Welfare of Competitive Price Discrimination with Captive Consumers*

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Abstract

We study the welfare effects of price discrimination in a duopoly market with both captive and contested consumers. Using a unified information design approach, we characterize the best and worst market segmentations for producer surplus, consumer surplus, and social surplus. The firm-optimal segmentation, which divides the market into two nested segments, consistently harms consumers compared to uniform pricing. The consumer-optimal segmentation, which divides the market into a symmetric segment and a nested segment, sometimes leads to a Pareto improvement. Social surplus, if monotone in firm profit, is often maximized either by the firm-optimal or consumer-optimal segmentation.

1 Introduction

With the rapid advancement of information technology and the proliferation of social media, the volume of consumer data available for market segmentation has grown exponentially. Firms now have numerous ways to classify consumers into different groups, leveraging data from social media platforms, mobile apps, and traditional offline

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sources. This increasingly detailed market segmentation facilitates third-degree price discrimination, where firms charge different prices to different consumer groups.

This development raises several important and policy-relevant research questions: What are the welfare effects of market segmentation? What segmentation generates the maximal (or minimal) producer surplus, consumer surplus, or social surplus? How do these welfare effects vary with different market structures and market configurations? When can market segmentation lead to a Pareto-improving market segmentation over uniform pricing?[1]

These questions are relevant in various contexts. For instance, data brokers like Acxiom, Corelogic, or Datalogix sell consumer data to competing firms, and the maximum producer surplus achievable through market segmentation can serve as a benchmark for the prices these data brokers can charge. Industry associations that collect consumer data might aim to maximize the aggregate welfare of their member firms. Similarly, consumer associations, regulators advocating for consumer welfare, or platforms seeking to enhance their appeal to consumers are interested in the segmentation that maximizes consumer surplus. The analysis of the welfare effects of market segmentation can also provide regulatory insights into how data brokers and third-party platforms may influence market competition through information provision in product markets, and whether consumers are necessarily harmed as firms pursue increasingly intricate market segmentation to boost profits.

Our paper builds on the seminal work of Bergemann, Brooks and Morris (2015) (BBM hereafter), who study the welfare effects of third-degree price discrimination in a monopoly setting and characterize the set of all possible combinations of consumer surplus and producer surplus achievable through market segmentation. One of the main challenges to extend BBM's analysis to oligopoly settings is to characterize all possible equilibria in the baseline pricing model. As observed by Armstrong and Vickers (2019), even for duopoly pricing models, "except in symmetric and other special cases ... the form of the equilibrium is not known." Hence, a stylized baseline model is often necessary for tractability.

We adopt the model of Armstrong and Vickers (2019) as our baseline model and partially extend BBM's analysis to a duopoly setting. In this model, two firms produce a homogeneous product and compete in prices. Each firm has its own captive consumers who can only buy from that firm.[2] There are also contested consumers who are loyal

[1] We call a segmentation a Pareto improvement over another if the former increases both (expected) consumer surplus and producer surplus relative to the latter.

[2] Consumers can become captive to firms due to market dynamics, psychological factors, and external barriers that hinder switching. For example, brand loyalty or brand-specific investments can make it costly for consumers to switch, leading to captivity.

to neither firm and will buy from the lower-priced firm. All consumers have the same downward-sloping demand.[3] A market segmentation, which divides the market into segments with different proportions of captive and contested consumers, allows firms to perform competitive (third-degree) price discrimination,[4] and a designer selects a segmentation to maximize welfare measures such as producer surplus, consumer surplus or social surplus.

In a duopoly, the designer can reveal different segmentations to each firm ("private segmentation") or the same segmentation to both firms ("public segmentation"), a distinction not present in the monopoly setting. In practice, data owners are often legally required to disclose the same information to different firms. For instance, in the credit market, credit bureaus provide standardized credit scores and histories to all lenders, ensuring transparency and fairness in lending practices by preventing any single lender from gaining an unfair advantage through exclusive data access. Similarly, in the energy sector, smart meter data is often shared among utility providers to promote competitive pricing and consumer choice. Another example comes from public records, such as property ownership or tax data, which are made available to various firms like real estate agents and insurers. These examples indicate that implementing private segmentation can be challenging in certain applications. Furthermore, if the objective of private segmentation does not align with firms' incentives, firms may have incentive to share their private information.[5] Therefore, we will focus on public segmentations to identify the best and worst market segmentations for producer surplus, consumer surplus, and social surplus, respectively. This restriction not only improves tractability but also allows for a more direct comparison to the existing literature (e.g., Armstrong and Vickers (2019)), where firms share the same exogenous full information.

Rather than solving the six optimization problems separately, we introduce a unified optimization problem with an objective that can be any monotone function of firm profit. We develop a procedure to solve this unified problem and then adapt it to our six specific optimization problems. Our analysis of the unified problem yields several key insights. First, any market segment can be decomposed into a symmetric segment and a nested segment, such that the decomposition replicates the welfare in the original segment.[6] Thus, we can focus on optimal market segmentation that consists only of

[3]This model framework was developed by Varian (1980) and Narasimhan (1988) for the case of unit demand, and later generalized by Armstrong and Vickers (2019) to the case of downward-sloping demand. This model has been a working horse in the marketing literature for studying promotional strategies. See for example, Chen, Narasimhan and Zhang (2001) and references therein.

