-hand side, the first ratio is weakly increasing in p and the second ratio is strictly increasing in p, so the above equation cannot hold for all $p \in [a_1, a_2]$, i.e., an interval such as $[a_1, a_2]$ cannot exist. $\square$

Next we provide more structure on the resulting competition under C type equilibrium. Specifically, type-1 firms will advertise only "high" prices while type-2 firms will advertise only "low" prices from type-2 consumers' perspective. Therefore, in equilibrium where type-2 consumers receive both product offers, the price offers of product 2 are always more attractive to type-2 consumers than the price offers of product 1. In other words, type-2 consumers buy product 1 in equilibrium only when they receive no offers of product 2. In particular, whenever type-1 firms want to cross-sell to type-2 consumers, they will advertise all prices between $\underline{p}_1$ and $\beta_2 u$, and the maximal price advertised by type-1 firms is $\bar{p}_2 = \underline{p}_1 + (1-\beta_2)u$.

Lemma 4. *For given ad prices, (b_1, b_2), if in equilibrium both product ads are sent and $\underline{p}_1 < \beta_2 u$, then $\pi_1^P(p) = 0$ for all $p \in [\underline{p}_1, \beta_2 u]$ and $\bar{p}_2 = \underline{p}_1 + (1-\beta_2)u$.*

Proof. Suppose by contradiction that $\pi_1^P(p) < 0$ for some $\tilde{p} \in [\underline{p}_1, \beta_2 u]$. By continuity of $A_1^P(p)$ and $A_1^P(p)$, $\pi_1^P(p)$ is also continuous. Therefore, there must exist an interval $[\underline{p}_1, \beta_2 u] \supset (a_1, a_2) \ni \tilde{p}$ such that $\pi_1^P(p) < 0$ for all $p \in (a_1, a_2)$ and $\pi_1^P(a_1) = 0$. Free entry of type-2 firms implies that all prices $p \in (a_1 + (1-\beta_2)u, a_2 + (1-\beta_2)u)$ must be advertised by type-2 firms. Type-2 firms must earn zero profit by sending ads with prices $p' + (1-\beta_2)u \in (a_1 + (1-\beta_2)u, a_2 + (1-\beta_2)u)$:

$$\pi_2^P(p' + (1-\beta_2)u) = \gamma_2 e^{-A_2^P(p'+(1-\beta_2)u)} e^{-A_1^P(a_1)}(p' + (1-\beta_2)u - c) - b_2 = 0.$$

If a type-1 firm sends ad with price p', the expected profit is

$$\begin{aligned}
\pi_1^P(p') &= \left[\gamma_2 e^{-A_2^P(p'+(1-\beta_2)u)} + \gamma_1\right] e^{-A_1^P(a_1)}(p' - c) - b_1 \\
&= \frac{\gamma_1 + \gamma_2 e^{-A_2^P(p'+(1-\beta_2)u)}}{\gamma_2 e^{-A_2^P(p'+(1-\beta_2)u)}} \frac{p' - c}{p' + (1-\beta_2)u - c} b_2 - b_1 \\
&> \frac{\gamma_1 + \gamma_2 e^{-A_2^P(a_1+(1-\beta_2)u)}}{\gamma_2 e^{-A_2^P(a_1+(1-\beta_2)u)}} \frac{a_1 - c}{a_1 + (1-\beta_2)u - c} b_2 - b_1 \\
&= 0,
\end{aligned}$$

where the inequality follows because both ratios in the expression is increasing in p'. A contradiction to free entry.

For the claim of $\bar{p}_2 = \underline{p}_1 + (1-\beta_2)u$, note that, by Lemma 3, $\bar{p} \le \underline{p}_1 + (1-\beta_2)u$. If $\bar{p} < \underline{p}_1 + (1-\beta_2)u$, a similar argument as in the proof of Lemma 1 would help establish that type-2 firms can earn strictly positive profit by sending ads with price $\bar{p} + \epsilon$ for some small $\epsilon > 0$. Hence, $\bar{p}_2 = \underline{p}_1 + (1-\beta_2)u$. □

For C type equilibrium, by Lemma 1 and Lemma 4, the support of $A_2^P(p)$ is always $[\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$. The support of $A_1^P(p)$ can be either $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ or $[\underline{p}_1, \beta_2 u]$.

For E type equilibrium, the ad price for one of the products is too high to let firms advertise.

Lemma 5. *For given ad prices, (b_1, b_2), in any equilibrium where only product 1 is advertised (E1), the support of $A_2^P(p)$ is empty and the support of $A_1^P(p)$ is $[\underline{p}_1, u]$ or $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ or $[\underline{p}_1, \beta_2 u]$. In equilibrium where only product 2 is advertised (E2), the support of $A_1^P(p)$ is empty and the support of $A_2^P(p)$ is $[\underline{p}_2, u]$.*

Proof. The proof of E2 equilibrium is relatively simple and follows directly from the proof of Lemma 1 that $\bar{p}_2 = u$. For E1 equilibrium, Lemma 1 shows that either $\bar{p}_1 = u$ or $\bar{p}_2 = \beta_2 u$.

Therefore, the support of $A_1^P(p)$ is either $[\underline{p}_1, u]$ or $[\underline{p}_1, \beta_2 u]$. However, notice that in the first case when $\underline{p}_1 < \beta_2 u$, similar to the proof of Lemma 4, $\pi_1^P(p) = e^{-A_1^P(p)}(p-c) - b_1 = 0$ for all $p \in [\underline{p}_1, \beta_2 u]$. For sufficiently small $\epsilon > 0$,

$$\begin{aligned}
\pi_1^P(\beta_2 u + \epsilon) &= \gamma_1 e^{-A_1^P(\beta_2 u + \epsilon)}(\beta_2 u + \epsilon - c) - b_1 \\
&= \gamma_1 e^{-A_1^P(\beta_2 u + \epsilon)}(\beta_2 u - c) - b_1 + \epsilon \gamma_1 e^{-A_1^P(\beta_2 u + \epsilon)} \\
&= \left(\gamma_1 e^{-A_1^P(\beta_2 u + \epsilon)} - e^{-A_1^P(\beta_2 u)}\right) b_1 + \epsilon \gamma_1 e^{-A_1^P(\beta_2 u + \epsilon)} \\
&< 0.
\end{aligned}$$

Therefore, prices just above $\beta_2 u$ will not be advertised and the support of $A_1^P(p)$ is $[\underline{p}_1, u]$ or $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ when $\underline{p}_1 < \beta_2 u$. $\square$

The following lemma simply consolidates the results of Lemmas 1–5.

Lemma 6. *The equilibrium supports of the advertising functions will take one of the following forms:*

- ***W equilibrium:*** *The support of $A_2^P(p)$ is $[\underline{p}_2, u]$ and the support of $A_1^P(p)$ is $[\underline{p}_1, u]$ with $\underline{p}_1 \geq \beta_2 u$.*
- ***C equilibrium:*** *The support of $A_2^P(p)$ is always $[\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$. The support of $A_1^P(p)$ can be either $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ (C1 type) or $[\underline{p}_1, \beta_2 u]$ (C2 type) with $\underline{p}_1 < \beta_2 u$ and $\beta_2 u \tilde{u} < u$.*
- ***E equilibrium:*** *In E1 equilibrium, the support of $A_2^P(p)$ is empty and the support of $A_1^P(p)$ is either $[\underline{p}_1, u]$ (W type) $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ (C1 type) or $[\underline{p}_1, \beta_2 u]$ (C2 type). In E2 equilibrium, the support of $A_1^P(p)$ is empty and the support of $A_2^P(p)$ is $[\underline{p}_2, u]$.*

7.2 Proof of Proposition 1

Define the sales function $S_{j \to i}^{\lambda}(p)$ as the total sales of product j to type i consumers at prices less than or equal to p under privacy mode $\lambda \in \{P, NP\}$.

There are a large number of infinitesimal firms in the market, so they will simply earn zero profit in any equilibrium. However, we can compare the market size in advertising market and product market when the privacy mode changes between NP and P. Under privacy mode $\lambda \in \{P, NP\}$, we define the market size of type-i product in advertising market as the total number of matched ads, represented by $\gamma_i A_i^{\lambda}(u)$ for type-i product if $\lambda = NP$, and $\gamma_1 A_1^{\lambda}(u) + \gamma_2 A_1^{\lambda}(\beta_2 u)$ for type-1 product and $\gamma_2 A_2^{\lambda}(u)$ for type-2 product if $\lambda = P$. The market size of type-i product in product market can be represented by $S_{i \to 1}^{\lambda}(u) + S_{i \to 2}^{\lambda}(u)$.

Under **no privacy**, the market size of type-i product in advertising market is

$$\gamma_i A_i^{NP}(u) = \gamma_i \left(\ln(u-c) - \ln b_i\right),$$

and the market size of type-i product in product market is

$$S_{i\to i}^{NP}(u) = \gamma_i \left(1 - e^{-A_i^{NP}(p)}\right) = \gamma_i \left(1 - \frac{b_i}{u-c}\right).$$

The average advertised price of type-i product is

$$\begin{aligned}
\frac{\int_{c+b_i}^{u} p dA_i^{NP}(p)}{A_i^{NP}(u)} &= \frac{\int_{c+b_i}^{u} \frac{p}{p-c} dp}{\ln(u-c) - \ln b_i} \\
&= \frac{u - c - b_i + c(\ln(u-c) - \ln b_i)}{\ln(u-c) - \ln b_i} \\
&= c + \frac{u-c-b_i}{\ln(u-c) - \ln b_i}
\end{aligned}$$

Under **privacy** and **W** or **E2 equilibrium**, the market size of type-i product in advertising market is

$$\gamma_i A_i^P(u) = \gamma_i (\ln \gamma_i + \ln(u-c) - \ln b_i),$$

and the market size of type-i product in product market is

$$S_{i\to 1}^{P}(u) + S_{i\to 2}^{P}(u) = \gamma_i \left(1 - e^{-A_i^{P}(u)}\right) = \gamma_i \left(1 - \frac{b_i}{\gamma_i(u-c)}\right).$$

The average advertised price of type-i product is

$$\begin{aligned}
\frac{\int_{c+b_i/\gamma_i}^{u} p dA_i^{P}(p)}{A_i^{P}(u)} &= \frac{\int_{c+b_i/\gamma_i}^{u} \frac{p}{p-c} dp}{\ln\gamma_i + \ln(u-c) - \ln b_i} \\
&= c + \frac{u-c-\frac{b_i}{\gamma_i}}{\ln(u-c) - \ln \frac{b_i}{\gamma_i}} \\
&> c + \frac{u-c-b_i}{\ln(u-c) - \ln b_i}
\end{aligned}$$

where the inequality follows from

$$\frac{\partial}{\partial x}\left(\frac{a-x}{\ln a - \ln x}\right) = \frac{-\ln\frac{a}{x} + \frac{a}{x} - 1}{(\ln a - \ln x)^2} > 0$$

for $x < a$. The market size of both products in both markets would shrink, while the average price goes up compared to no privacy environment.

Under **privacy** and **E1 equilibrium**, the total number of matched ads of type-1 product in advertising market is

$$\gamma_1 A_1^P(u) = \gamma_1 (\ln\gamma_1 + \ln(u-c) - \ln b_1) = \gamma_1 \ln\gamma_1 + \gamma_1(\ln(u-c) - \ln b_1),$$

if $\gamma_1 > \rho$, or

$$\gamma_1 A_1^P(\beta_2 u) = \gamma_1 \left(\ln(\beta_2 u - c) - \ln b_1\right),$$

if $\gamma_1 \leq \rho$. In both cases, the total number of matched ads of type-1 product in advertising market shrinks. The market size of type-1 product in product market is

$$\begin{aligned} S^P_{1\to 1}(u) + S^P_{1\to 2}(\beta_2 u) &= \gamma_1\left(1 - e^{-A_1^P(u)}\right) + \gamma_2\left(1 - e^{-A_1^P(\beta_2 u)}\right) \\ &= \gamma_1\left(1 - \frac{b_1}{\gamma_1(u-c)}\right) + \gamma_2\left(1 - \frac{b_1}{\beta_2 u - c}\right), \end{aligned}$$

if $\gamma_1 > \rho$, or

$$S^P_{1\to 1}(\beta_2 u) + S^P_{1\to 2}(\beta_2 u) = 1 - e^{-A_1^P(\beta_2 u)} = 1 - \frac{b_1}{\beta_2 u - c}.$$

if $\gamma_1 \leq \rho$. The market size shrinks if

$$\begin{aligned} &\gamma_1\left(1 - \frac{b_1}{\gamma_1(u-c)}\right) + \gamma_2\left(1 - \frac{b_1}{\beta_2 u - c}\right) < \gamma_1\left(1 - \frac{b_1}{u-c}\right) \\ \Leftrightarrow\quad & b_1 > \frac{\rho}{1+\rho}(u-c) \end{aligned}$$

when $\gamma_1 > \rho$, or

$$\begin{aligned} &1 - \frac{b_1}{\beta_2 u - c} < \gamma_1\left(1 - \frac{b_1}{u-c}\right) \\ \Leftrightarrow\quad & b_1 > \frac{\rho - \gamma_1\rho}{1-\gamma_1\rho}(u-c) \end{aligned}$$

when $\gamma_1 \leq \rho$. Therefore, the market size of type-1 product in product market may expand compared to no privacy environment when b_1 is small enough.

The average advertised price of type-1 product is

$$\begin{aligned} \frac{\int_{c+b_1}^{u} p\, dA_1^P(p)}{A_1^P(u)} &= \frac{\int_{c+b_1}^{\beta_2 u} \frac{p}{p-c} dp + \int_{\tilde{u}}^{u} \frac{p}{p-c} dp}{\ln\gamma_1 + \ln(u-c) - \ln b_1} \\ &= c + \frac{\beta_2 u - c - b_1 + u - \tilde{u}}{\ln\gamma_1 + \ln(u-c) - \ln b_1} \end{aligned}$$

if $\gamma_1 > \rho$, where $\tilde{u}$ is given by $\gamma_1(\tilde{u} - c) = \beta_2 u - c$. Therefore, the advertised price is lower than under no privacy if

$$\begin{aligned} & c + \frac{\beta_2 u - c - b_1 + u - \tilde{u}}{\ln\gamma_1 + \ln(u-c) - \ln b_1} < c + \frac{u - c - b_1}{\ln(u-c) - \ln b_1} \\ \Leftrightarrow\quad & -\frac{\gamma_2}{\gamma_1}(\beta_2 u - c)\ln\frac{u-c}{b_1} - (u - c - b_1)\ln\gamma_1 < 0. \end{aligned}$$

The partial derivative with respect to b_1 of the left hand side is

$$\begin{aligned} & \frac{\gamma_2}{\gamma_1}\frac{\beta_2 u - c}{b_1} + \ln\gamma_1 \\ > \quad & \frac{1}{\gamma_1} - 1 + \ln\gamma_1 > 0. \end{aligned}$$

Therefore,

$$-\frac{\gamma_2}{\gamma_1}(\beta_2 u - c)\ln\frac{u-c}{b_1} - (u-c-b_1)\ln\gamma_1 < -\frac{\gamma_2}{\gamma_1}(\beta_2 u - c)\ln\frac{u-c}{u-c} - (u-c-(u-c))\ln\gamma_1 = 0.$$

The average advertised price of type-1 product is

$$\begin{aligned} \frac{\int_{c+b_1}^{\beta_2 u} p dA_1^P(p)}{A_1^P(\beta_2 u)} &= \frac{\int_{c+b_1}^{\beta_2 u} \frac{p}{p-c}dp}{\ln(\beta_2 u - c) - \ln b_1} \\ &= c + \frac{\beta_2 u - c - b_1}{\ln(\beta_2 u - c) - \ln b_1} \end{aligned}$$

if $\gamma_1 \leq \rho$. Therefore, the advertised price is lower than under no privacy if

$$\begin{aligned} & c + \frac{\beta_2 u - c - b_1}{\ln(\beta_2 u - c) - \ln b_1} < c + \frac{u - c - b_1}{\ln(u-c) - \ln b_1} \\ \Leftrightarrow \quad & (\beta_2 u - c - b_1)\ln\frac{u-c}{b_1} < (u-c-b_1)\ln\frac{\beta_2 u - c}{b_1} \\ \Leftrightarrow \quad & (1-\beta_2)\, u \ln\frac{\beta_2 u - c}{b_1} + (\beta_2 u - c - b_1)\ln\rho > 0 \end{aligned}$$

The partial derivative with respect to b_1 of the left hand side is

$$\begin{aligned} & -\frac{(1-\beta_2)\, u}{b_1} - \ln\rho \\ < \quad & -\frac{(1-\beta_2)\, u}{\beta_2 u - c} - \ln\rho \\ = \quad & -\left(\frac{1}{\rho} - 1 + \ln\rho\right) < 0. \end{aligned}$$

Therefore,

$$(1-\beta_2)\, u \ln\frac{\beta_2 u - c}{b_1} + (\beta_2 u - c - b_1)\ln\rho > (1-\beta_2)\, u\ln\frac{\beta_2 u - c}{\beta_2 u - c} + (\beta_2 u - c - (\beta_2 u - c))\ln\rho = 0.$$

Under **privacy** and **C-equilibrium**, the market size of type-2 product in advertising market is

$$\gamma_2 A_2^P(u) = \gamma_2\left(\ln\gamma_2 + \ln\left(\underline{p}_1 + (1-\beta_2)u - c\right) - \ln b_2\right),$$

which is smaller than under no privacy. The market size of type-2 product in product market is

$$S_{2\to1}^P(u) + S_{2\to2}^P(u) = \gamma_2\left(1 - e^{-A_2^P(u)}\right) = \gamma_2\left(1 - \frac{b_2}{\gamma_2\left(\underline{p}_1 + (1-\beta_2)u - c\right)}\right),$$

which is also smaller than under no privacy.