[4]Henceforth, we will use market segmentation and price discrimination interchangeably.

[5]There is extensive literature on information sharing in oligopoly. See Raith (1996) for a comprehensive survey and a unified approach to information sharing among firms.

[6]A symmetric segment contains equal fractions of captive consumers for both firms, while a nested

symmetric and nested segments. Second, a more symmetric market fosters stronger competition. Within segments with a fixed total fraction of captive consumers, a nested segment maximizes producer surplus due to its maximal asymmetry, while a symmetric segment maximizes consumer surplus and minimizes producer surplus. Third, since consumers are risk averse due to downward-sloping demand, the consumer-optimal segmentation must balance between the mean and variance of consumer payoff across segments. If consumers are highly risk-averse, reducing the variability of consumer payoff across segments may be prioritized over increasing the average payoff.

We find that the optimal market segmentation often takes a simple form for all three welfare measures. Consider a prior (or aggregate) market (γ_1, γ_2) with a normalized market size of one, where γ_i is the share of consumers captive to firm i and $1 - \gamma_1 - \gamma_2$ is the share of contested consumers. Let $\ell = \gamma_1 + \gamma_2$ denote the total share of captive consumers. Three types of market segmentation are of particular interest:

- Nested Segmentation: Divides the market into two nested segments $(\ell, 0)$ and $(0, \ell)$ with sizes γ_1/ℓ and γ_2/ℓ, respectively.

- Field-Leveling Segmentation: Divides the market into a "maximal" symmetric segment of $(\gamma_2/(1 - \gamma_1 + \gamma_2), \gamma_2/(1 - \gamma_1 + \gamma_2))$ with size $1 - \gamma_1 + \gamma_2$ and the remainder of $(1, 0)$ with size $\gamma_1 - \gamma_2$.

- Perfect Segmentation: Divides the market into three perfect segments, each containing only one type of consumers: $(1, 0)$, $(0, 1)$ and $(0, 0)$, with sizes γ_1, γ_2 and $1 - \gamma_1 - \gamma_2$, respectively.

To illustrate nested segmentation and field-leveling segmentation, consider a toy example with two firms competing for 36 consumers. Each consumer is a member of an association and has an association email account. In the two figures below, each consumer is represented by a colored email address—different colors indicate different consumer types. The association collects members' information and learns that 16 consumers are captive to firm 1 (yellow), 8 are captive to firm 2 (green), and the remaining 12 are contested (red). The normalized prior market is $(4/9, 2/9)$.

The association organizes the 36 email addresses into distinct email lists (i.e., submarkets), disclosing publicly the size and compositions of each list without revealing the specific color of each consumer. It also manages the firms' access to these lists.[7] Each firm sends exactly one price offer per email list through the association.

The segmentation illustrated in the left panel of Figure 1 divides contested consumers proportionally to captive consumers to form two submarkets. It represents a

segment contains only one firm's captive consumers.

[7] If the association creates different lists for different firms, it is an example of private segmentation.

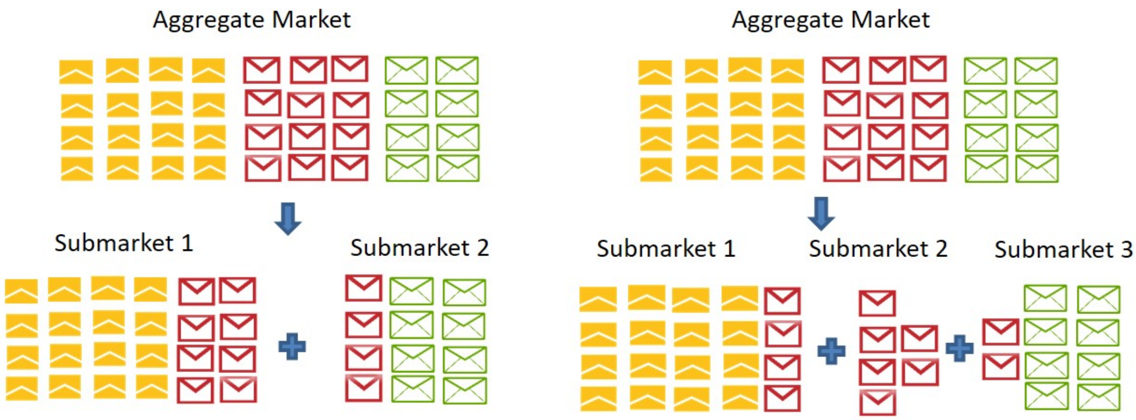

Figure 1: Nested segmentation and its modification

nested segmentation because it can be normalized as $2/3(2/3, 0) + 1/3(0, 2/3)$. Starting with the nested segmentation and extracting some of the contested consumers from the two submarkets in proportion to their size to form a third submarket, we obtain a "modified" nested segmentation, as illustrated in the right panel.

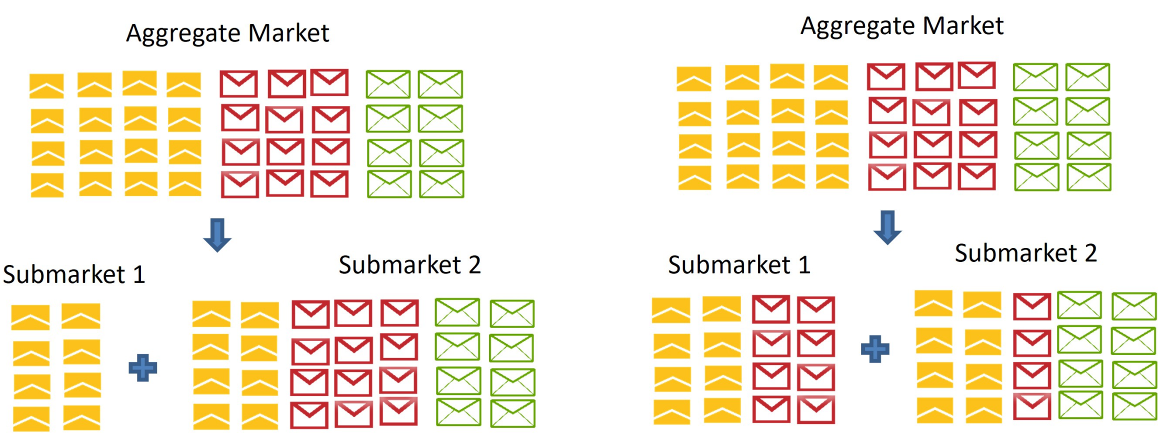

Figure 2: Field-leveling segmentation and its modification

The segmentation in the left panel of Figure 2 separates the market into a "maximal" symmetric submarket and the remainder. After normalization, this segmentation can be written as $2/9(1, 0) + 7/9(2/7, 2/7)$, which represents a field-leveling segmentation. The right panel's segmentation is a "modified" version of the field-leveling segmentation. It also consists of a symmetric segment and a nested segment, but the symmetric segment is not maximal and the nested segment is not "extreme".

Table 1 summarizes our main findings on optimal market segmentation. Producer surplus is uniquely maximized by nested segmentation and minimized by either per-

	Nested	Modified N	Field-Leveling	Modified FL	Perfect
P-Max	✓				
P-Min			✓		✓
C-Max			Low risk averse	High risk averse	
C-Min	High $\gamma_1 + \gamma_2$	Low $\gamma_1 + \gamma_2$			
S-Max	$s(\pi) \uparrow \pi$		$s(\pi) \downarrow \pi$ Low risk averse	$s(\pi) \downarrow \pi$ High risk averse	
S-Min	$s(\pi) \downarrow \pi$ High $\gamma_1 + \gamma_2$	$s(\pi) \downarrow \pi$ Low $\gamma_1 + \gamma_2$			$s(\pi) \uparrow \pi$

Table 1: The best and worst market segmentation

fect segmentation or field-leveling segmentation. Although nested segmentation does not always minimize consumer surplus, it consistently harms consumers compared to uniform pricing.

Consumer surplus is uniquely maximized by field-leveling segmentation if consumers are not highly risk-averse. Otherwise, a modified field-leveling segmentation can be optimal because the modification reduces the volatility of consumer payoffs, benefiting risk averse consumers. This modified segmentation may also increase producer surplus compared to uniform pricing, resulting in a Pareto improvement.

Consumer surplus is uniquely minimized by nested segmentation when the share of captive consumers is high, and by a modified nested segmentation otherwise. In a modified nested segmentation, the fraction of captive consumers in the nested segments is higher, leading to a consumer loss that may outweigh the consumer gain from the third segment of contested consumers.

The analysis of social surplus mirrors that of producer or consumer surplus. If social surplus $s(\pi)$ increases with firm profit π, it is also maximized by nested segmentation and minimized by perfect segmentation. If social surplus decreases with firm profit, the segmentation that maximizes (or minimizes) social surplus resembles that of consumer surplus, though they may not coincide.

Now, we discuss how our results relate to and differ from those in the existing literature. Armstrong and Vickers (2019) compare consumer welfare under perfect segmentation and uniform pricing, showing that perfect segmentation harms consumers relative to uniform pricing if the market is sufficiently symmetric, but benefits consumers if the market is sufficiently asymmetric. In contrast, we adopt an information design approach similar to BBM and demonstrate that nested segmentation maximizes producer surplus. Furthermore, we show that nested segmentation always reduces consumer surplus relative to uniform pricing for all prior market configurations, in contrast to the ambiguous result in Armstrong and Vickers (2019).