The analysis of market size of type-1 product varies by case. For C2-equilibrium, the market size of type-1 product in advertising market is

$$\gamma_1 A_1^P(\beta_2 u) = \gamma_1\left(\ln\left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right) + \ln(\beta_2 u - c) - \ln b_1\right),$$

and for C1-equilibrium, the market size of type-1 product in advertising market is

$$\gamma_1 A_1^P(u) = \gamma_1(\ln\gamma_1 + \ln(u-c) - \ln b_1).$$

In both equilibrium, the total number of matched ads decreases under full privacy. For C2 equilibrium, the total number of sales of type-1 product in product market is

$$\begin{aligned}
S_{1\to1}^P(\beta_2 u) + S_{1\to2}^P(\beta_2 u) &= \left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right)\left(1 - e^{-A_1^P(\beta_2 u)}\right) \\
&= \left(\gamma_1 + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c}\right)\left(1 - \frac{b_1}{\left(\gamma_1 + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c}\right)(\beta_2 u - c)}\right) \\
&= \gamma_1 + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c} - \frac{b_1}{\beta_2 u - c},
\end{aligned}$$

which is smaller than under no privacy if and only if

$$\begin{aligned}
&\gamma_1 + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c} - \frac{b_1}{\beta_2 u - c} < \gamma_1 - \frac{\gamma_1 b_1}{u-c} \\
\Leftrightarrow\quad & \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c} - \frac{b_1}{\beta_2 u - c} + \frac{\gamma_1 b_1}{u-c} < 0.
\end{aligned}$$

The partial derivatives of the left hand side with respect to b_1, b_2 are

$$\begin{aligned}
b_1: &\quad -\frac{b_2 \frac{\partial \underline{p}_1}{\partial b_1}}{\left(\underline{p}_1 + (1-\beta_2)u - c\right)^2} - \frac{1}{\beta_2 u - c} + \frac{\gamma_1}{u-c} < 0, \\
b_2: &\quad \frac{\underline{p}_1 + (1-\beta_2)u - c - b_2\frac{\partial \underline{p}_1}{b_2}}{\left(\underline{p}_1 + (1-\beta_2)u - c\right)^2} > 0.
\end{aligned}$$

When we decrease b_1 or increase b_2, the equilibrium continues to be C2 equilibrium until

$\underline{p}_1 = c + b_2/\gamma_2$ where the equilibrium turns to E1 equilibrium. Therefore, the total number of sales may go up under full privacy for small b_1.

For C1 equilibrium, the market size of type-1 product in product market is

$$\begin{aligned}
S^P_{1\to 1}(u) + S^P_{1\to 2}(u) &= \gamma_1\left(1 - e^{-A^P_1(u)}\right) + \gamma_2 e^{-A^P_2\left(\underline{p}_1 + (1-\beta_2)u\right)}\left(1 - e^{-A^P_1(\beta_2 u)}\right) \\
&= \gamma_1\left(1 - \frac{b_1}{\gamma_1(u-c)}\right) + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c} - \frac{\frac{b_2}{\underline{p}_1 + (1-\beta_2)u-c}}{\gamma_1 + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u-c}}\frac{b_1}{\beta_2 u - c} \\
&= \gamma_1\left(1 - \frac{b_1}{\gamma_1(u-c)}\right) + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c} - \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c}\frac{\underline{p}_1 - c}{\beta_2 u - c} \\
&= \gamma_1\left(1 - \frac{b_1}{\gamma_1(u-c)}\right) + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c}\frac{\beta_2 u - \underline{p}_1}{\beta_2 u - c}
\end{aligned}$$

which is smaller than under no privacy if and only if

$$\begin{aligned}
& \gamma_1 - \frac{b_1}{u-c} + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c}\frac{\beta_2 u - \underline{p}_1}{\beta_2 u - c} < \gamma_1 - \frac{\gamma_1 b_1}{u-c} \\
\Leftrightarrow\quad & -\frac{\gamma_2 b_1}{u-c} + \frac{b_2}{\underline{p}_1 + (1-\beta_2)u - c}\frac{\beta_2 u - \underline{p}_1}{\beta_2 u - c} < 0
\end{aligned}$$

The partial derivatives of the left hand side with respect to b_1, b_2 are

$$\begin{aligned}
b_1: &\qquad -\frac{\gamma_2}{u-c} - \frac{b_2\frac{\partial \underline{p}_1}{\partial b_1}}{\rho\left(\underline{p}_1 + (1-\beta_2)u - c\right)^2} < 0, \\
b_2: &\qquad \frac{1}{\beta_2 u - c}\frac{\beta_2 u - \underline{p}_1}{\underline{p}_1 + (1-\beta_2)u - c} - \frac{b_2\frac{\partial \underline{p}_1}{\partial b_2}}{\rho\left(\underline{p}_1 + (1-\beta_2)u - c\right)^2} > 0.
\end{aligned}$$

When we decrease b_1 or increase b_2, the equilibrium continues to be C2 equilibrium until $\underline{p}_1 = c + b_2/\gamma_2$ where the equilibrium turns to E1 equilibrium. Therefore, the total number of sales may go up under full privacy for small b_1.

The average advertised price of type-2 product is lower than under no privacy as high prices are no longer advertised while the density of each advertised price stays the same.

For C1-equilibrium, the average advertised price of type-1 product is

$$\begin{aligned}
\frac{\int_{\underline{p}_1}^{u} p\, dA^P_1(p)}{A^P_1(u)} &= \frac{\int_{\underline{p}_1}^{\beta_2 u}\frac{p}{p-c}dp + \int_{\tilde{u}}^{u}\frac{p}{p-c}dp}{\ln\gamma_1 + \ln(u-c) - \ln b_1} \\
&= c + \frac{\beta_2 u - \underline{p}_1 + u - \tilde{u}}{\ln\gamma_1 + \ln(u-c) - \ln b_1} \\
&< c + \frac{\beta_2 u - c - b_1 + u - \tilde{u}}{\ln\gamma_1 + \ln(u-c) - \ln b_1}
\end{aligned}$$

$$< \quad c + \frac{u - c - b_1}{\ln(u-c) - \ln b_1}$$

where $\tilde{u}$ is given by $\gamma_1(\tilde{u} - c) = \beta_2 u - c$. Therefore, the advertised price is lower than under no privacy.

For C2-equilibrium, the average advertised price of type-1 product is

$$\begin{aligned}
\frac{\int_{\underline{p}_1}^{\beta_2 u} p dA_1^P(p)}{A_1^P(\beta_2 u)} &= \frac{\int_{\underline{p}_1}^{\beta_2 u} \frac{p}{p-c} dp}{\ln(\beta_2 u - c) - \ln\left(\underline{p}_1 - c\right)} \\
&= c + \frac{\beta_2 u - \underline{p}_1}{\ln(\beta_2 u - c) - \ln\left(\underline{p}_1 - c\right)}
\end{aligned}$$

Therefore, the advertised price is lower than under no privacy if

$$\begin{aligned}
& c + \frac{\beta_2 u - \underline{p}_1}{\ln(\beta_2 u - c) - \ln\left(\underline{p}_1 - c\right)} < c + \frac{u - c}{\ln(u-c) - \ln b_1} \\
\Leftrightarrow \quad & \left(\beta_2 u - \underline{p}_1\right) \ln \frac{u-c}{b_1} - (u - c - b_1) \ln \frac{\beta_2 u - c}{\underline{p}_1 - c} < 0
\end{aligned}$$

The partial derivative with respect to b_2 of the left hand side is

$$\frac{\partial \underline{p}_1}{\partial b_2} \left(\frac{u - c - b_1}{\underline{p}_1 - c} - \ln \frac{u-c}{b_1} \right).$$

As $\frac{\partial \underline{p}_1}{\partial b_2} < 0$, and $\frac{u-c-b_1}{\underline{p}_1 - c} - \ln \frac{u-c}{b_1}$ is strictly increasing in b_2, the above partial derivative is either always negative or always positive or positive first and then negative. Therefore, the left hand side of inequality reaches minimum at either end. As the change in b_2 is bound by the conditions that $c + b_1 < \underline{p}_1 < \beta_2 u$,

$$\begin{aligned}
& \left(\left(\beta_2 u - \underline{p}_1\right) \ln \frac{u-c}{b_1} - (u - c - b_1) \ln \frac{\beta_2 u - c}{\underline{p}_1 - c} \right) \Big|_{\underline{p}_1 = \beta_2 u} = 0 \\
& \left(\left(\beta_2 u - \underline{p}_1\right) \ln \frac{u-c}{b_1} - (u - c - b_1) \ln \frac{\beta_2 u - c}{\underline{p}_1 - c} \right) \Big|_{\underline{p}_1 = c + b_1} \\
= \quad & (\beta_2 u - c - b_1) \ln \frac{u-c}{b_1} - (u - c - b_1) \ln \frac{\beta_2 u - c}{b_1} \\
= \quad & \ln \frac{u-c}{b_1} \ln \frac{\beta_2 u - c}{b_1} \left(\frac{\beta_2 u - c - b_1}{\ln(\beta_2 u - c) - \ln b_1} - \frac{u - c - b_1}{\ln(u-c) - \ln b_1} \right) < 0
\end{aligned}$$

Therefore, the left hand side of inequality is either strictly decreasing or first increasing and then decreasing. When b_2 is small such that $\underline{p}_1$ is close to $\beta_2 u$,

$$\lim_{\underline{p}_1 \to \beta_2 u} \frac{u-c-b_1}{\underline{p}_1 - c} - \ln \frac{u-c}{b_1} = \frac{u-c-b_1}{\beta_2 u - c} - \ln \frac{u-c}{b_1} < 0$$

for small b_1. Therefore, it is possible that the left hand side of the original inequality is positive. The advertised price could be higher or lower than under no privacy.

7.3 Proof of Proposition 2

Consider first the welfare of type-1 consumers. Since type-1 consumers are picky, they only buy product 1 and $S^{\lambda}_{2\to1}(p) = 0$ for all equilibrium prices p. Note that the *measure* of type 1 consumers who receive at least one ads of product 1 with price less than or equal to p is $\gamma_1(1 - e^{-A^{\lambda}_1(p)})$, and these consumers will buy as long as $p \le u$. Hence, total sales of product 1 to type 1 consumers with price less than or equal to p are $S^{\lambda}_{1\to1}(p) = \gamma_1(1 - e^{-A^{\lambda}_1(p)})$. In both type W and type C equilibrium, the gain of a type-1 consumers from having privacy is

$$\begin{aligned}
& \int_{c+b_1/\gamma_1}^{u} (u-p) d\frac{S^{P}_{1\to1}(p)}{\gamma_1} - \int_{c+b_1}^{u} (u-p) d\frac{S^{NP}_{1\to1}(p)}{\gamma_1} \\
= & \ \frac{1}{\gamma_1}\left(\int_{c+b_1/\gamma_1}^{u} S^{P}_{1\to1}(p)\, dp - \int_{c+b_1}^{u} S^{NP}_{1\to1}(p)\, dp \right) \\
< & \ 0
\end{aligned}$$

where the equality follows by integration by parts, and the inequality follows from the fact that $A^{NP}_1(p) > A^{P}_1(p)$ and hence $S^{NP}_{1\to1}(p) > S^{P}_{1\to1}(p)$ for all $p \in [c + b_1/\gamma_1, u]$. As type-1 consumers never buys type-2 products, type-2 ads do not matter for them. It follows immediately that type-1 consumers prefer no privacy in E1-equilibrium as in either type W and type C equilibria and prefer no privacy in E2-equilibrium to be served. Hence, type-1 consumers always prefer no privacy.

Type-2 consumers only buy product 2 in equilibrium under no privacy; hence the sales functions are $S^{NP}_{1\to2}(p) = 0$ and $S^{NP}_{2\to2}(p) = \gamma_2\left(1 - e^{-A^{NP}_2(p)}\right)$ with $p \in [c + b_2, u]$. Under full privacy, we need to consider the type-W, type-C, type-E1 and type-E2 equilibrium separately. For type-W equilibrium, there is no cross-product competition in equilibrium. The welfare comparison for type-2 consumers is the same as the one for type-1 consumers, so type-2 consumers are also better off with no privacy. For type-C equilibrium, type-2 consumers may buy both products. The sales functions are $S^{P}_{2\to2}(p) = \gamma_2\left(1 - e^{-A^{P}_2(p)}\right)$ with $p \in [c + b_2/\gamma_2, \underline{p}_1 + (1-\beta_2)u]$ and $S^{P}_{1\to2}(p) = \gamma_2 e^{-A^{P}_2(\underline{p}_1 + (1-\beta_2)u)}\left(1 - e^{-A^{P}_1(p)}\right)$ with $p \in [\underline{p}_1, \beta_2 u]$. When the privacy mode changes from full privacy to no privacy, the change in consumer welfare for a type-2 consumer is

$$\int_{c+b_2}^{u} (u-p) d\frac{S^{NP}_{2\to2}(p)}{\gamma_2} - \left(\int_{c+b_2/\gamma_2}^{\underline{p}_1 + (1-\beta_2)u} (u-p) d\frac{S^{P}_{2\to2}(p)}{\gamma_2} + \int_{\underline{p}_1}^{\beta_2 u} (\beta_2 u - p) d\frac{S^{P}_{1\to2}(p)}{\gamma_2} \right)$$

$$
\begin{aligned}
&= \frac{1}{\gamma_2}\int_{c+b_2}^{u} S_{2\to 2}^{NP}(p)dp - \frac{1}{\gamma_2}\Bigg((u-p)\, S_{2\to 2}^{P}(p)\Big|_{c+b_2/\gamma_2}^{\underline{p}_1+(1-\beta_2)u} + \int_{c+b_2/\gamma_2}^{\underline{p}_1+(1-\beta_2)u} S_{2\to 2}^{P}(p)dp \\
&\quad + \int_{\underline{p}_1}^{\beta_2 u} S_{1\to 2}^{P}(p)dp\Bigg) \\
&= \int_{c+b_2}^{u}\left(1-e^{-A_2^{NP}(p)}\right)dp - \left(\beta_2 u - \underline{p}_1\right)\left(1-e^{-A_2^{P}\left(\underline{p}_1+(1-\beta_2)u\right)}\right) \\
&\quad - \int_{c+b_2/\gamma_2}^{\underline{p}_1+(1-\beta_2)u}\left(1-e^{-A_2^{P}(p)}\right)dp - \int_{\underline{p}_1+(1-\beta_2)u}^{u} e^{-A_2^{P}\left(\underline{p}_1+(1-\beta_2)u\right)}\left(1-e^{-A_1^{P}(p-(1-\beta_2)u)}\right)dp \\
&= \frac{\gamma_1}{\gamma_2}b_2 - b_2\ln\frac{u-c}{b_2} + \frac{b_2}{\gamma_2}\ln\frac{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}{b_2} + \frac{b_2}{\gamma_2}\frac{\underline{p}_1-c}{\underline{p}_1+(1-\beta_2)u-c}\ln\frac{\beta_2 u - c}{\underline{p}_1-c} \\
&= \frac{b_2}{\gamma_2}g(b_1,b_2)
\end{aligned}
$$

where we use integration by parts for the first equality and a change of variable for the second equality, and

$$
g(b_1,b_2) = \gamma_1 - \gamma_2\ln\frac{u-c}{b_2} + \ln\frac{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}{b_2} + \frac{\underline{p}_1-c}{\underline{p}_1+(1-\beta_2)u-c}\ln\frac{\beta_2 u-c}{\underline{p}_1-c}.
$$

The partial derivatives are

$$
\begin{aligned}
\frac{\partial g(b_1,b_2)}{\partial b_1} &= \frac{1}{\underline{p}_1+(1-\beta_2)u-c}\frac{\partial \underline{p}_1}{\partial b_1} + \frac{(1-\beta_2)u}{\left(\underline{p}_1+(1-\beta_2)u-c\right)^2}\ln\frac{\beta_2 u-c}{\underline{p}_1-c}\frac{\partial \underline{p}_1}{\partial b_1} \\
&\quad - \frac{1}{\underline{p}_1+(1-\beta_2)u-c}\frac{\partial \underline{p}_1}{\partial b_1} \\
&= \frac{(1-\beta_2)u}{\left(\underline{p}_1+(1-\beta_2)u-c\right)^2}\ln\frac{\beta_2 u-c}{\underline{p}_1-c}\frac{\partial \underline{p}_1}{\partial b_1} \\
&> 0, \\
\frac{\partial g(b_1,b_2)}{\partial b_2} &= \frac{\gamma_2}{b_2} - \frac{1}{b_2} + \frac{1}{\underline{p}_1+(1-\beta_2)u-c}\frac{\partial \underline{p}_1}{\partial b_2} + \frac{(1-\beta_2)u}{\left(\underline{p}_1+(1-\beta_2)u-c\right)^2}\ln\frac{\beta_2 u-c}{\underline{p}_1-c}\frac{\partial \underline{p}_1}{\partial b_2} \\
&\quad - \frac{1}{\underline{p}_1+(1-\beta_2)u-c}\frac{\partial \underline{p}_1}{\partial b_2} \\
&= -\frac{\gamma_1}{b_2} + \frac{(1-\beta_2)u}{\left(\underline{p}_1+(1-\beta_2)u-c\right)^2}\ln\frac{\beta_2 u-c}{\underline{p}_1-c}\frac{\partial \underline{p}_1}{\partial b_2} \\
&< 0,
\end{aligned}
$$

where $\partial \underline{p}_1/\partial b_1 > 0$ and $\partial \underline{p}_1/\partial b_2 < 0$ follows the derivation of (15) [15]. Therefore, $g(b_1,b_2) > 0$ if

[15] For details, see Section 7.5.2 below.