The optimality of nested segmentation has been previously identified by Bergemann, Brooks and Morris (2020) and Albrecht (2020), using the unit demand version of Armstrong and Vickers (2019) as their baseline model. With unit demand, consumers are always served in every possible segmentation, resulting in constant total social surplus across segmentations. Consequently, what benefits firms must harm consumers and vice versa, making the analysis of consumer surplus a straightforward corollary of producer surplus. However, with downward-sloping demand, the interaction between firms and consumers is no longer zero-sum, and a Pareto-improving market segmentation over uniform pricing may be possible. Hence, our analysis of consumer welfare is more nuanced.[8]

The BBM framework has been extended to various monopoly contexts, such as multiproduct monopoly (Ichihashi (2020), Haghpanah and Siegel (2022), Haghpanah and Siegel (2023), Hidir and Vellodi (2021)), lemons market with interdependent values (Kartik and Zhong (2023)), and revenue-maximizing data brokers (Yang (2022)).[9] In a bilateral trade setting where a seller posts a price to a buyer who may or may not be better informed, Kartik and Zhong (2023) characterize the set of payoff vectors achievable across all information structures and find that the buyer-optimal information structure must minimize the seller's payoff, which generalizes the findings of both BBM and Roesler and Szentes (2017). We also find that the consumer-optimal segmentation minimizes producer surplus when consumers are not highly risk averse. However, when consumers are highly risk averse, this is no longer the case, and the consumer-optimal segmentation may strictly improve producer surplus over no segmentation, resulting in a Pareto improvement. Our latter result is related to Haghpanah and Siegel (2023), who show that a Pareto-improving segmentation exists whenever a firm-optimal segmentation leads to inefficiency. However, our notion of Pareto improvement differs from and is weaker than theirs.

All the aforementioned papers take consumer demand as given and focus on designing information structures to influence firms' learning. Alternatively, one can design information structures to affect consumer learning. Roesler and Szentes (2017) consider a monopoly model with privately informed consumers and derive consumer-optimal information structures. Armstrong and Zhou (2022) extend this analysis to a duopoly

[8]Elliott et al. (2024) extends the analysis of BBM to an oligopoly setting with unit demand. They provide a necessary and sufficient condition for a firm-optimal segmentation to extract the full surplus and characterize a consumer-optimal segmentation which induces an efficient allocation and delivers minimal profit to each firm.

[9]See also Ali, Lewis and Vasserman (2023) for an analysis of how consumer data control can affect consumer welfare by influencing the learning of and the competition between firms.

setting, characterizing firm-optimal and consumer-optimal information structures. Assuming firms, rather than the designer, choose information structures, Ivanov (2013) and Hwang, Kim and Boleslavsky (2023) derive equilibrium information structures in games where firms compete in both pricing and advertising.

2 The Model

Our baseline model is taken from Armstrong and Vickers (2019). There are two firms who can produce a homogeneous product at zero cost and compete for a unit mass of consumers in prices. There are three types of consumers: consumers who are captive to (and hence can only buy from) firm 1, consumers who are captive to firm 2, and contested consumers who will buy from the firm that charges a lower price. Let γ_1 and γ_2 denote the share of consumers captive to firm 1 and firm 2, respectively, and the share of contested consumers is then $1-\gamma_1-\gamma_2$. Without loss of generality, we assume that $\gamma_2 \leq \gamma_1$.

Consumers have quasilinear preferences and their demand $D(p)$ is downward sloping and twice-differentiable. If a consumer buys from a firm that charges a price p, this consumer will buy $D(p)$ units of the product, making a profit of $\pi(p) \equiv pD(p)$ for the firm. As in Armstrong and Vickers (2019), we impose the following assumption:

Assumption 1 *The elasticity of demand $\eta(p) \equiv -pD'(p)/D(p)$ is strictly increasing.*

Remark 1 *The downward sloping demand model can nest a unit demand specification with a random taste shock. Let $i = 0, 1, 2$, denote the types of consumers who are contested, captive to firm 1, and captive to firm 2, respectively. Suppose that each consumer has a unit demand and that a type i consumer's valuation for product j is*

$$u_{ij} = \theta_{ij} + \varepsilon_i$$

where θ_{ij} is the normalized mean utility of type i consumers for firm j's product ($j = 1, 2$), and ε_i is type i's taste shock which is randomly drawn from a common distribution Ψ with density ψ. The normalized utility θ_{ij} takes the value of 0 if $i = 0$ or $i = j$ and the value of $-\infty$ otherwise, so captive consumers ($i = 1, 2$) will only buy from their favorite firms. Furthermore, the taste shock ε_i is common across products, so contested consumers ($i = 0$) will buy from the firm that offers the lower price. The taste shock is realized upon receiving product offers. If a type 1 consumer is offered product 1 at price p, this consumer will buy if $\varepsilon_i \geq p$, which happens with probability $1 - \Psi(p)$. If we define the "demand function" as $D(p) = 1 - \Psi(p)$, then Assumption 1 is equivalent *to the requirement that $p\psi(p)/[1-\Psi(p)]$ is strictly increasing in p.*

Under Assumption 1, $\pi(p)$ is single-peaked and hence is strictly increasing for all $p \in [0, p^*]$ where p^* is the revenue-maximizing price $p^* = \arg\max \pi(p)$. Moreover, consumer surplus $v(\pi)$ as a function of profit π is strictly decreasing, twice-differentiable, and strictly concave in $[0, \pi^*]$, where $\pi^* \equiv p^* D(p^*)$ is the maximal profit. To rule out triviality, we assume that $\pi^* > 0$ and $v(\pi^*) > 0$.

The overall duopoly market, referred to as the prior market, can be segmented into different submarkets or segments which may have different relative shares of captive and contested consumers. We will use the terms of "submarket" and "segment" interchangeably. In a segment (q_1, q_2), q_1 and q_2 are the fraction of consumers captive to firm 1 and firm 2, respectively, and $1 - q_1 - q_2$ is the fraction of contested consumers. The set of possible segments is

$$\mathcal{M} = \left\{ (q_1, q_2) \in [0,1]^2 : 0 \le q_1 + q_2 \le 1 \right\}.$$

A market segmentation can be represented as a probability distribution $m(q_1, q_2) \in \Delta\mathcal{M}$ of different segments such that

$$\gamma_i = \sum_{(q_1,q_2)\in\mathcal{M}} m(q_1, q_2) q_i, \quad i = 1, 2. \tag{1}$$

Once the designer selects a market segmentation, both firms can use this information to implement price discrimination. Given a market segmentation m, firms determine what prices to charge for each submarket (q_1, q_2) within the support of m to maximize their profit. It is clear that if a prior market (γ_1, γ_2) does not contain any contested consumers (i.e., $\gamma_1 + \gamma_2 = 1$), both firms attain the maximal profit π^* for every segment, and all market segmentations yield the same payoffs for firms and consumers. Therefore, from now on, we assume that $\gamma_1 + \gamma_2 < 1$.

We first characterize the unique equilibrium for a generic segment (q_1, q_2) with $q_1 \ge q_2 \ge 0$. Since profit π is strictly increasing in price $p \in [0, p^*]$, it is without loss to focus on profits in $[0, \pi^*]$, for which there is one-to-one mapping between profit and price. As in Armstrong and Vickers (2019), it is more convenient to consider firms as choosing the per-consumer profit π rather than the price p they charge consumers. Consumers then select the firm with the smallest π from the set of firms they consider.

Before stating the equilibrium characterization, we follow Armstrong and Vickers (2019) and define firm 1's *captive-to-reach* ratio $\rho(q_1, q_2)$ as

$$\rho(q_1, q_2) = \frac{q_1}{1 - q_2} \in [0, 1], \tag{2}$$

where q_1 is the fraction of consumers in segment (q_1, q_2) who are capped by firm 1, and $1-q_2$ is the fraction of consumers who can be reached by firm 1. When the underlying segment (q_1, q_2) is clear, we often omit its dependence on (q_1, q_2) and write ρ directly. Intuitively, a higher ρ indicates a lower incentive for firm 1 to undercut firm 2 in order to attract contest consumers. The following equilibrium characterization is taken from Narasimhan (1988) and Armstrong and Vickers (2019). We omit the proof.

Lemma 1 *In the unique equilibrium for segment (q_1, q_2) with $q_1 \geq q_2$ and $q_1 + q_2 < 1$, both firm 1 and firm 2 play mixed strategies on a common support $[\rho\pi^*, \pi^*]$. Firm 1 chooses per-consumer profit according to distribution*

$$F_1(\pi) = \frac{1-q_1}{1-q_1-q_2}\left(1 - \frac{\rho\pi^*}{\pi}\right)$$

with an atom of size $(q_1 - q_2)/(1-q_2)$ at $\pi = \pi^*$, and firm 2 chooses per-consumer profit according to distribution

$$F_2(\pi) = \frac{1-q_2}{1-q_1-q_2}\left(1 - \frac{\rho\pi^*}{\pi}\right)$$

with no atom. The equilibrium profits are $\pi_1 = q_1\pi^*$ and $\pi_2 = (1-q_1)\rho\pi^*$.

In this game, the unique Nash equilibrium is in mixed strategy. Firms randomize on a common support, and the firm with more captive consumer has a mass point at the bottom. Note that the above lemma does not allow $q_1 + q_2 = 1$, in which case each firm's mixed strategy is reduced to a pure strategy of π^* and operates as a monopoly. Furthermore, if $q_2 > q_1$, we can simply swap the role of the two firms.

Now we can use the equilibrium characterization in Lemma 1 to compute three welfare measures for the segment (q_1, q_2). Let $G(\pi; q_1, q_2)$ denote the equilibrium probability that a consumer in segment (q_1, q_2) is offered a minimum profit weakly lower than π. Since firm i's profit offer is considered only by consumers captive to firm i and contested consumers, we have

$$\begin{aligned} G(\pi; q_1, q_2) &= (1-q_2) F_1(\pi) + (1-q_1) F_2(\pi) - (1-q_1-q_2) F_1(\pi) F_2(\pi) \\ &= \frac{(1-q_1)(1-q_2)}{1-q_1-q_2}\left(1 - \frac{q_1^2}{(1-q_2)^2}\left(\frac{\pi^*}{\pi}\right)^2\right) \end{aligned} \tag{3}$$

with an atom of size $q_1(q_1 - q_2)/(1-q_2)$ at $\pi = \pi^*$. It follows from Lemma 1 that the equilibrium producer surplus obtained in segment (q_1, q_2) is

$$P(q_1, q_2) = \int_{\rho\pi^*}^{\pi^*} \pi dG(\pi; q_1, q_2) \tag{4}$$

The corresponding equilibrium consumer surplus is

$$C(q_1, q_2) = \int_{\rho\pi^*}^{\pi^*} v(\pi) dG(\pi; q_1, q_2), \tag{5}$$

and the equilibrium social surplus is

$$S(q_1, q_2) = \int_{\rho\pi^*}^{\pi^*} s(\pi) dG(\pi; q_1, q_2), \tag{6}$$

where $s(\pi) = v(\pi) + \pi$ is strictly concave due to the strict concavity of $v(\pi)$.