$g(b_1', b_2') > 0$ for some $b_1' \leq b_1$ and $b_2' \geq b_2$. However, the decrease of b_1 and increase of b_2 cannot be arbitrary. For type C equilibrium to exist, we need $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$ and $\underline{p}_1 \leq \beta_2 u$, $c + b_2/\gamma_2 \leq \underline{p}_1 + (1-\beta_2)u$. When b_1 continues to decrease, the first three constraints still hold, but the last one may be violated. When we decrease b_1 to the point where $c + b_2/\gamma_2 = \underline{p}_1 + (1-\beta_2)u$, this will be E1 type equilibrium where no type-1 product ads are sent. Therefore, in order to show that type-2 consumers prefer no privacy in C type equilibrium, i.e. $g(b_1, b_2) > 0$, it suffices to show that type-2 consumers prefer no privacy in any E1 type equilibrium.

Now consider E1 and E2 type equilibrium. First for E2-equilibrium, we have $\underline{p}_2 = c + b_2/\gamma_2$ and $A_2^P(p) = \ln\gamma_2 + \ln(p-c) - \ln b_2$ with $p \in [c + b_2/\gamma_2, u]$. It follows that $A_2^{NP}(p) > A_2^P(p)$ and hence $S_{2\to 2}^{NP}(p) > S_{2\to 2}^P(p)$ for all $p \in [c + b_2/\gamma_2, u]$. Type-2 consumers would prefer no privacy as only type-2 products are advertised.

In E1-equilibrium, the advertising function $A_1^P(p)$ takes the form as in either W or C type equilibrium. For W type form, all the advertised prices of product 1 are not accepted by type-2 consumers and hence type-2 consumers are obviously worse off. For C-type form, we have $\underline{p}_1 = c + b_1$ and $A_1^P(p) = \ln(p-c) - \ln b_1$ with $p \in [c + b_1, u]$. The sales function is $S_{1\to 2}^P(p) = \gamma_2\left(1 - e^{-A_1^P(p)}\right)$ with $p \in [c + b_1, \beta_2 u]$. When the privacy mode changes from full privacy to no privacy, the change in consumer welfare for a type-2 consumer is

$$
\begin{aligned}
& \int_{c+b_2}^{u} (u-p)d\frac{S_{2\to 2}^{NP}(p)}{\gamma_2} - \int_{c+b_1}^{\beta_2 u} (\beta_2 u - p)d\frac{S_{1\to 2}^{P}(p)}{\gamma_2} \\
= \quad & \frac{1}{\gamma_2}\int_{c+b_2}^{u} S_{2\to 2}^{NP}(p)dp - \frac{1}{\gamma_2}\int_{c+b_1}^{\beta_2 u} S_{1\to 2}^{P}(p)dp \\
= \quad & \int_{c+b_2}^{u}\left(1 - e^{-A_2^{NP}(p)}\right)dp - \int_{c+b_1+(1-\beta_2)u}^{u}\left(1 - e^{-A_1^P(p-(1-\beta_2)u)}\right)dp \\
= \quad & b_1 + (1-\beta_2)u - b_2 - b_2\ln\frac{u-c}{b_2} + b_1\ln\frac{\beta_2 u - c}{b_1}
\end{aligned}
$$

The partial derivative with respect b_1 is

$$
1 + \ln\frac{\beta_2 u - c}{b_1} - 1 = \ln\frac{\beta_2 u - c}{b_1} > 0.
$$

Therefore, with the condition that $b_1/b_2 \geq (\beta_2 u - c)/(u-c)$, we have

$$
\begin{aligned}
& \int_{c+b_2}^{u} (u-p)d\frac{S_{2\to 2}^{NP}(p)}{\gamma_2} - \int_{c+b_1}^{\beta_2 u} (\beta_2 u - p)d\frac{S_{1\to 2}^{P}(p)}{\gamma_2} \\
\geq \quad & \frac{\beta_2 u - c}{u-c}b_2 + (1-\beta_2)u - b_2 - b_2\ln\frac{u-c}{b_2} + \frac{\beta_2 u - c}{u-c}b_2\ln\frac{u-c}{b_2} \\
= \quad & (1-\beta_2)u\left[1 - \frac{b_2}{u-c}\left(1 + \ln\frac{u-c}{b_2}\right)\right] \\
> \quad & 0
\end{aligned}
$$

where the last inequality follows that

$$\frac{\partial(x(1-\ln x))}{\partial x}=1-\ln x-1=-\ln x>0$$

for $x<1$ and hence $x(1-\ln x)<1$ for $x<1$.

In short, regardless of the type of equilibrium, W, C or E, type-2 consumers prefer no privacy as long as $b_1/b_2 \geq \rho$.

7.4 Proof of Proposition 3

Here we show the marginal effect of ad price increase on advertising quantity and average ad price in C equilibrium. The effect in the other cases are straightforward. First we consider the effect of ad price increase on type-2 products. When b_1 increases, it has an indirect effect on advertising function through $\underline{p}_1$. The details on how to derive $\underline{p}_1$ is deferred to section 7.5.2. Here we use

$$\begin{aligned}
\frac{\partial \underline{p}_1}{\partial b_1^P} &= \frac{\underline{p}_1+(1-\beta_2)u-c}{\sqrt{\Delta}}>0,\\
\frac{\partial \underline{p}_1}{\partial b_2^P} &= -\frac{\underline{p}_1-c}{\sqrt{\Delta}}<0,
\end{aligned}$$

where

$$\begin{aligned}
\Delta &= \left(b_1^P-b_2^P-\gamma_1(1-\beta_2)u\right)^2+4\gamma_1 b_1^P(1-\beta_2)u,\\
\underline{p}_1 &= c+\frac{b_1^P-b_2^P-\gamma_1(1-\beta_2)u+\sqrt{\Delta}}{2\gamma_1}
\end{aligned}$$

The advertising function of type-2 product is given by

$$A_2^P(p)=\ln\gamma_2+\ln(p-c)-\ln b_2 \quad \text{with} \quad p\in\left[c+b_2/\gamma_2,\underline{p}_1+(1-\beta_2)u\right].$$

An increase in b_2 will raise the lower bound and lower the upper bound of the support. Type-2 consumers receive fewer ads with high prices and fewer ads with low prices as well. An increase in b_1 will expose type-2 consumers with more type-2 ads with high prices. However, the increase or decrease in type-2 ads may be at the benefit of more or at the cost of fewer type-1 ads. We will come back at this after we analyze the effect of ad price change on type-1 ads.

C2-equilibrium: The advertising function of type-1 product is given by

$$A_1^P(p)=\ln\left(\gamma_1+\gamma_2 e^{-A_2^P\left(\underline{p}_1+(1-\beta_2)u\right)}\right)+\ln(p-c)-\ln b_1 \quad \text{if} \quad p\in[\underline{p}_1,\beta_2 u].$$

An increase in b_1 would increase $\underline{p}_1$ and not affect the upper bound $\beta_2 u$. Therefore, for type-1 consumers, an increase in ad price b_1 would increase the average product price by reducing the amount of low prices advertised for product-1. Similarly, an increase in ad price b_2 would decrease the average price by increasing the amount of low prices advertised for product-1.

Now we analyze type- consumers. Notice that this range of prices are the only ones of type-1 ads that may cross-sell to type-2. When b_1 increases, an increase in $\underline{p}_1$ implies more type-2 ads and fewer type-1 ads are sent. For type-2 consumers, the increase of type-2 ads has density $1/(p-c)$ while the decrease of "utility-equivalent" type-1 ads has density $1/(p-(1-\beta_2)u-c)$. Therefore, type-2 consumers would receive fewer ads with relatively high prices. When b_2 increases, an decrease in $\underline{p}_1$ implies fewer type-2 ads and more type-1 ads are sent. The total amount of relevant ads for type-2 consumers is

$$\begin{aligned}
A_1^P\left(\beta_2 u\right)+A_2^P\left(\underline{p}_1+(1-\beta_2)u\right) &= \ln\left(\gamma_1+\gamma_2 e^{-A_2^P\left(\underline{p}_1+(1-\beta_2)u\right)}\right)+\ln\left(p-c\right)-\ln b_1 \\
&\quad +\ln\gamma_2+\ln\left(p-c\right)-\ln b_2 \\
&= \ln\gamma_2-\ln\left(\frac{b_1}{\underline{p}_1-c}-\gamma_1\right)+\ln\left(\beta_2 u-c\right)-\ln\left(\underline{p}_1-c\right) \\
&= \ln\gamma_2+\ln\left(\beta_2 u-c\right)-\ln\left(b_1-\gamma_1\left(\underline{p}_1-c\right)\right)
\end{aligned}$$

which is decreasing in b_2. Therefore, an increase in b_2 would reduce the amount of lower product prices, increase the amount of higher product prices, and reduce the total amount of ads for type-2 consumers. The average price paid by type-2 consumers, as a result, would increase.

C1-equilibrium: On top of the above interval of advertised prices, the remaining of the advertising function is given by

$$A_1^P\left(p\right)=\begin{cases} \ln\left(\gamma_1+\gamma_2 e^{-A_2^P\left(\underline{p}_1+(1-\beta_2)u\right)}\right)+\ln\left(\beta_2 u-c\right)-\ln b_1 & \text{if} \quad p\in\left(\beta_2 u,\tilde{u}\right) \\ \ln\gamma_1+\ln\left(p-c\right)-\ln b_1 & \text{if} \quad p\in\left[\tilde{u},u\right] \end{cases}$$

where $\tilde{u}$ is determined by $A_1^P(\beta_2 u)=A_1^P(\tilde{u})$. An increase in b_1 would increase $\underline{p}_1$ and hence decrease $A_1^P(\beta_2 u)$ and $\tilde{u}$. Once again, it has no effect on the density at each advertised price. Therefore, combining the three segments of advertised prices, an increase in b_1 would reduce the amount of lowest product prices, increase the amount of higher product prices, and reduce the total amount of ads. Therefore, the average price paid by type 1 consumers will also increase. An increase in b_2 will reduce $\underline{p}_1$ and hence increase $\tilde{u}$. The total amount of type-1 ads is unaffected. Therefore, the average price paid by type 1 consumers will decrease.

7.5 Proof of Proposition 5

Proposition 5 is a summary of Lemmas 7-11 below.

7.5.1 W1 and W2 equilibrium

If β_2 is small, type-2 consumers view product 1 as a poor substitute of product 2 and hence equilibrium in the product market features only within-product competition. In this case, the support of $A_2^P(p)$ is $[\underline{p}_2, u]$ and the support of $A_1^P(p)$ is $[\underline{p}_1, u]$ with $\underline{p}_1 \geq \beta_2 u$. We consider two subcases, depending on whether $\underline{p}_1 > \beta_2 u$ or $\underline{p}_1 = \beta_2 u$.

If $\underline{p}_1 > \beta_2 u$, the marginal profit of type 1 firms by one additional ad at price $p \in [\underline{p}_1, u]$ is

$$\pi_1^P \left(p; A_1^P, A_2^P\right) = \gamma_1 e^{-A_1^P(p)} (p - c) - b_1^P,$$

and the marginal profit of type 2 firms generated by one additional ad at price $p \in [\underline{p}_2, u]$ is

$$\pi_2^P \left(p; A_1^P, A_2^P\right) = \gamma_2 e^{-A_2^P(p)} (p - c) - b_2^P.$$

The zero-profit condition $\pi_i^P \left(p; A_1^P, A_2^P\right) = 0$ for all $p \in [\underline{p}_i, u]$ with $\underline{p}_1 > \beta_2 u$ implies that

$$A_i^P (p) = \ln \gamma_i + \ln (p - c) - \ln b_i^P.$$

It follows from $A_i^P(\underline{p}_i) = 0$ that

$$\underline{p}_i = c + b_i^P / \gamma_i.$$

If $\underline{p}_1 = \beta_2 u$, the marginal profit of type 1 firms by one additional ad at price $p = \beta_2 u$ is

$$\pi_1^P \left(\beta_2 u; A_1^P, A_2^P\right) = \left(\gamma_1 + \gamma_2 e^{-A_2^P(u)}\right) e^{-A_1^P(\beta_2 u)} (\beta_2 u - c) - b_1^P.$$

The zero-profit condition $\pi_1^P \left(\beta_2 u; A_1^P, A_2^P\right) = 0$, together with the constraint $A_1^P (\beta_2 u) = 0$, implies that

$$\ln \left(\gamma_1 + \gamma_2 e^{-A_2^P(u)}\right) + \ln (\beta_2 u - c) - \ln b_1^P = 0.$$

Therefore, the constraint $\underline{p}_1 = \beta_2 u$ or $A_1^P (\beta_2 u) = 0$ is equivalent to

$$b_1^P = \gamma_1 (\beta_2 u - c) + b_2^P \frac{\beta_2 u - c}{u - c} = \rho \left(\gamma_1 (u - c) + b_2^P\right). \tag{9}$$

The total demand for ads of product i is

$$A_i^P (u) = \ln \gamma_i + \ln (u - c) - \ln b_i^P.$$

Given ad prices b_1^P and b_2^P, the platform's ad revenue $b_1^P A_1^P (u) + b_2^P A_2^P (u)$ can be rewritten as

$$R_W^P = b_1^P \left(\ln \gamma_1 + \ln (u - c) - \ln b_1^P\right) + b_2^P \left(\ln \gamma_2 + \ln (u - c) - \ln b_2^P\right) \tag{10}$$

The optimization problem of the monopoly platform is to choose b_1^P and b_2^P to maximize R_W^P

subject to $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$ and

$$\underline{p}_1 \geq \beta_2 u. \tag{11}$$

We consider two sub-cases, depending on whether constraint (11) is binding.

If constraint (11) is not binding, the solutions to the optimization problem without any constraints are

$$b_i^P = b_i^* \equiv \frac{\gamma_i (u-c)}{e}.$$

The equilibrium advertising functions are

$$A_i^P(p) = \ln(p-c) - \ln(u-c) + 1.$$

It is easy to verify that constraints $A_1^P(u) \geq 0$ and $A_2^P(u) \geq 0$ are satisfied. Constraint (11) is equivalent to

$$c + \frac{u-c}{e} \geq \beta_2 u \Longleftrightarrow \rho \leq \frac{1}{e}.$$

The equilibrium sales functions are

$$S_{i\to i}^P(p) = \gamma_i - \frac{b_i^P}{p-c} = \gamma_i - \frac{\gamma_i}{e}\frac{u-c}{p-c}.$$

The total number of matched ads of type-i product in advertising market is

$$\gamma_i A_i^P(u) = \gamma_i \left(\ln \gamma_i + \ln(u-c) - \ln b_i^P\right) = \gamma_i,$$

and the market size of type-i product in product market is

$$S_{i\to i}^P(u) = \gamma_i \left(1 - \frac{b_i^P}{\gamma_i(u-c)}\right) = \gamma_i \frac{e-1}{e}.$$

The consumer acquisition cost per consumer for type-i firm is

$$\frac{b_i^P A_i^P(u)}{S_i^P(u)} = \frac{u-c}{e-1}.$$

Hence, by comparing to what we obtain in Section 5.1, we conclude that the two privacy modes generate identical equilibrium advertising functions, sales functions, market sizes, and customer acquisition costs for every product.