The designer's problem is to choose market segmentation $m(q_1, q_2) \in \Delta\mathcal{M}$ to maximize (or minimize) the expected producer surplus

$$\sum_{(q_1,q_2)\in\mathcal{M}} m(q_1, q_2)P(q_1, q_2),$$

or the expected consumer surplus

$$\sum_{(q_1,q_2)\in\mathcal{M}} m(q_1, q_2)C(q_1, q_2),$$

or the expected social surplus

$$\sum_{(q_1,q_2)\in\mathcal{M}} m(q_1, q_2)S(q_1, q_2),$$

subject to the consistency constraint (1). We call a market segmentation P-Max (P-Min, respectively) if it maximizes (minimizes, respectively) the expected producer surplus. The C-Max, C-Min, S-Max and S-Min segmentations are similarly defined.

The objective of the paper is to identify these six optimal market segmentations. To achieve this objective, we need to solve six different optimization problems separately. Instead of addressing each problem separately, we adopt a unified approach. Noting the similarity in welfare measures (4)-(6), we introduce a unified optimization problem that encompasses all six optimization problems.

3 A Unified Approach

We begin this section by introducing the unified problem. We then demonstrate how a key observation can significantly simplify the problem and improve its tractability. Finally, we present a solution that can be applied to obtain the six optimal market segmentations.

3.1 A unified optimization problem

Let $a(\pi)$ be a twice continuously differentiable function $a : [0, \pi^*] \to \mathbb{R}$ and define

$$A(q_1, q_2) = \int_{\rho\pi^*}^{\pi^*} a(\pi)\, dG(\pi; q_1, q_2).$$

Now consider the following optimization problem:

$$\max_{m \in \Delta\mathcal{M}} \sum_{(q_1,q_2)\in\mathcal{M}} m(q_1, q_2) A(q_1, q_2) \tag{OPT}$$

subject to the consistency constraint (1). Throughout of the paper, we will refer to this problem as Problem (OPT). By choosing the function $a(\pi)$ appropriately, Problem (OPT) can nest the six optimization problems as special cases:

$$\begin{aligned}
\text{P-Max} &: a(\pi) = \pi \\
\text{P-Min} &: a(\pi) = -\pi \\
\text{C-Max} &: a(\pi) = v(\pi) \\
\text{C-Min} &: a(\pi) = -v(\pi) \\
\text{S-Max} &: a(\pi) = s(\pi) \\
\text{S-Min} &: a(\pi) = -s(\pi)
\end{aligned}$$

We can use expression (3) for $G(q_1, q_2)$ to rewrite $A(q_1, q_2)$ as

$$A(q_1, q_2) = \frac{q_1(q_1 - q_2)}{1 - q_2} a(\pi^*) + \frac{2q_1^2(1 - q_1)}{(1 - q_2)(1 - q_1 - q_2)} (\pi^*)^2 \int_{\rho\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi. \tag{7}$$

The function $A(q_1, q_2)$ will be central to our analysis. As discussed in Kamenica and Gentzkow (2011) and Bergemann and Morris (2019), finding the optimal segmentation is equivalent to finding the concave envelope of the function $A(q_1, q_2)$, denoted as $A^{\#}(q_1, q_2)$. However, finding $A^{\#}(q_1, q_2)$ can be quite challenging because $A(q_1, q_2)$ is defined on a two-dimensional space and because there are almost no restrictions on the set of segments that can be included in the support of the segmentation $m(q_1, q_2)$. Our strategy is to leverage the problem's structure to significantly narrow down the types of market segments that may be included in the solution to Problem (OPT).

3.2 Solution characterization

We first argue that it is sufficient to consider segmentations that consist of at most three types of simple segments which are defined below.

Definition 1 *An L-nested segment is a segment of the form* $(q_1, 0)$*. An R-nested segment is a segment of the form* $(0, q_2)$*. A symmetric segment is a segment of the form* (q, q)*.*

The key to our argument is the following simple observation: We can decompose any segment (q_1, q_2) with $q_1 \geq q_2$ into a symmetric segment and an L-nested segment as follows:

$$\frac{(1+q_1-q_2)\,q_2}{q_1}\left(\frac{q_1}{1+q_1-q_2}, \frac{q_1}{1+q_1-q_2}\right) + \frac{(q_1-q_2)(1-q_2)}{q_1}\left(\frac{q_1}{1-q_2}, 0\right). \tag{8}$$