If constraint (11) is binding (i.e., $\rho \geq 1/e$), then it is equivalent to (9). Hence, we can use it to rewrite the platform's optimization problem as

$$\max_{b_2^P} \left\{ \begin{array}{c} \rho\left(\gamma_1(u-c) + b_2^P\right)\left(\ln\gamma_1 + \ln(u-c) - \ln\left(\rho\left(\gamma_1(u-c)+b_2^P\right)\right)\right) \\ +b_2^P\left(\ln\gamma_2 + \ln(u-c) - \ln b_2^P\right) \end{array} \right\}$$

subject to $A_1^P(u) \geq 0$ and $A_2^P(u) \geq 0$ which are equivalent to

$$b_2^P \leq \min\left\{\frac{1-\rho}{\rho}\gamma_1(u-c), \gamma_2(u-c)\right\}.$$

The derivative of ad revenue R_W^P with respect to b_2^P is given by

$$\frac{dR_W^P}{db_2^P} = \rho\left(\ln\gamma_1 + \ln(u-c) - \ln\left(\rho\left(\gamma_1(u-c)+b_2^P\right)\right) - 1\right) + \ln\gamma_2 + \ln(u-c) - \ln b_2^P - 1$$

It is easy to see R_W^P is concave in b_2^P and hence the optimal ad price b_2^P is implicitly determined by the first-order condition $\frac{dR_W^P}{db_2^P} = 0$. To ensure $A_1^P(u) \geq 0$, we need

$$\frac{dR_W^P}{db_2^P}|_{b_2^P=\frac{1-\rho}{\rho}\gamma_1(u-c)} \leq 0 \Longleftrightarrow -\rho + \ln\gamma_2 - \ln\frac{1-\rho}{\rho}\gamma_1 - 1 \leq 0 \Longleftrightarrow \frac{\gamma_2}{\gamma_1} \leq \frac{1-\rho}{\rho}e^{1+\rho}.$$

To ensure $A_2^P(u) \geq 0$, we need

$$\frac{dR_W^P}{db_2^P}|_{b_2^P=\gamma_2(u-c)} \leq 0 \Longleftrightarrow \rho\left(\ln\frac{\gamma_1}{\rho} - 1\right) - 1 \leq 0 \Longleftrightarrow \gamma_1 \leq \rho e^{1+\frac{1}{\rho}},$$

which is always satisfied since $\rho \geq 1/e$. Finally, we would like to compare the optimal ad prices b_i^P with $\underline{p}_1 = \beta_2 u$ to the welfare-neutral prices b_i^*. Since $\rho \geq 1/e$, we have

$$\frac{dR_W^P}{db_2^P}|_{b_2^P=b_2^*} = -\rho\ln\left(\rho e + \frac{\gamma_2}{\gamma_1}\rho\right) < 0,$$

which implies that $b_2^P < b_2^*$. On the other hand,

$$b_1^P = \rho\left(\gamma_1(u-c) + b_2^P\right) > \rho\gamma_1(u-c) = \rho e b_1^* \geq b_1^*.$$

Now we are ready to compare consumer welfare under full privacy with consumer welfare under no privacy when $\underline{p}_1 = \beta_2 u$. Since the equilibrium features only within-product competition, a lower ad price for product i implies more ads from type-i ads of any prices and hence higher welfare for type-i consumers under full privacy relative to no privacy. Given that $b_2^P < b_2^*$ and $b_1^P > b_1^*$, we conclude that type-2 consumers are better off and type-1 consumers are worse off under full privacy.

The total number of matched ads of type-i product in advertising market is

$$\gamma_i A_i^P(u) = \gamma_i\left(\ln\gamma_i + \ln(u-c) - \ln b_i^P\right) = \gamma_i \ln\frac{b_i^*}{b_i^P},$$

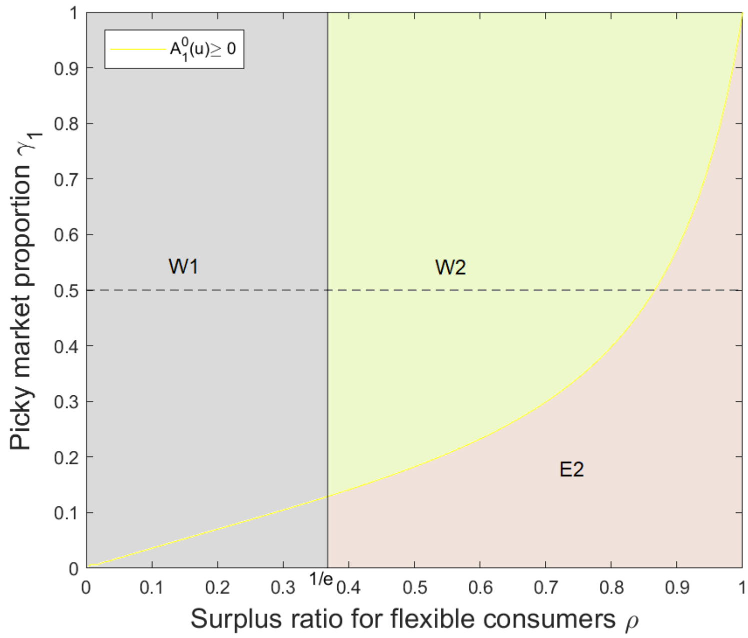

Figure 3: Parameter regions for existence of W2 equilibrium

and the market size of type-i product in product market is

$$S^P_{i\to i}(u) = \gamma_i\left(1 - \frac{b_i^P}{\gamma_i(u-c)}\right) = \gamma_i\frac{e - b_i^P/b_i^*}{e}.$$

The consumer acquisition cost per consumer for type-i firms is

$$\frac{b_i^P A_i^P(u)}{S^P_{i\to i}(u)} = \frac{b_i^P \ln\frac{\gamma_i(u-c)}{b_i^P}}{\gamma_i\left(1 - \frac{b_i^P}{\gamma_i(u-c)}\right)} = (u-c)\frac{b_i^P}{eb_i^* - b_i^P}\ln\frac{eb_i^*}{b_i^P},$$

which is strictly increasing in b_i^P. To see this, note that

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{(u-c)\ln x}{x-1}\right) &= (u-c)\frac{\frac{1}{x}(x-1) - \ln x}{(x-1)^2} \\ &= (u-c)\frac{1 - \frac{1}{x} - \ln x}{(x-1)^2} < 0 \quad \text{for} \quad x > 1,\end{aligned}$$

and replace x by eb_i^*/b_i^P. Therefore, with $b_1^P > b_1^*$ and $b_2^P < b_2^*$, the market size of type-1 product in both advertising and product market shrinks, and the consumer acquisition cost of type-1 product rises under privacy, vice versa for type-2 product.

Lemma 7. *For W-equilibrium,*

1. ***W1 equilibrium:*** *optimal for the platform by setting $b_i^P = b_i^*$ when $\rho \leq 1/e$. In equilibrium,*

the two privacy modes generate identical equilibrium advertising functions, sales functions, market sizes, and customer acquisition costs for every product, and hence all parties — both types of consumers and the platform — are indifferent between the two privacy modes.

2. ***W2 equilibrium:*** *possible when $\rho \geq 1/e$ and $\frac{\gamma_2}{\gamma_1} \leq \frac{1-\rho}{\rho} e^{1+\rho}$. The optimal ad prices satisfy $b_2^P < b_2^*$ and $b_1^P > b_1^*$, and in equilibrium flexible consumers are better off and picky consumers are worse off under full privacy. The market size of flexible consumer product in both advertising and product market expands and the consumer acquisition cost declines, vice versa for picky consumer product.*

7.5.2 C1 equilibrium

Suppose β_2 is large so that $\rho > 1/e$. In this case, if product 1 is priced competitively against product 2, type-2 consumers may buy product 1 rather than product 2. Hence, the equilibrium in the market of type-2 consumers may feature cross-product competition. Such an equilibrium must satisfy $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$ and constraints (6) and (7):

$$A_1^P(\beta_2 u) \geq 0,$$

and

$$A_1^P(u) \geq A_1^P(\beta_2 u).$$

In what follows, we assume that the platform wants to induce equilibrium with cross-product competition. In any such equilibrium, type-2 firms will advertise prices $p \in [\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$. Free entry implies that $\pi_2^P\left(p; A_1^P, A_2^P\right) = 0$ for all $p \in [\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$, or equivalently

$$\gamma_2 e^{-A_2^P(p)}(p-c) - b_2^P = 0 \tag{12}$$

which implies that

$$A_2^P(p) = \ln \gamma_2 + \ln(p-c) - \ln b_2^P.$$

In particular, we have

$$A_2^P(u) = A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right) = \ln \gamma_2 + \ln\left(\underline{p}_1 - c + (1-\beta_2)u\right) - \ln b_2^P. \tag{13}$$

For the remaining analysis, we need to consider separately two cases: either $\bar{p}_1 = u$ or $\bar{p}_1 = \beta_2 u$.

Consider first the case of $\bar{p}_1 = u$. The set of prices that type-1 firms will advertise in equilibrium takes the form of $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ with $\underline{p}_1 < \beta_2 u < \tilde{u} < u$. The marginal profit of type-1 firms by sending one additional ad at price $p \leq \beta_2 u$ is

$$\pi_1^P\left(p; A_1^P, A_2^P\right) = \left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right) e^{-A_1^P(p)}(p-c) - b_1^P.$$

For any equilibrium price $p \leq \beta_2 u$ advertised by type-1 firms, zero-profit condition implies that

$$\left[\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right] e^{-A_1^P(p)} (p - c) - b_1^P = 0. \tag{14}$$

By setting $p = \underline{p}_1 + (1-\beta_2)u$ in (12) and $p = \underline{p}_1$ in (14) and cancelling out $e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}$, we obtain $\underline{p}_1$ implicitly as solution to

$$\frac{b_1^P}{\underline{p}_1 - c} = \gamma_1 + \frac{b_2^P}{\underline{p}_1 + (1-\beta_2)u - c}. \tag{15}$$

We can solve $\underline{p}_1$ explicitly as

$$\underline{p}_1 = c + \frac{b_1^P - b_2^P - \gamma_1(1-\beta_2)u + \sqrt{\Delta}}{2\gamma_1} \tag{16}$$

where

$$\Delta = \left(b_1^P - b_2^P - \gamma_1(1-\beta_2)u\right)^2 + 4\gamma_1 b_1^P (1-\beta_2)u.$$

The marginal profit of type-1 firms by sending one additional ad at price $p \in [\tilde{u}, u]$ is

$$\pi_1^P\left(p; A_1^P, A_2^P\right) = \gamma_1 e^{-A_1^P(p)} (p - c) - b_1^P.$$

By setting $p = u$ and using the zero-profit condition $\pi_1^P\left(p; A_1^P, A_2^P\right) = 0$, we obtain

$$A_1^P(u) = \ln \gamma_1 + \ln(u - c) - \ln b_1^P. \tag{17}$$

The platform's optimization problem is to choose b_1^P and b_2^P to maximize its ad revenue

$$\begin{aligned} R_{C1}^P &= b_1^P A_1^P(u) + b_2^P A_2^P(u) \\ &= b_1^P \left[\ln \gamma_1 + \ln(u - c) - \ln b_1^P\right] + b_2^P \left[\ln \gamma_2 + \ln\left(\underline{p}_1 - c + (1-\beta_2)u\right) - \ln b_2^P\right] \end{aligned}$$

subject to $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$ and constraints (6) and (7). Consider the relaxed problem without any constraints. The first-order conditions are

$$\ln \gamma_1 + \ln(u - c) - \ln b_1^P + \frac{b_2^P \frac{\partial \underline{p}_1}{\partial b_1^P}}{\underline{p}_1 - c + (1-\beta_2)u} - 1 = 0, \tag{18}$$

$$\ln \gamma_2 + \ln\left(\underline{p}_1 + (1-\beta_2)u - c\right) - \ln b_2^P + \frac{b_2^P \frac{\partial \underline{p}_1}{\partial b_2^P}}{\underline{p}_1 - c + (1-\beta_2)u} - 1 = 0. \tag{19}$$

where

$$\frac{\partial \underline{p}_1}{\partial b_1^P} = \frac{\underline{p}_1 + (1-\beta_2)u - c}{\sqrt{\Delta}} > 0,$$
$$\frac{\partial \underline{p}_1}{\partial b_2^P} = -\frac{\underline{p}_1 - c}{\sqrt{\Delta}} < 0.$$

It follows that

$$\ln \gamma_1 + \ln (u-c) - \ln b_1^P - 1 < 0,$$
$$\ln \gamma_2 + \ln \left(\underline{p}_1 + (1-\beta_2)u - c\right) - \ln b_2^P - 1 > 0,$$

and hence

$$b_1^P > b_1^* \text{ and } b_2^P < \frac{\gamma_2(\underline{p}_1 - c + (1-\beta_2)u)}{e}. \tag{20}$$

Since $\underline{p}_1 \leq \beta_2 u$, the second part of (20) immediately implies that

$$b_2^P < b_2^*.$$

Therefore, b_2^P is lower than the welfare-neutral ad price b_2^* while b_1^P is higher than the welfare-neutral ad price b_1^*.

Before we proceed to verify that the solution to the relaxed problem satisfies all dropped constraint, we need to first drive $A_i^P(p)$ for all equilibrium prices p. For $p \in [c+b_2^p/\gamma_2, \underline{p}_1+(1-\beta_2)u]$, the zero profit condition for type-2 firms is

$$\gamma_2 e^{-A_2^P(p)} (p-c) - b_2^P = 0 \implies A_2^P(p) = \ln \gamma_2 + \ln(p-c) - \ln b_2^P.$$

For $p \in [\underline{p}_1, \beta_2 u]$, the zero-profit condition for type-1 firms is

$$\left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1+(1-\beta_2)u\right)}\right) e^{-A_1^P(p)} (p-c) - b_1^P = 0.$$

We substitute $A_2^P(p)$ in and obtain

$$\begin{aligned} A_1^P(p) &= \ln\left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1+(1-\beta_2)u\right)}\right) + \ln(p-c) - \ln b_1^P \\ &= \ln\left(\gamma_1 + \frac{b_2^P}{\underline{p}_1 - c + (1-\beta_2)u}\right) + \ln(p-c) - \ln b_1^P \\ &= \ln(p-c) - \ln(\underline{p}_1 - c) \end{aligned}$$

where the last equality follows from (15). For $p \in [\tilde{u}, u]$, the zero-profit condition for type-1 firms is

$$\gamma_1 e^{-A_1^P(p)} (p-c) - b_1^P = 0 \implies A_1^P(p) = \ln \gamma_1 + \ln(p-c) - \ln b_1^P,$$

where $\tilde{u}$ is given by type-1 firms' indifference condition between advertising $\beta_2 u$ and $\tilde{u}$:

$$\ln(\beta_2 u - c) - \ln(\underline{p}_1 - c) = \ln\gamma_1 + \ln(\tilde{u} - c) - \ln b_1^P. \tag{21}$$

Now we are ready to derive conditions under which the solution to the relaxed problem satisfies all dropped constraints. Constraint (7) of $A_1^P(u) \geq A_1^P(\beta_2 u)$ can be rewritten as

$$\ln\gamma_1 + \ln(u - c) - \ln b_1^P \geq \ln\left(\gamma_1 + \frac{b_2^P}{\underline{p}_1 - c + (1-\beta_2)u}\right) + \ln(\beta_2 u - c) - \ln b_1^P$$

which is equivalent to

$$b_2^P \leq \frac{\gamma_1(1-\rho)}{\rho}\left(\underline{p}_1 - c + (1-\rho)(u-c)\right). \tag{22}$$

Constraint (6) of $A_1^P(\beta_2 u) \geq 0$ is equivalent to

$$\underline{p}_1 \leq \beta_2 u, \tag{23}$$

which is also equivalent to

$$\left(\gamma_1 + \frac{b_2^P}{\underline{p}_1 - c + (1-\beta_2)u}\right)(\beta_2 u - c) \geq b_1^P \Longleftrightarrow \gamma_1 \geq \frac{b_1^P}{\rho(u-c)} - \frac{b_2^P}{\underline{p}_1 - c + (1-\rho)(u-c)}. \tag{24}$$

Note that the constraints $A_i^P(u) \geq 0$ are equivalent to

$$\begin{aligned} A_1^P(u) &\geq 0 \Longleftrightarrow b_1^P \leq \gamma_1(u-c), \\ A_2^P(u) &\geq 0 \Longleftrightarrow b_2^P \leq \gamma_2\left(\underline{p}_1 - c + (1-\beta_2)u\right). \end{aligned}$$

The first inequality is implied by (22) and (24), and the second is implied by (20). Therefore, if γ_1 and ρ satisfy conditions (22) and (23), then the ad prices given by (18) and (19) are indeed optimal for the platform.

Next we would like to argue that in equilibrium with cross-product competition and $\bar{p}_1 = u$, type-2 consumers are better off while type 1 consumers are worse off with full privacy. Consider first the welfare of type-2 consumers. Under full privacy, they receive product 2 price offers distributed according to $A_2^P(p)$ with $p \in [\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$ as well as product 1 price offers distributed according to $A_1^P(p)$ with $p \in [\underline{p}_1, \beta_2 u]$. A product 1 offer at price $p - (1-\beta_2)u$ generates the same surplus for type-2 consumers as a product 2 offer at price p. Under no privacy, they receive only product 2 offers with prices distributed according to $A_2^{NP}(p)$ for $p \in [b_2^*/\gamma_2 + c, u]$. Note that $\underline{p}_2 = b_2^P/\gamma_2 + c < b_2^*/\gamma_2 + c$, so a sufficient condition for type-2 consumers to prefer full privacy is

$$\begin{cases} A_2^P(p) > A_2^{NP}(p) & \text{for} \quad p \in [\underline{p}_2, \underline{p}_1 + (1-\beta_2)u] \\ \frac{\partial A_1^P(p-(1-\beta_2)u)}{\partial p} \geq \frac{\partial A_2^{NP}(p)}{\partial p} & \text{for} \quad p \in [\underline{p}_1 + (1-\beta_2)u, u] \end{cases}.$$

The first part of the condition says that type-2 consumers receive more ads from type-2 firms under full privacy with prices no higher than p for every p in the range of prices advertised by type-2 firms in equilibrium under full privacy. The second part of the condition implies that type-2 consumers receive more ads from type-1 firms under full privacy which generate the same surplus as ads from type-2 firms of prices p for every p that is advertised by type-2 firms under no privacy but not under full privacy. The first part is implied by $b_2^P < b_2^*$ and the second part is always true. Therefore, type-2 consumers are better off with full privacy.

For type-1 consumers who buy only product 1, the sales distribution function $S_{1\to1}^P(p)$ is

$$S_{1\to1}^P(p) = \gamma_1\left(1 - e^{-A_1^P(p)}\right) = \begin{cases} \gamma_1\left(1 - \frac{\underline{p}_1 - c}{p-c}\right) & \text{if} \quad p \in [\underline{p}_1, \beta_2 u] \\ \gamma_1\left(1 - \frac{\underline{p}_1 - c}{\rho(u-c)}\right) & \text{if} \quad p \in (\beta_2 u, \tilde{u}] \\ \gamma_1\left(1 - \frac{b_1^P}{\gamma_1(p-c)}\right) & \text{if} \quad p \in (\tilde{u}, u] \end{cases}$$

The consumer surplus for a type-1 consumer is given by

$$\begin{aligned} & \int_{\underline{p}_1}^{u} (u-p) d\left(\frac{S_{1\to1}^P(p)}{\gamma_1}\right) \\ = \ & \frac{1}{\gamma_1}\int_{\underline{p}_1}^{u} S_{1\to1}^P(p)\, dp \\ = \ & \int_{\underline{p}_1}^{\beta_2 u}\left(1 - \frac{\underline{p}_1 - c}{p-c}\right) dp + \int_{\beta_2 u}^{\tilde{u}}\left(1 - \frac{\underline{p}_1 - c}{\rho(u-c)}\right) dp + \int_{\tilde{u}}^{u}\left(1 - \frac{b_1^P}{\gamma_1(p-c)}\right) dp \\ = \ & \left(u - \underline{p}_1\right) - (\underline{p}_1 - c)\ln\frac{\beta_2 u - c}{\underline{p}_1 - c} - \frac{\underline{p}_1 - c}{\rho(u-c)}(\tilde{u} - \beta_2 u) - \frac{b_1^P}{\gamma_1}\ln\frac{u-c}{\tilde{u}-c} \\ = \ & \left(u - \underline{p}_1\right) + (\underline{p}_1 - c)\ln\frac{\underline{p}_1 - c}{\rho(u-c)} - (\underline{p}_1 - c)\left(\frac{b_1^P}{\gamma_1(\underline{p}_1 - c)} - 1\right) - \frac{b_1^P}{\gamma_1}\ln\frac{\gamma_1(\underline{p}_1 - c)}{\rho b_1^P} \\ = \ & (u-c) - \frac{b_1^P}{\gamma_1} + \frac{b_1^P}{\gamma_1}\ln\frac{b_1^P}{\gamma_1(u-c)} - \left(\frac{b_1^P}{\gamma_1} - (\underline{p}_1 - c)\right)\ln\frac{\underline{p}_1 - c}{\rho(u-c)} \end{aligned}$$

Type-1 consumers would prefer no privacy if and only if

$$\frac{1}{\gamma_1}\int_{\underline{p}_1}^{u} S_{1\to1}^P(p)\, dp \leq (u-c)\left(1 - \frac{2}{e}\right)$$

or equivalently

$$\frac{b_1^P}{\gamma_1(u-c)} - \frac{b_1^P}{\gamma_1(u-c)}\ln\frac{b_1^P}{\gamma_1(u-c)} + \left(\frac{b_1^P}{\gamma_1(u-c)} - \frac{\underline{p}_1 - c}{u-c}\right)\ln\frac{\underline{p}_1 - c}{\rho(u-c)} \geq \frac{2}{e}.$$

It can be verified that this inequality holds under conditions (22) and (23).

The total number of matched ads of type-1 product in advertising market is

$$\gamma_1 A_1^P(u) = \gamma_1 \left(\ln \gamma_1 + \ln (u-c) - \ln b_1^P\right) = \gamma_1 \ln \frac{b_1^*}{b_1^P} < \gamma_1,$$

and the total number of matched ads of type-2 product in advertising market is

$$\gamma_2 A_2^P(u) = \gamma_2 \left(\ln \gamma_2 + \ln \left(\underline{p}_1 + (1-\beta_2)u - c\right) - \ln b_2^P\right) > \gamma_2.$$

The market size of type-1 product in product market is

$$\begin{aligned}
S_{1\to 1}^P(u) + S_{1\to 2}^P(\beta_2 u) &= \gamma_1 \left(1 - e^{-A_1^P(u)}\right) + \gamma_2 e^{-A_2^P(\underline{p}_1+(1-\beta_2)u-c)} \left(1 - e^{-A_1^P(\beta_2 u)}\right) \\
&= \gamma_1 \left(1 - \frac{b_1^P}{\gamma_1(u-c)}\right) \\
&\quad + \frac{b_2^P}{\underline{p}_1 - c + (1-\beta_2)u} \left(1 - \frac{b_1^P}{\left(\gamma_1 + \frac{b_2^P}{\underline{p}_1 - c + (1-\beta_2)u}\right)(\beta_2 u - c)}\right) \\
&= \gamma_1 \left(1 - \frac{b_1^P}{\gamma_1(u-c)}\right) + \frac{b_2^P}{\underline{p}_1 - c + (1-\beta_2)u} \left(1 - \frac{\underline{p}_1 - c}{\beta_2 u - c}\right) \\
&= \gamma_1 \left(1 - \frac{b_1^P}{\gamma_1(u-c)}\right) + \left(\frac{b_1^P}{\underline{p}_1 - c} - \gamma_1\right) \left(1 - \frac{\underline{p}_1 - c}{\beta_2 u - c}\right) \\
&= \frac{b_1^P}{\underline{p}_1 - c} - \frac{b_1^P}{u-c} - \frac{b_1^P}{\beta_2 u - c} + \gamma_1 \frac{\underline{p}_1 - c}{\beta_2 u - c}.
\end{aligned}$$

Plugging in b_1^P and b_2^P, it can be numerically shown that it is smaller than the total number of sales under no privacy $\gamma_1(e-1)/e$ if and only if $\rho < g_1(\gamma_1)$ for some weakly increasing function g_1. The market size of type-2 product in product market is

$$\begin{aligned}
S_{2\to 2}^P(\beta_2 u) &= \gamma_2 \left(1 - e^{-A_2^P(\underline{p}_1+(1-\beta_2)u-c}\right) \\
&= \gamma_2 \left(1 - \frac{b_2^P}{\gamma_2(\underline{p}_1 + (1-\beta_2)u - c)}\right) \\
&> \gamma_2 \frac{e-1}{e}.
\end{aligned}$$

The consumer acquisition cost per consumer for type-1 firms is

$$\frac{b_1^P A_1^P(u)}{S_{1\to 1}^P(u) + S_{1\to 2}^P(\beta_2 u)} = \frac{b_1^P \ln \frac{\gamma_1(u-c)}{b_1^P}}{\frac{b_1}{\underline{p}_1 - c} - \frac{b_1}{u-c} - \frac{b_1}{\beta_2 u - c} + \gamma_1 \frac{\underline{p}_1 - c}{\beta_2 u - c}},$$

which can be shown numerically smaller than the consumer acquisition cost under no privacy

$(u-c)/(e-1)$ if and only if $\rho < g_2(\gamma_1)$ for some weakly increasing function g_2. The consumer acquisition cost per consumer for type-2 firms is

$$\begin{aligned}
\frac{b_2^P A_2^P(\underline{p}_1 + (1-\beta_2)u - c)}{S_{2\to 2}^P(\underline{p}_1 + (1-\beta_2)u - c)} &= \frac{b_2^P \ln \frac{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}{b_2^P}}{\gamma_2\left(1 - \frac{b_2^P}{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}\right)} \\
&= (\underline{p}_1 + (1-\beta_2)u - c)\frac{\ln \frac{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}{b_2^P}}{\frac{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}{b_2^P} - 1} \\
&< \frac{\underline{p}_1 + (1-\beta_2)u - c}{e-1} \\
&< \frac{u-c}{e-1}
\end{aligned}$$

from condition 20.

We summarize the above findings in the following lemma.

Lemma 8. *Suppose $\rho \geq 1/e$ and conditions (22) and (23) hold, and that the platform wants to induce equilibrium with cross product competition and $\bar{p}_1 = u$. Then the optimal ad prices must satisfy $b_1^P > b_1^*$ and $b_2^P < b_2^*$, and*

- ***Consumers:*** *flexible consumers are better off while picky consumers are worse off with full privacy.*
- ***Advertising market:*** *the total number of matched ads is lower for type-1 product and higher for type-2 product under full privacy.*
- ***Product market:*** *under full privacy, the total number of sales is higher for type-2 product with lower consumer acquisition cost; for type-1 product, the total number of sales is higher when $\rho > g_1(\gamma_1)$, and the consumer acquisition cost is higher when $\rho > g_2(\gamma_1)$.*

It can be verified that condition (22) can be represented as $\rho \leq h_1(\theta)$ where $h_1(\theta)$ is strictly increasing while condition (23) is redundant.

7.5.3 C2 equilibrium

Next consider the case of $\bar{p}_1 = \beta_2 u$. In this case, the optimal solution to the platform's problem lies on the boundary of $A_1^P(u) = A_1^P(\beta_2 u)$. In other words, type-1 firms never advertise prices above $\beta_2 u$. In particular, they would incur a loss if they were to advertise at price $p = u$. The set of prices that type-1 firms will advertise in equilibrium takes the form of $[\underline{p}_1, \beta_2 u]$. The set of prices that type-2 firms will advertise in equilibrium again takes the form of $[\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$. The expressions for $A_2^P(u)$ and for $\underline{p}_1$ are the same as in the case of $\bar{p}_1 = u$, given by (13) and

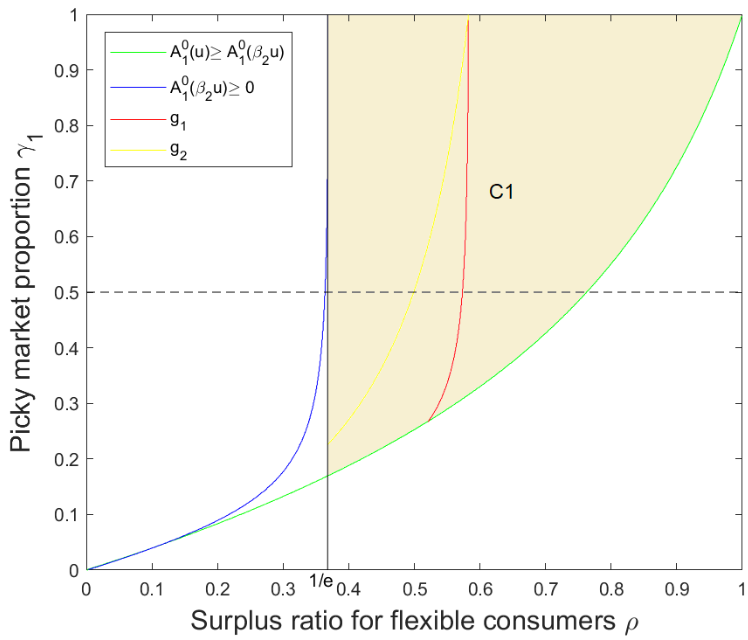

Figure 4: Parameter regions for existence of C1 equilibrium

(16) and replicated here:

$$A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right) = \ln \gamma_2 + \ln\left(\underline{p}_1 - c + (1-\beta_2)u\right) - \ln b_2^P,$$

and

$$\underline{p}_1 = c + \frac{b_1^P - b_2^P - \gamma_1(1-\beta_2)u + \sqrt{\Delta}}{2\gamma_1}$$

where

$$\Delta = \left(b_1^P - b_2^P - \gamma_1(1-\beta_2)u\right)^2 + 4\gamma_1 b_1^P(1-\beta_2)u.$$

The zero-profit conditions $\pi_1^P\left(\beta_2 u; A_1^P, A_2^P\right) = 0$ and $\pi_1^P\left(\underline{p}_1; A_1^P, A_2^P\right) = 0$ can be written as

$$\begin{aligned}\left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right) e^{-A_1^P(\beta_2 u)}\left(\beta_2 u - c\right) - b_1^P &= 0\\ \left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right)\left(\underline{p}_1 - c\right) - b_1^P &= 0\end{aligned}$$

which imply that

$$A_1^P\left(\beta_2 u\right) = \ln\left(\beta_2 u - c\right) - \ln(\underline{p}_1 - c). \tag{25}$$

The optimization problem of the platform is to choose ad prices b_1^P and b_2^P to maximize the total ad revenue

$$R_{C2}^P = b_1^P A_1^P\left(\beta_2 u\right) + b_2^P A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)$$

$$= \quad b_1^P \left[\ln(\beta_2 u - c) - \ln(\underline{p}_1 - c)\right] + b_2^P \left[\ln\gamma_2 + \ln(\underline{p}_1 - c + (1-\beta_2)u) - \ln b_2^P\right].$$

subject to $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$ and constraints (6) and (7). Consider the relaxed problem without the two constraints. We can use (15) to write the first-order conditions as

$$\ln(\beta_2 u - c) - \ln(\underline{p}_1 - c) - \gamma_1 \frac{\partial \underline{p}_1}{\partial b_1^P} = 0, \tag{26}$$

$$-\gamma_1 \frac{\partial \underline{p}_1}{\partial b_2^P} + \ln\gamma_2 + \ln(\underline{p}_1 - c + (1-\beta_2)u) - \ln b_2^P - 1 = 0. \tag{27}$$

It follows from (27) and $\frac{\partial \underline{p}_1}{\partial b_2^P} < 0$ that

$$b_2^P > \frac{\gamma_2(\underline{p}_1 - c + (1-\beta_2)u)}{e}. \tag{28}$$

Next we show that $\gamma_1 \partial\underline{p}_1/\partial b_1^P < 1$. To see this,

$$\begin{aligned}
& \gamma_1 \frac{\partial \underline{p}_1}{\partial b_1^P} < 1 \\
\Leftrightarrow\quad & \underline{p}_1 + (1-\beta_2)u - c < \frac{\sqrt{\Delta}}{\gamma_1} \\
\Leftrightarrow\quad & b_1^P - b_2^P + \gamma_1(1-\beta_2)u < \sqrt{\Delta} \\
\Leftrightarrow\quad & \left(b_1^P - b_2^P + \gamma_1(1-\beta_2)u\right)^2 - \left(b_1^P - b_2^P + \gamma_1(1-\beta_2)u\right)^2 < 4\gamma_1 b_1^P(1-\beta_2)u \\
\Leftrightarrow\quad & b_1^P - b_2^P < b_1^P.
\end{aligned}$$

Therefore, it follows from from (26) that

$$\frac{\underline{p}_1 - c}{\beta_2 u - c} > \frac{1}{e}. \tag{29}$$

We need to verify that the solution to (26) and (27) satisfy the dropped constraints $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$ and constraints (6) and (7). Note that any price $p \in [c + b_2^p/\gamma_2, \underline{p}_1 + (1-\beta_2)u]$ advertised by type-2 firms, we have

$$\gamma_2 e^{-A_2^P(p)}(p - c) - b_2^P = 0,$$

which implies that

$$A_2^P(p) = \ln\gamma_2 + \ln(p - c) - \ln b_2^P.$$

For any price $p \in [\underline{p}_1, \beta_2 u]$ advertised by type-1 firms, we have

$$\left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right) e^{-A_1^P(p)}(p - c) - b_1^P = 0$$

which, together with the zero-profit condition of

$$\pi_1^P\left(\underline{p}_1; A_1^P, A_2^P\right) = \left(\gamma_1 + \gamma_2 e^{-A_2^P\left(\underline{p}_1 + (1-\beta_2)u\right)}\right)(\underline{p}_1 - c) - b_1^P = 0$$

implies

$$A_1^P(p) = \ln(p-c) - \ln(\underline{p}_1 - c).$$

Therefore, constraints $A_1^P(u) \geq 0$ is equivalent to constraint (6) of

$$\underline{p}_1 \leq \beta_2 u,$$

which is implied by first-order condition (26). Constraint $A_2^P(u) \geq 0$ is equivalent to

$$b_2^P \leq \gamma_2(\underline{p}_1 + (1-\beta_2)u - c). \tag{30}$$

Binding constraint (7) of $A_1^P(u) = A_1^P(\beta_2 u)$ is equivalent to $\pi_1^P(u) \leq 0$:

$$\gamma_1 e^{-A_1^P(\beta_2 u)}(u-c) - b_1^P \leq 0,$$

or equivalently

$$b_1^P \geq \frac{\gamma_1(\underline{p}_1 - c)}{\rho}. \tag{31}$$

Therefore, if conditions (30) and (31) are satisfied, then the solution to (26) and (27) is indeed optimal for the platform.