There are many other ways of decomposing (q_1, q_2) into a symmetric segment and a L-nested segment, but the above decomposition uniquely ensures that the two new segments share the same captive-to-reach ratio ρ for firm 1, defined in (2), as the original segment:

$$\rho\,(q_1, q_2) = \rho\left(\frac{q_1}{1+q_1-q_2}, \frac{q_1}{1+q_1-q_2}\right) = \rho\left(\frac{q_1}{1-q_2}, 0\right).$$

We illustrate this decomposition in Figure 3. All segments (q_1, q_2) with $q_1 \geq q_2$ lie within the shaded lower triangle formed by points $(0,0)$, $(1/2, 1/2)$ and $(1, 0)$. The symmetric segment

$$\left(\frac{q_1}{1+q_1-q_2}, \frac{q_1}{1+q_1-q_2}\right) = \left(\frac{\rho}{1+\rho}, \frac{\rho}{1+\rho}\right)$$

lies on the edge linking $(0,0)$ with $(1/2, 1/2)$, while the L-nested segment

$$\left(\frac{q_1}{1-q_2}, 0\right) = (\rho, 0)$$

lies on the edge linking $(0,0)$ to $(1,0)$. It is easy to verify that all points on the line segment linking $(\rho, 0)$ and $\left(\frac{\rho}{1+\rho}, \frac{\rho}{1+\rho}\right)$ share the same captive-to-reach ratio ρ. Moreover, an extension of this line segment passes the point $(0, 1)$. An analogous

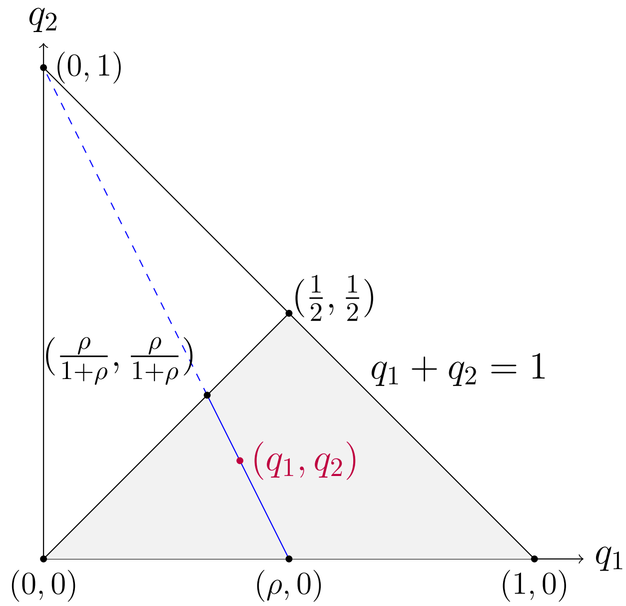

Figure 3: Decomposition into symmetric and nested segments

decomposition can divide a segment (q_1, q_2) with $q_1 \leq q_2$ in the upper triangle into a symmetric segment lying on the edge of (0,0)-(1/2,1/2) and an R-nested segment lying on the edge of (0,0)-(0,1).

The significance of this decomposition is that it preserves the objective value of Problem (OPT). Therefore, as stated in the following lemma, it is without loss of generality to focus on optimal segmentation that consists only of symmetric and nested segments. The proofs of this and subsequent lemmas and propositions are collected in the appendix.

Lemma 2 *Problem (OPT) admits an optimal segmentation that consists only of symmetric segments and nested segments.*

Therefore, when solving Problem (OPT), we only need to consider the concave envelopes of symmetric segments and nested segments, i.e., $A^{\#}(q, q)$, $A^{\#}(q_1, 0)$ and $A^{\#}(0, q_2)$. In fact, because all concave envelopes are concave, we need to select at most one point from each of the three concave envelopes. Let m_S denote the size of the symmetric segment (x, x), m_L the size of the L-nested segment $(y, 0)$, and m_R the size of the R-nested segment $(0, z)$. Problem (OPT) can be simplified as:

$$\max_{\substack{m_S, m_L, m_R \in [0,1] \\ x \in [0,1/2], y \in [0,1], z \in [0,1]}} \quad m_S A^{\#}(x, x) + m_L A^{\#}(y, 0) + m_R A^{\#}(0, z) \tag{OPT-S}$$

subject to

$$\begin{aligned} m_S + m_L + m_R &= 1, \\ m_S x + m_L y &= \gamma_1, \\ m_S x + m_R z &= \gamma_2. \end{aligned}$$

So far, we have not imposed any restriction on the shape of the function $a(\pi)$. However, for all the six optimization problems that we are interested in, $a(\pi)$ is strictly monotone. For strictly monotone $a(\pi)$, we can further simplify Problem (OPT-S) by excluding one of the three segments in the optimal segmentation, as stated in the following lemma.