Now consider the consumer welfare. For type-1 consumers who buy only product 1, the sales distribution function is given by

$$S_{1\to1}^P(p) = \gamma_1\left(1 - e^{-A_1^P(p)}\right) = \gamma_1\left(1 - \frac{\underline{p}_1 - c}{p-c}\right) \text{ for } p \in [\underline{p}_1, \beta_2 u].$$

For type-2 consumers who may buy both products, the sales functions are, for $p \in [\underline{p}_2, \underline{p}_1 + (1-\beta_2)u]$,

$$S_{2\to2}^P(p) = \gamma_2\left(1 - e^{-A_2^P(p)}\right) = \gamma_2\left(1 - \frac{b_2^P}{\gamma_2(p-c)}\right)$$

and for $p \in [\underline{p}_1, \beta_2 u]$,

$$S_{1\to2}^P(p) = \gamma_2 e^{-A_2(\underline{p}_1 + (1-\beta_2)u)}\left(1 - e^{-A_1^P(p)}\right) = \frac{b_2^P}{\underline{p}_1 + (1-\beta_2)u - c}\left(1 - \frac{\underline{p}_1 - c}{p-c}\right).$$

The consumer surplus of a type-1 consumers under full privacy is

$$\int_{\underline{p}_1}^{\beta_2 u}(u-p)d\left(\frac{S_{1\to1}^P(p)}{\gamma_1}\right)$$

$$
\begin{aligned}
&= \frac{1}{\gamma_1}\left[(u-\beta_2 u)S^P_{1\to 1}(\beta_2 u) + \int_{\underline{p}_1}^{\beta_2 u} S^P_{1\to 1}(p)\,dp\right] \\
&= (u-\beta_2 u)\left(1-\frac{\underline{p}_1 - c}{\beta_2 u - c}\right) + \int_{\underline{p}_1}^{\beta_2 u}\left(1-\frac{\underline{p}_1 - c}{p-c}\right)dp \\
&= (u-\beta_2 u)\left(1-\frac{\underline{p}_1 - c}{\beta_2 u - c}\right) + \left(\beta_2 u - \underline{p}_1\right) - (\underline{p}_1 - c)\ln\frac{\beta_2 u - c}{\underline{p}_1 - c} \\
&= (u-c)\left[1-\frac{\underline{p}_1 - c}{u-c}\left(\frac{1}{\rho} - \ln\frac{\underline{p}_1 - c}{\rho\,(u-c)}\right)\right]
\end{aligned}
$$

The consumer surplus of a type-2 consumer under full privacy is

$$
\begin{aligned}
&\int_{\underline{p}_2}^{\underline{p}_1+(1-\beta_2)u}(u-p)d\left(\frac{S^P_{2\to 2}(p)}{\gamma_2}\right) + \int_{\underline{p}_1}^{\beta_2 u}(\beta_2 u - p)d\left(\frac{S^P_{1\to 2}(p)}{\gamma_2}\right) \\
=\;& \frac{1}{\gamma_2}\left[\left(\beta_2 u - \underline{p}_1\right)S^P_{2\to 2}\left(\underline{p}_1 + (1-\beta_2)u\right) + \int_{\underline{p}_2}^{\underline{p}_1+(1-\beta_2)u} S^P_{2\to 2}(p)\,dp\right] + \frac{1}{\gamma_2}\int_{\underline{p}_1}^{\beta_2 u} S^P_{1\to 2}(p)\,dp \\
=\;& \left(\beta_2 u - \underline{p}_1\right)\left(1-\frac{b_2^P}{\gamma_2(\underline{p}_1 + (1-\beta_2)u - c)}\right) + \int_{\underline{p}_2}^{\underline{p}_1+(1-\beta_2)u}\left(1-\frac{b_2^P}{\gamma_2(p-c)}\right)dp \\
&+\frac{1}{\gamma_2}\int_{\underline{p}_1}^{\beta_2 u}\frac{b_2^P}{\underline{p}_1 + (1-\beta_2)u - c}\left(1-\frac{\underline{p}_1 - c}{p-c}\right)dp \\
=\;& \left(\beta_2 u - \underline{p}_1\right)\left(1-\frac{b_2^P}{\gamma_2(\underline{p}_1 + (1-\beta_2)u - c)}\right) + \underline{p}_1 + (1-\beta_2)u - \underline{p}_2 \\
&-\frac{b_2^P}{\gamma_2}\ln\frac{\underline{p}_1 + (1-\beta_2)u - c}{\underline{p}_2 - c} + \frac{1}{\gamma_2}\frac{b_2^P}{\underline{p}_1 + (1-\beta_2)u - c}\left[\left(\beta_2 u - \underline{p}_1\right) - \left(\underline{p}_1 - c\right)\ln\frac{\beta_2 u - c}{\underline{p}_1 - c}\right] \\
=\;& u - c - \frac{b_2^P}{\gamma_2} + \frac{b_2^P}{\gamma_2}\ln\frac{b_2^P}{\gamma_2(\underline{p}_1 + (1-\beta_2)u - c)} - \frac{b_2^P(\underline{p}_1 - c)}{\gamma_2(\underline{p}_1 + (1-\beta_2)u - c)}\ln\frac{\beta_2 u - c}{\underline{p}_1 - c} \\
=\;& (u-c)\left[1-\frac{b_2^P}{eb_2^*} + \frac{b_2^P}{eb_2^*}\ln\frac{b_2^P}{\gamma_2(\underline{p}_1 + (1-\beta_2)u - c)} - \frac{b_2^P}{eb_2^*}\frac{\underline{p}_1 - c}{\underline{p}_1 + (1-\beta_2)u - c}\ln\frac{\beta_2 u - c}{\underline{p}_1 - c}\right]
\end{aligned}
$$

Since the consumer surplus for both types of consumers is $(u-c)(1-2/e)$, type-1 consumers would prefer no privacy if and only if

$$
\frac{\underline{p}_1 - c}{u-c}\left(\frac{1}{\rho} - \ln\frac{\underline{p}_1 - c}{\rho\,(u-c)}\right) \geq \frac{2}{e}, \tag{32}
$$

and type-2 consumers would prefer full privacy if and only if

$$
1 - \ln\frac{b_2^P}{\gamma_2(\underline{p}_1 + (1-\beta_2)u - c)} + \frac{\underline{p}_1 - c}{\underline{p}_1 + (1-\beta_2)u - c}\ln\frac{\beta_2 u - c}{\underline{p}_1 - c} \leq 2\frac{b_2^*}{b_2^P}. \tag{33}
$$

It can be numerically verified that, under constraints (30) and (31), the first inequality holds if and only if $\gamma_1 \leq h_1(\rho)$ and the second inequality holds if and only if $\gamma_1 \leq h_2(\rho)$. The results are shown in Figure 5.

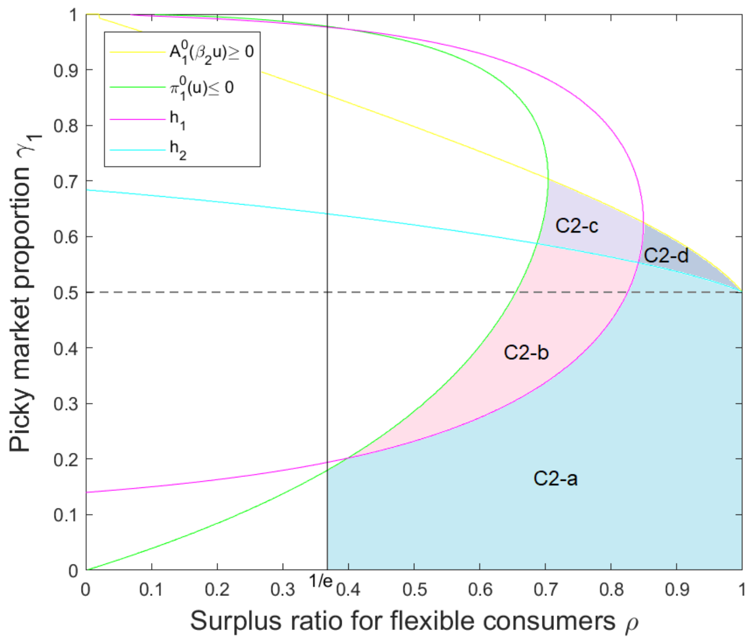

Figure 5: Parameter regions for existence of C2 equilibrium

The total number of matched ads of type-1 product in advertising market is

$$\gamma_1 A_1^P(\beta_2 u) = \gamma_1 \left(\ln(\beta_2 u - c) - \ln\left(\underline{p}_1 - c\right)\right) < \gamma_1,$$

where the inequality follows from (29). The total number of matched ads of type-2 product in advertising market is

$$\gamma_2 A_2^P(\underline{p}_1 + (1-\beta_2)u - c) = \gamma_2 \left(\ln \gamma_2 + \ln\left(\underline{p}_1 + (1-\beta_2)u - c\right) - \ln b_2^P\right) < \gamma_2,$$

where the inequality follows from (28). The market size of type-1 product in product market is

$$\begin{aligned} S_{1\to 1}^P(\beta_2 u) + S_{1\to 2}^P(\beta_2 u) &= \left(\gamma_1 + \gamma_2 e^{-A_2^P(\underline{p}_1 + (1-\beta_2)u - c)}\right)\left(1 - e^{-A_1^P(\beta_2 u)}\right) \\ &= \gamma_1 + \frac{b_2^P}{\underline{p}_1 - c + (1-\beta_2)u} - \frac{b_1^P}{\beta_2 u - c} \\ &= \frac{b_1^P}{\underline{p}_1 - c} - \frac{b_1^P}{\beta_2 u - c}, \end{aligned}$$

which can be shown numerically larger than the total sales under no privacy $\gamma_1(1 - 1/e)$. The market size of type-2 product in product market is

$$\begin{aligned}
S^P_{2\to 2}(\beta_2 u) &= \gamma_2\left(1 - e^{-A^P_2(\underline{p}_1+(1-\beta_2)u-c)}\right)\\
&= \gamma_2\left(1 - \frac{b^P_2}{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}\right)\\
&< \gamma_2\frac{e-1}{e}.
\end{aligned}$$

from condition 28. The consumer acquisition cost per consumer for type-1 firms is

$$\begin{aligned}
\frac{b^P_1 A^P_1(\beta_2 u)}{S^P_{1\to 1}(\beta_2 u)+S^P_{1\to 2}(\beta_2 u)} &= \frac{b^P_1 \ln\frac{\beta_2 u-c}{\underline{p}_1-c}}{\frac{b^P_1}{\underline{p}_1-c}-\frac{b^P_1}{\beta_2 u-c}}\\
&= (\beta_2 u-c)\,\frac{\ln\frac{\beta_2 u-c}{\underline{p}_1-c}}{\frac{\beta_2 u-c}{\underline{p}_1-c}-1},
\end{aligned}$$

which can be shown numerically smaller than the consumer acquisition cost under no privacy $(u-c)/(e-1)$ if and only if $\rho < g_3(\gamma_1)$ for some strictly increasing function g_3. The consumer acquisition cost per consumer for type-2 firms is

$$\frac{b^P_2 A^P_2(\underline{p}_1+(1-\beta_2)u-c)}{S^P_{2\to 2}(\underline{p}_1+(1-\beta_2)u-c)} = \frac{b^P_2\ln\frac{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}{b^P_2}}{\gamma_2\left(1-\frac{b^P_2}{\gamma_2(\underline{p}_1+(1-\beta_2)u-c)}\right)}$$

which can be shown numerically smaller than the consumer acquisition cost under no privacy $(u-c)/(e-1)$ if and only if $\gamma_1 < g_4(\rho)$ for some strictly decreasing function g_4.

Lemma 9. *Suppose $\rho \geq 1/e$ and conditions (30) and (31) hold. Suppose that the platform wants to induce equilibrium with cross product competition and $\bar{p}_1 = \beta_2 u$.*

- ***Consumers:*** *picky consumers are worse off with full privacy if $\gamma_1 < h_1(\rho)$ and better off otherwise, while flexible consumers are better off if $\gamma_1 < h_2(\rho)$ and worse off otherwise.*
- ***Advertising market:*** *the total number of matched ads decreases for both types of product.*
- ***Product market:*** *the total number of sales increases for type-1 product and decreases for type-2 product. The consumer acquisition cost for for type-1 product is smaller when $\rho < g_3(\gamma_1)$ and cost for type-2 product is smaller when $\gamma_1 \leq g_4(\rho)$.*

7.5.4 E1 and E2 equilibrium

In this subsection, we consider the equilibrium with exclusion where the platform sets ad prices such that only one of the products is produced and advertised.

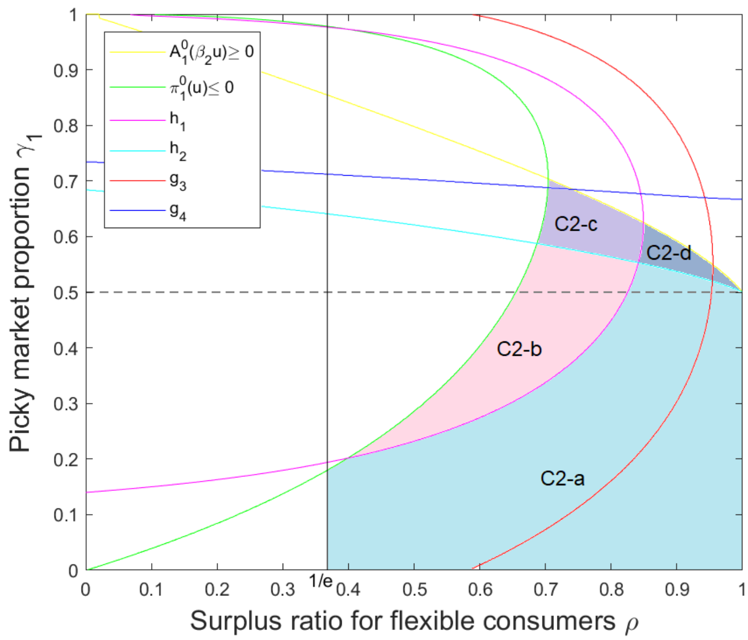

Figure 6: Consumer acquisition cost comparison under C2 equilibrium

E1 equilibrium: Consider first the platform wants to sell ads of product 1 only (E1). Note that if a W1 equilibrium exists (i.e. $\rho \leq 1/e$), it will dominate all other types of equilibrium. Hence, we will focus on the case with $\rho > 1/e$ and hence the support of $A_1^P(p)$ must be $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ (mimicking C1), or $[\underline{p}_1, u]$ with $\underline{p}_1 > \beta_2 u$ (mimicking W1), or $[\underline{p}_1, \beta_2 u]$ (mimicking C2).

Consider first the E1 equilibrium with support $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$. The zero-profit condition $\pi_1^P(p; A_1^P, A_2^P) = 0$ for $p \in [\tilde{u}, u]$ implies that

$$\gamma_1 e^{-A_1^P(p)}(p-c) - b_1^P = 0 \Longrightarrow A_1^P(p) = \ln \gamma_1 + \ln(p-c) - \ln b_1^P.$$

For any equilibrium price $p \in [\underline{p}_1, \beta_2 u]$ advertised by type-1 firms, the zero-profit condition $\pi_1^P(p; A_1^P, A_2^P) = 0$ implies that $\underline{p}_1 = b_1^P + c$ and

$$e^{-A_1^P(p)}(p-c) - b_1^P = 0 \Longrightarrow A_1^P(p) = \ln(p-c) - \ln b_1^P.$$

The platform's ad revenue is

$$R_{E1}^P = b_1^P A_1^P(u) = b_1^P \left(\ln \gamma_1 + \ln(u-c) - b_1^P\right)$$

and the optimal ad price is

$$b_1^P = \frac{\gamma_1 (u-c)}{e} = b_1^*.$$

In order for this to be an equilibrium, we need to make sure that

$$A_1^P(u) > A_1^P(\beta_2 u)$$

Hence, a necessary condition for E1 with $p \in [\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$ to be an equilibrium (in the product market) is $A_1^P(u) > A_1^P(\beta_2 u)$, or equivalently

$$\ln \gamma_1 + \ln(u - c) - b_1^P > \ln(\beta_2 u - c) - b_1^P \iff \gamma_1 > \rho.$$

The following lemma shows that only the C2 form of support is possible for E1 equilibrium.

Lemma 10. *It is never optimal for the platform to induce an E1 equilibrium with support $[\underline{p}_1, u]$ with $\underline{p}_1 > \beta_2 u$ or with support $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$.*

Proof. The first part that platform will not induce an E1 equilibrium with support $[\underline{p}_1, u]$ with $\underline{p}_1 > \beta_2 u$ is easy to see. In this case, type-2 consumers are not served. The platform can earn additional ad revenue from type-2 firms by charging $b_2 = \epsilon$ for some sufficiently small $\epsilon > 0$.

Next, we argue that it is never optimal for the platform to induce an E1 equilibrium with support $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$. To see this, suppose the platform induces a C1 equilibrium by selling both product ads. Recall that in a C1 equilibrium, $\underline{p}_1$ is implicitly determined by condition (15):

$$\frac{b_1^P}{\underline{p}_1 - c} = \gamma_1 + \frac{b_2^P}{\underline{p}_1 + (1 - \beta_2)u - c}.$$

We note that $\underline{p}_1$ is decreasing in b_2^P and that $\underline{p}_1 = b_1^P + c$ when $b_2^P = \gamma_2 \left(b_1^P + (1 - \beta_2)u\right)$. Now suppose the platform charges ad prices at

$$b_1^P = b_1^* \quad \text{and} \quad b_2^P = \gamma_2(b_1^* + (1 - \beta_2)u - \varepsilon)$$

where $\varepsilon > 0$ is small. Then $\underline{p}_1$ is above but can be made arbitrarily close to $(b_1^* + c)$. The support of $A_1^P(p)$ under the C1 equilibrium takes the form of $[\underline{p}_1, \beta_2 u] \cup [\tilde{u}', u]$ where $\beta_2 u < \tilde{u}' < u$ determined by condition (15)

$$\ln(\beta_2 u - c) - \ln(\underline{p}_1 - c) = \ln \gamma_1 + \ln(\tilde{u}' - c) - \ln b_1^*.$$

Therefore, for a sufficiently small $\varepsilon > 0$, $\tilde{u}' < \tilde{u}$ but can be arbitrarily close to $\tilde{u}$. Therefore, such $\tilde{u}'$ is possible with sufficiently small ε. As long as $\beta_2 u < \tilde{u}' < u$, the total demand for ads of product 1 is unchanged at

$$A_1^P(u) = \ln \gamma_1 - \ln(u - c) - \ln b_1^*.$$

The ad revenue from type-1 firms will be the same while the platform earns additional ad revenue from type-2 firms. Hence, the platform will never induce an E1 equilibrium with support

$[\underline{p}_1, \beta_2 u] \cup [\tilde{u}, u]$. $\square$

It remains to consider the E1 equilibrium with price support $[\underline{p}_1, \beta_2 u]$. The zero-profit condition $\pi_1^P\left(p; A_1^P, A_2^P\right) = 0$ for any equilibrium price $p \in [\underline{p}_1, \beta_2 u]$ can be written as

$$e^{-A_1^P(p)}(p-c) - b_1^P = 0 \Longrightarrow A_1^P(p) = \ln(p-c) - \ln b_1^P,$$

with $\underline{p}_1 = b_1^P + c$. The platform's ad revenue is

$$R_{E1}^P = b_1^P A_1^P(\beta_2 u) = b_1^P\left(\ln(\beta_2 u - c) - b_1^P\right),$$

and the optimal ad price is

$$b_1^P = \frac{\beta_2 u - c}{e}.$$

A necessary condition for E1 with $p \in [\underline{p}_1, \beta_2 u]$ to be an equilibrium (in the product market) is $\pi_1^P\left(u; A_1^P, A_2^P\right) \leq 0$. That is,

$$\gamma_1 e^{-A_1^P(\beta_2 u)}(u-c) - b_1^P \leq 0 \Longleftrightarrow \gamma_1 \leq \rho.$$

Therefore, the platform's revenue by inducing E1 with price support $[\underline{p}_1, \beta_2 u]$ is $R_{E1}^P = b_1^P$. For type-1 consumers, the sales distribution function $S_{1\to1}^P(p)$ is

$$S_{1\to1}^P(p) = \gamma_1\left(1 - e^{-A_1^P(p)}\right) = \gamma_1\left(1 - \frac{b_1^P}{p-c}\right) \text{ with } p \in [b_1^P + c, \beta_2 u].$$

The consumer surplus for a type-1 consumer is

$$\begin{aligned}
& \int_{b_1^P+c}^{\beta_2 u}(u-p)d\left(\frac{S_{1\to1}^P(p)}{\gamma_1}\right) \\
= \;& \frac{1}{\gamma_1}\left((u-\beta_2 u)S_{1\to1}^P(\beta_2 u) + \int_{b_1^P+c}^{\beta_2 u} S_{1\to1}^P(p)\,dp\right) \\
= \;& (u-\beta_2 u)\left(1 - \frac{b_1^P}{\beta_2 u - c}\right) + \beta_2 u - b_1^P - c - b_1^P \ln\frac{\beta_2 u - c}{b_1^P} \\
= \;& (u - c - (\beta_2 u - c))\left(1 - \frac{1}{e}\right) + \left(1 - \frac{2}{e}\right)(\beta_2 u - c) \\
= \;& (u-c)\left(1 - \frac{1}{e} - \frac{\rho}{e}\right)
\end{aligned}$$

Hence, type-1 consumers are better off with full privacy in an E1 equilibrium with price support $[\underline{p}_1, \beta_2 u]$ because

$$(u-c)\left(1 - \frac{1+\rho}{e}\right) \geq \left(1 - \frac{2}{e}\right)(u-c)$$

For type-2 consumers, the sales function $S^P_{1\to 2}(p)$ is

$$S^P_{1\to 2}(p) = \gamma_2\left(1 - e^{-A^P_1(p)}\right) = \gamma_2\left(1 - \frac{b^P_1}{p-c}\right) \text{ with } p \in [b^P_1 + c, \beta_2 u].$$

The consumer surplus for a type-2 consumer is

$$\begin{aligned} & \int_{b^P_1+c}^{\beta_2 u} (\beta_2 u - p) d\left(\frac{S^P_{1\to 2}(p)}{\gamma_2}\right) \\ = & \frac{1}{\gamma_2}\int_{b^P_1+c}^{\beta_2 u} S^P_{1\to 2}(p)\, dp \\ = & \beta_2 u - b^P_1 - c - b^P_1 \ln \frac{\beta_2 u - c}{b^P_1} \\ = & \left(1 - \frac{2}{e}\right)(\beta_2 u - c) \end{aligned}$$

Hence, type-2 consumers are worse off with full privacy in an E1 equilibrium with price support $[\underline{p}_1, \beta_2 u]$ because

$$\left(1 - \frac{2}{e}\right)(\beta_2 u - c) \leq \left(1 - \frac{2}{e}\right)(u - c).$$

The total number of matched ads of type-1 product in advertising market is

$$\gamma_1 A^P_1(\beta_2 u) = \gamma_1\left(\ln(\beta_2 u - c) - \ln b^P_1\right) = \gamma_1,$$

and the market size of type-1 product in product market is

$$\begin{aligned} S^P_{1\to 1}(\beta_2 u) + S^P_{1\to 2}(\beta_2 u) &= 1 - e^{-A^P_1(\beta_2 u)} \\ &= 1 - \frac{b^P_1}{\beta_2 u - c} \\ &= \frac{e-1}{e}. \end{aligned}$$

The consumer acquisition cost per consumer for type-1 firms is

$$\frac{b^P_1 A^P_1(\beta_2 u)}{S^P_{1\to 1}(\beta_2 u) + S^P_{1\to 2}(\beta_2 u)} = \frac{\beta_2 u - c}{e-1}.$$

E2 equilibrium: Now consider the E2 equilibrium where only type-2 product is advertised. In this case, the support of $A^P_2(p)$ must be $[\underline{p}_2, u]$. The zero-profit condition $\pi^P_2\left(p; A^P_1, A^P_2\right) = 0$ for $p \in [\underline{p}_2, u]$ implies that

$$\gamma_2 e^{-A^P_2(p)}(p-c) - b^P_2 = 0.$$

By setting $p = u$ we obtain

$$A_2^P(u) = \ln \gamma_2 + \ln(u-c) - \ln b_2^P.$$

The platform's ad revenue is

$$R_{E2}^P = b_2^P \left(\ln \gamma_2 + \ln(u-c) - b_2^P \right)$$

and the optimal ad price is

$$b_2^P = \frac{\gamma_2 (u-c)}{e} = b_2^*.$$

Hence, the platform's revenue by inducing E2 is $R_{E2}^P = b_2^*$. It is clear that in the E2 equilibrium, type-2 consumers are indifferent between the two privacy modes and type-1 consumers are worst off with full privacy.

The total number of matched ads of type-2 product in advertising market is

$$\gamma_2 A_2^P(\beta_2 u) = \gamma_2 \left(\ln \gamma_2 + \ln(u-c) - \ln b_2^P \right) = \gamma_2,$$

and the market size of type-2 product in product market is

$$\begin{aligned} S_{2\to 2}^P(u) &= \gamma_2 \left(1 - e^{-A_2^P(u)} \right) \\ &= \gamma_2 \left(1 - \frac{b_2^P}{\gamma_2 (u-c)} \right) \\ &= \gamma_2 \frac{e-1}{e}. \end{aligned}$$

The consumer acquisition cost per consumer for type-2 firms is

$$\frac{b_2^P A_2^P(u)}{S_{2\to 2}^P(u)} = \frac{u-c}{e-1}.$$

Finally, we compare the platform's ad revenue under an E2 equilibrium and under an E1 equilibrium. If $\gamma_1 \geq \gamma_2$, then an E1 equilibrium dominates an E2 equilibrium since

$$R_{E1}^P = \frac{\gamma_1 (u-c)}{e} \cdot \mathbf{1}\{\gamma_1 > \rho\} + \frac{\beta_2 u - c}{e} \cdot \mathbf{1}\{\gamma_1 \leq \rho\} \geq \frac{\gamma_1 (u-c)}{e}.$$

If $\gamma_1 < \gamma_2 \leq \rho$, then an E1 equilibrium again dominates an E2 equilibrium because

$$R_{E1}^P - R_{E2}^P = \frac{\beta_2 u - c}{e} - \frac{\gamma_2 (u-c)}{e} = \frac{u-c}{e} (\rho - \gamma_2).$$

The remaining cases are $\rho < \gamma_1 < \gamma_2$ and $\gamma_1 \leq \rho < \gamma_2$, which can be combined as $\gamma_1 < 1/2$ and $\gamma_1 + \rho < 1$. We show the results in Figure 7.

Lemma 11. *For E-equilibrium,*

1. ***E1 equilibrium:*** *all equilibrium prices must be below $\beta_2 u$. Flexible consumers are worse off and picky consumers are better off with full privacy. Picky consumer product has the same number of matched ads but expands in product market with lower consumer acquisition cost under full privacy. Flexible consumer product is excluded.*

2. ***E2 equilibrium:*** *platform chooses over E1 equilibrium when $\gamma_1 < 1/2$ and $\gamma_1 + \rho < 1$. Flexible consumers are indifferent with the two privacy modes and picky consumers are worse off with full privacy. Flexible consumer product has the same market size in both advertising and product market with the same consumer acquisition cost, while picky consumer product is excluded under full privacy.*

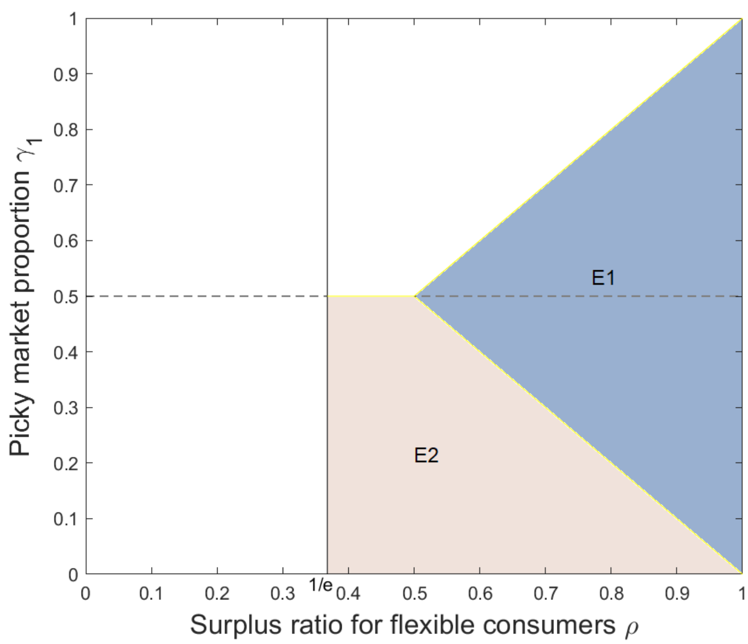

Figure 7: Parameter regions for existence of E1 and E2 equilibrium

7.6 Platform's choice of equilibrium

In this section, we list all the graphs when the platform compare the profits under different types of equilibrium.

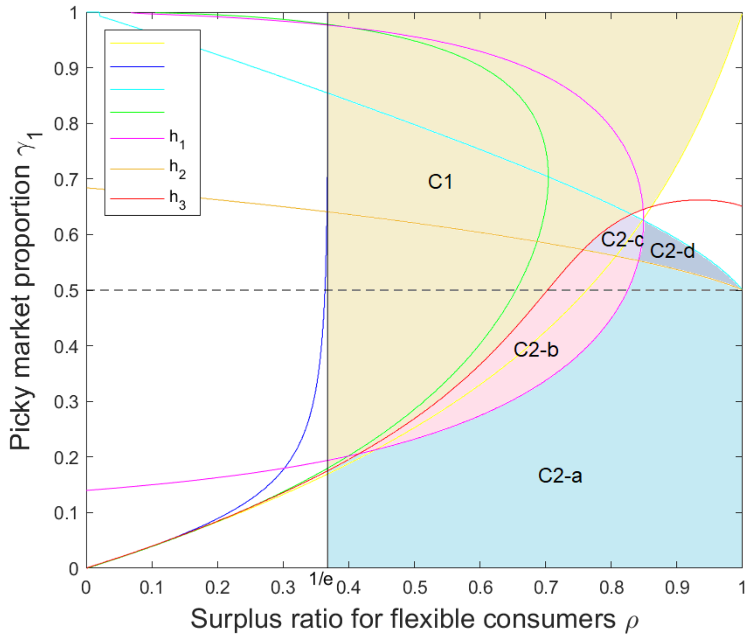

Figure 8: Platform's decision over C1 and C2 equilibrium

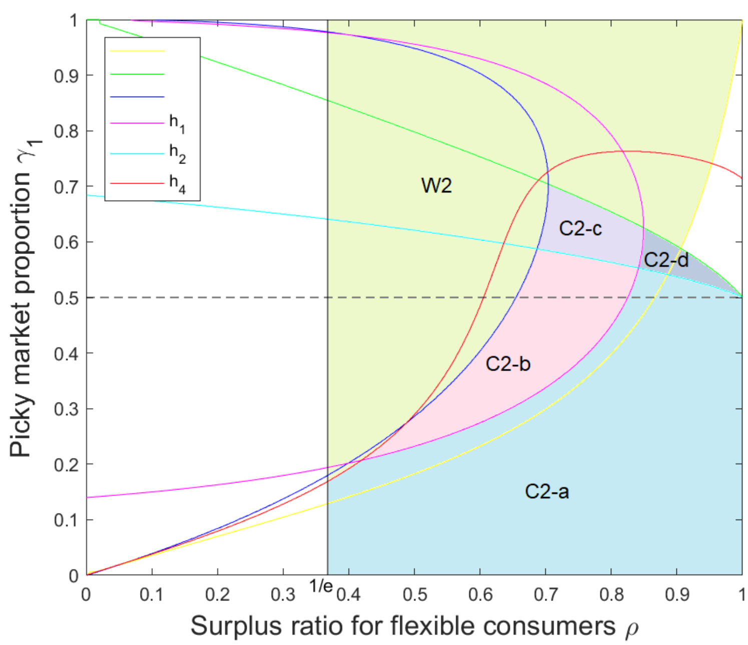

Figure 9: Platform's decision over C2 and W2 equilibrium

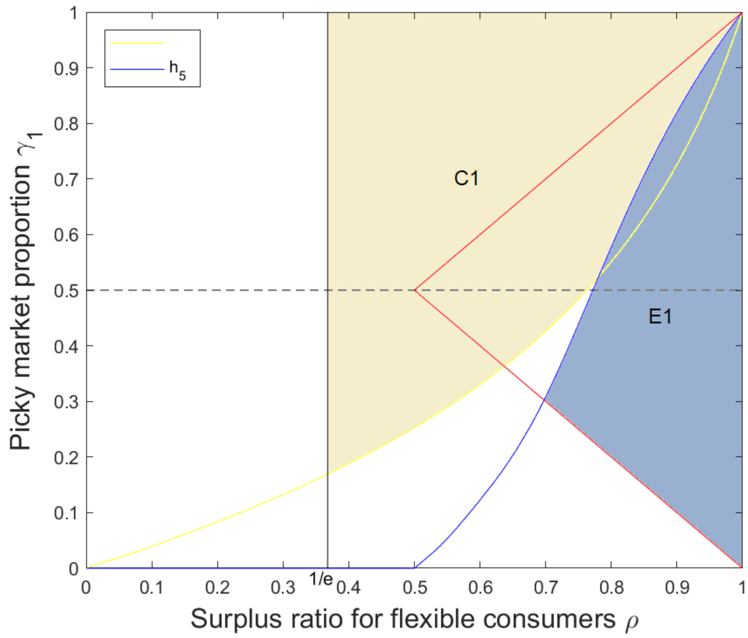

Figure 10: Platform's decision over C1 and E1 equilibrium

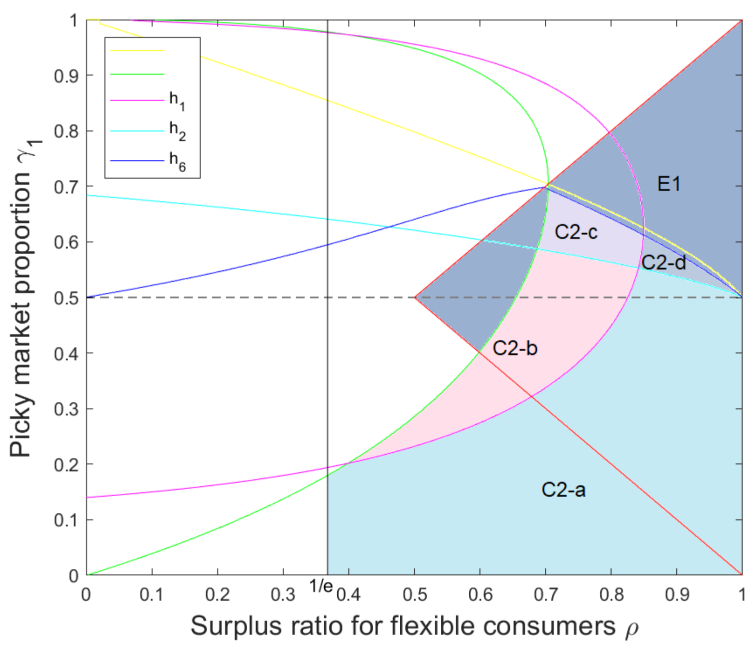

Figure 11: Platform's decision over C2 and E1 equilibrium

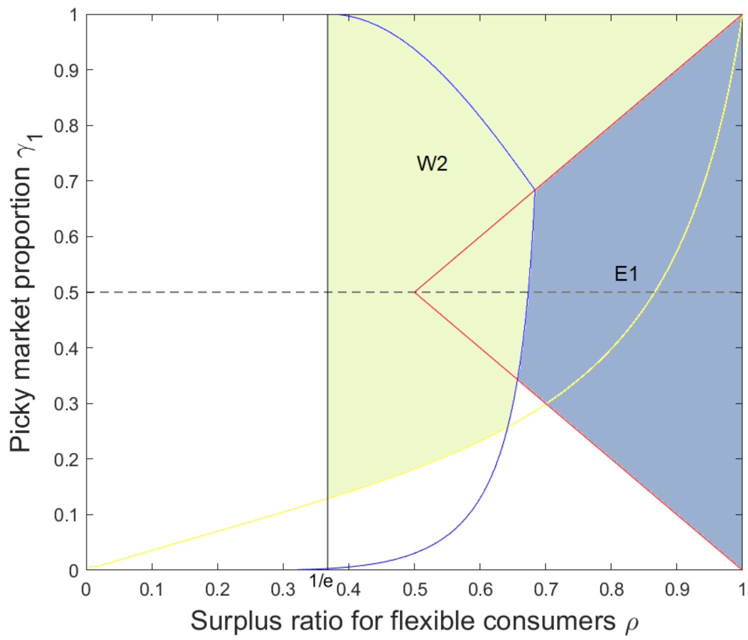

Figure 12: Platform's decision over W2 and E1 equilibrium

7.7 Two flexible consumers

In this subsection, we consider the case that both types of consumers are flexible and try to extend our results that flexible consumers benefit from privacy. We want to show that both types of consumers can benefit from privacy by "hiding behind" each other. Suppose both β_1 and β_2 are large so that $(\beta_i u - c)/(u - c) > 1/e$ for both $i = 1, 2$. In this case, if product $-i$ is priced competitively against product i, type-i consumers may buy product $-i$ rather than product i. Hence, the equilibrium in both consumer markets may feature cross-product competition. In general, the equilibrium structure varies depending on the parameters of β_1, β_2 and γ_1 and a full characterization is too complicated to solve. Instead, we are going to use a special case to illustrate the idea. We focus on the case that $\beta_1 = \beta_2 = 0.5$ and $\gamma_1 = 0.5$. For simplicity of notation, assume that $u = 1$ and $c = 0$.

In general, we can write the support of advertising function $A_i^P(p)$ as $[\underline{p}_i, 0.5] \cup [\tilde{u}_i, \bar{p}_i]$ where $\tilde{u}_i \geq 0.5$ for $i = 1, 2$. Following similar proofs as in Lemmas 3-4, we can get that $\underline{p}_i = \bar{p}_{-i} - 0,5$ for $i = 1, 2$. To see this, for type-2 firms and prices $p' \in (\tilde{u}_2, \bar{p}_2)$:

$$\pi_2^P(p') = \frac{1}{2} e^{-A_2^P(p')} e^{-A_1^P(a_1)} p' - b_2^P.$$

If a type-1 firm sends ad with price $p' - 0.5$, the expected profit is

$$\pi_1^P(p' - 0.5) = \left[\frac{1}{2} + \frac{1}{2} e^{-A_2^P(p')}\right] e^{-A_1^P(p')}(p' - 0.5) - b_1^P$$

$$= \quad \frac{1+e^{-A_2^P(p')}}{e^{-A_2^P(p')}}\frac{p'-0.5}{p'}\left(\pi_2^P(p')+b_2^P\right)-b_1^P,$$

where both ratios in the expression are increasing in p'. Therefore, there cannot be an interval $[a_1, a_2]$ such that $\pi_1^P(p) = 0$ and $\pi_2^P(p+0.5) = 0$ for all $p \in [a_1, a_2]$. In addition, there will be a cutoff $\hat{p}$ such that $\bar{p}_2 \leq \hat{p} \leq \underline{p}_1 + 0.5$.

If $\bar{p}_2 < \underline{p}_1 + 0.5$, a similar argument as in the proof of Lemma 1 would help establish that type-2 firms can earn strictly positive profit by sending ads with price $\bar{p} + \epsilon$ for some small $\epsilon > 0$. Hence, $\underline{p}_1 = \bar{p}_2 - 0.5$. Similarly, we have $\underline{p}_2 = \bar{p}_1 - 0.5$.

In what follows, we first assume that the platform wants to induce equilibrium with cross-product competition. In any such equilibrium, the set of prices that type-i firms will advertise in equilibrium takes the form of $[\bar{p}_{-i} - 0.5, 0.5] \cup [\tilde{u}_i, \bar{p}_i]$ with $0.5 \leq \tilde{u}_i \leq \bar{p}_i$. The marginal profit of type-1 firms by sending one 1 additional ad at price $p \geq \tilde{u}_i$ is

$$\pi_1^P\left(p; A_1^P, A_2^P\right) = \frac{1}{2}e^{-A_1^P(p)}(p) - b_1^P,$$

as $p - 0.5 \leq \bar{p}_1 - 0.5$. For any equilibrium price $p \geq \tilde{u}_i$ advertised by type-1 firms, zero-profit condition implies that

$$\frac{1}{2}e^{-A_1^P(p)}(p) - b_1^P = 0. \tag{34}$$

The marginal profit of type-2 firms by sending one 1 additional ad at price $p \leq 0.5$ is

$$\pi_2^P\left(p; A_1^P, A_2^P\right) = \left[\frac{1}{2}e^{-A_1^P(p+0.5)} + \frac{1}{2}\right]e^{-A_2^P(p)}(p) - b_2^P.$$

For any equilibrium price $p \leq 0.5$ advertised by type-2 firms, zero-profit condition implies that

$$\left[\frac{1}{2}e^{-A_1^P(p+0.5)} + \frac{1}{2}\right]e^{-A_2^P(p)}(p) - b_2^P = 0. \tag{35}$$

By setting $p = \bar{p}_1$ in (34) and $p = \bar{p}_1 - 0.5$ in (35) and cancelling out $e^{-A_1^P(\bar{p}_1)}$, we obtain $\bar{p}_1$ implicitly as solution to

$$\frac{b_2^P}{\bar{p}_1 - 0.5} = \frac{1}{2} + \frac{b_1^P}{\bar{p}_1}. \tag{36}$$

We can solve $\bar{p}_1$ explicitly as

$$\bar{p}_1 = b_2^P - b_1^P + \frac{1}{4} + \sqrt{\left(b_2^P - b_1^P + \frac{1}{4}\right)^2 + b_1^P}. \tag{37}$$

We obtain the total demand of type-1 ads as

$$A_1^P\left(\bar{p}_1\right) = \ln \bar{p}_1 - \ln 2 - \ln b_1^P. \tag{38}$$

Similarly, the total demand of type-2 ads is

$$A_2^P(\bar{p}_2) = \ln \bar{p}_2 - \ln 2 - \ln b_2^P \tag{39}$$

with

$$\bar{p}_2 = b_1^P - b_2^P + \frac{1}{4} + \sqrt{\left(b_1^P - b_2^P + \frac{1}{4}\right)^2 + b_2^P}. \tag{40}$$

The platform's optimization problem is to choose b_1^P and b_2^P to maximize its ad revenue

$$\begin{aligned} R_{TF}^P &= b_1^P A_1^P(u) + b_2^P A_2^P(u) \\ &= b_1^P \left[\ln \bar{p}_1 - \ln 2 - \ln b_1^P\right] + b_2^P \left[\ln \bar{p}_2 - \ln 2 - \ln b_2^P\right] \end{aligned}$$

subject to $A_1^P(u) \geq 0$, $A_2^P(u) \geq 0$ and constraints (6) and (7) for both types of products. Consider the relaxed problem without any constraints. The first-order conditions are

$$\ln \bar{p}_1 - \ln 2 - \ln b_1^P + \frac{b_1^P \frac{\partial \bar{p}_1}{\partial b_1^P}}{\bar{p}_1} + \frac{b_2^P \frac{\partial \bar{p}_2}{\partial b_1^P}}{\bar{p}_2} - 1 = 0, \tag{41}$$

$$\ln \bar{p}_2 - \ln 2 - \ln b_2^P + \frac{b_1^P \frac{\partial \bar{p}_1}{\partial b_2^P}}{\bar{p}_1} + \frac{b_2^P \frac{\partial \bar{p}_2}{\partial b_2^P}}{\bar{p}_2} - 1 = 0, \tag{42}$$

where

$$\begin{aligned} \frac{\partial \bar{p}_1}{\partial b_1^P} &= \frac{-\bar{p}_1 + 0.5}{\sqrt{\left(b_2^P - b_1^P + \frac{1}{4}\right)^2 + b_1^P}}, \\ \frac{\partial \bar{p}_1}{\partial b_2^P} &= \frac{\bar{p}_1}{\sqrt{\left(b_2^P - b_1^P + \frac{1}{4}\right)^2 + b_1^P}}, \\ \frac{\partial \bar{p}_2}{\partial b_1^P} &= \frac{\bar{p}_2}{\sqrt{\left(b_1^P - b_2^P + \frac{1}{4}\right)^2 + b_2^P}}, \\ \frac{\partial \bar{p}_2}{\partial b_2^P} &= \frac{-\bar{p}_2 + 0.5}{\sqrt{\left(b_1^P - b_2^P + \frac{1}{4}\right)^2 + b_2^P}}. \end{aligned}$$

Assuming symmetry, we have

$$b_1^P = \left(\bar{p}_1 - \frac{1}{4}\right)^2 - \frac{1}{16} = \bar{p}_1\left(\bar{p}_1 - \frac{1}{2}\right)$$

where $\bar{p}_1$ can be solved by

$$\ln \bar{p}_1 - \ln 2 - \ln\left(\bar{p}_1\left(\bar{p}_1 - \frac{1}{2}\right)\right) + \frac{\bar{p}_1\left(\bar{p}_1 - \frac{1}{2}\right)}{2\bar{p}_1\left(\bar{p}_1 - \frac{1}{4}\right)} - 1 = 0$$

$$\Longrightarrow \quad -\ln(2\bar{p}_1 - 1) = \frac{\bar{p}_1}{2\bar{p}_1 - \frac{1}{2}}$$

We can get $\bar{p}_1 = 0.734274$ and hence $b_1^P = 0.172021$, $A_1^P = 0.758117$. We can further pin down $\tilde{u}_1$ by

$$\begin{aligned} \ln\left(\frac{1}{2} + \frac{1}{2}e^{-A_2^P(\bar{p}_2)}\right) + \ln\frac{1}{2} - \ln b_1^P &= \ln\frac{1}{2} + \ln\tilde{u}_1 - \ln b_1^P \\ \Longrightarrow \quad \tilde{u}_1 = \frac{1}{2} + \frac{1}{2}e^{-A_2^P(\bar{p}_2)} &= \frac{1}{2} + \frac{b_1^P}{\bar{p}_1} = \bar{p}_1 \end{aligned}$$

Now we want to argue that this is indeed the equilibrium. First the optimal support under this C1 type equilibrium is just $[\bar{p}_{-i} - 0.5, 0.5]$ for type-i product, which is the C2 type of equilibrium, and it can be easily shown that

$$\pi_1^P\left(1; A_1^P, A_2^P\right) = \frac{1}{2}e^{-A_2^P(0.5)}e^{-A_1^P(0.5)} - \ln b_1^P < 0.$$

Second, the profit for the platform is

$$\begin{aligned} R_{TF}^P &= b_1^P A_1^P(u) + b_2^P A_2^P(u) \\ &= 0.260824 \end{aligned}$$

which is larger than $1/(2e)$ the platform profit under E1 and E2 type of equilibrium. W1 equilibrium is not feasible while W2 equilibrium is a worse for the platform when C1 equilibrium is feasible.

Next we would like to argue that in equilibrium with cross-product competition, both type consumers are better off.

The sales distribution function $S_{1\to1}^P(p)$ is

$$S_{1\to1}^P(p) = \gamma_1\left(1 - e^{-A_1^P(p)}\right) = \frac{1}{2}\left(1 - \frac{\bar{p}_1 - 0.5}{p}\right)$$

The sales distribution function $S_{2\to1}^P(p)$ is

$$S_{2\to1}^P(p) = \gamma_1 e^{-A_1^P(0.5)}\left(1 - e^{-A_2^P(p)}\right) = \frac{1}{2}\frac{\bar{p}_1 - 0.5}{0.5}\left(1 - \frac{\bar{p}_2 - 0.5}{p}\right)$$

The consumer surplus of a type-2 consumer under full privacy is

$$\begin{aligned} &\int_{\bar{p}_2-0.5}^{0.5}(1-p)d\left(\frac{S_{1\to1}^P(p)}{\gamma_1}\right) + \int_{\bar{p}_1-0.5}^{0.5}(0.5-p)d\left(\frac{S_{2\to1}^P(p)}{\gamma_1}\right) \\ = \quad &\frac{1}{2}2S_{1\to1}^P(0.5) + \int_{\bar{p}_2-0.5}^{0.5} 2S_{1\to1}^P(p)\,dp + \int_{\bar{p}_1-0.5}^{0.5} 2S_{2\to1}^P(p)\,dp \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(1-\frac{\bar{p}_1-0.5}{0.5}\right)+\int_{\bar{p}_2-0.5}^{0.5}\left(1-\frac{\bar{p}_1-0.5}{p}\right)dp+\int_{\bar{p}_1-0.5}^{0.5}2\left(\bar{p}_1-0.5\right)\left(1-\frac{\bar{p}_2-0.5}{p}\right)dp \\
&= 2\left(1-\bar{p}_1\right)\left(0.5+\bar{p}_1\right)+2\bar{p}_1(\bar{p}_1-0.5)\ln\left(2\bar{p}_1-1\right) \\
&= 0.395133 \\
&> 1-\frac{2}{e}
\end{aligned}
$$

Therefore, both type consumers would prefer full privacy.

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