Lemma 3 *If $a(\pi)$ is strictly increasing, Problem (OPT-S) has a solution with $m_S = 0$. If $a(\pi)$ is strictly decreasing, Problem (OPT-S) has a solution with $m_R = 0$.*

This lemma is proved by construction. For a strictly increasing $a(\pi)$, we decompose a symmetric segment (q, q) with $q \leq 1/2$ into two nested segments

$$(q, q) = \frac{1}{2}(2q, 0) + \frac{1}{2}(0, 2q)$$

and establish that $A(q, q) \leq A(2q, 0)$. Therefore, by replacing a symmetric segment with an R-nested and an L-nested segment, we can weakly increase the objective value of (OPT-S). For a strictly decreasing $a(\pi)$, we can replace an R-nested segment and an L-nested segment by a symmetric segment, leading to a weakly higher objective value of (OPT-S). Formally, we prove the following inequality: for any $x \in [0, 1/2], y \in [0, 1], z \in [0, 1]$,

$$A\left(\frac{yz}{y+z}, \frac{yz}{y+z}\right) \geq \frac{z}{y+z} A(y, 0) + \frac{y}{y+z} A(0, z).$$

If $a(\pi)$ is strictly increasing, it follows from Lemma 3 that $m_S = 0$, and therefore Problem (OPT-S) can be rewritten as

$$\max_{m_L \in [\gamma_1, 1-\gamma_2]} m_L A^{\#}\left(\frac{\gamma_1}{m_L}, 0\right) + (1 - m_L) A^{\#}\left(\frac{\gamma_2}{1 - m_L}, 0\right).$$

By concavity of $A^{\#}(q_1, 0)$, this objective is bounded above by $A^{\#}(\gamma_1 + \gamma_2, 0)$. Furthermore, this upper-bound is attained by $m_L = \gamma_1/(\gamma_1 + \gamma_2)$.

If $a(\pi)$ is strictly decreasing, the exact solution to Problem (OPT-S) is not yet attainable due to its dependence on the shape of $a(\pi)$. Nevertheless, $m_R = 0$ by

Lemma 3, and thus Problem (OPT-S) is reduced to a single-variable maximization problem. The following proposition summarizes the above discussion.

Proposition 1 *If $a(\pi)$ is strictly increasing,*

$$(m_S, m_L, m_R) = \left(0, \frac{\gamma_1}{\gamma_1+\gamma_2}, \frac{\gamma_2}{\gamma_1+\gamma_2}\right) \tag{OPT-I}$$

solves Problem (OPT-S) and attains an objective value of

$$\frac{\gamma_1}{\gamma_1+\gamma_2} A^{\#}(\gamma_1+\gamma_2, 0) + \frac{\gamma_2}{\gamma_1+\gamma_2} A^{\#}(0, \gamma_1+\gamma_2).$$

If $a(\pi)$ is strictly decreasing, Problem (OPT-S) is reduced to

$$\max_{m_S \in [2\gamma_2, 1-\gamma_1+\gamma_2]} m_S A^{\#}\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1-m_S) A^{\#}\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right). \tag{OPT-D}$$

Proposition 1 provides a partial characterization of the solution to Problem (OPT). The only obstacle in applying Proposition 1 to solve our six design problems is computing envelope functions of the form $A^{\#}(q, q)$, $A^{\#}(q_1, 0)$ and $A^{\#}(0, q_2)$, i.e., the concave envelopes for symmetric and nested segments. These will be derived in the next subsection.

3.3 Concave envelopes for symmetric and nested segments

We now characterize the concave envelopes for symmetric segments and nested segments. Since $A(q, q)$, $A(q_1, 0)$ and $A(0, q_2)$ are functions of a single variable, finding their concave envelopes is much easier than finding the concave envelope $A^{\#}(q_1, q_2)$. We will focus on the cases where $a(\pi)$ is strictly monotone and is either concave or convex. This gives us four possible cases: (i) $a(\pi)$ is strictly decreasing and *strictly* concave; (ii) $a(\pi)$ is strictly decreasing and *weakly* convex; (iii) $a(\pi)$ is strictly increasing and *weakly* concave; and (iv) $a(\pi)$ is strictly increasing and strictly convex. We will label these cases as (d-scav), (d-wcvx), (i-wcav), and (i-scvx), respectively. Notably, Case (d-wcvx) allows $a(\pi)$ to be linearly decreasing, while Case (i-wcav) permits $a(\pi)$ to be linearly increasing.

For symmetric segments, we argue that the concavity/convexity of $a(\pi)$ directly translates into the concavity/convexity of $A(q, q)$. The argument was first sketched in Armstrong and Vickers (2019). We provide a formal proof in the Appendix for completeness. Given the concavity/convexity of $A(q,q)$, the characterization of its concave envelope $A^{\#}(q,q)$ is then straightforward.

Lemma 4 *$A(q,q)$ is weakly concave (convex, respectively) if $a(\pi)$ is weakly concave (convex, respectively). Therefore, the concave envelope $A^{\#}(q,q)$ is given by*

$$A^{\#}(q,q)=\begin{cases} A(q,q) & \text{if} \quad a(\pi) \quad \text{is weakly concave} \\ 2qA(\frac{1}{2},\frac{1}{2})+(1-2q)A(0,0) & \text{if} \quad a(\pi) \quad \text{is weakly convex} \end{cases}$$

For nested segments, we will focus our analysis on $A(q_1,0)$ since the analysis for $A(0,q_2)$ is analogous. In contrast to $A(q,q)$, $A(q_1,0)$ can be neither concave nor convex.

Lemma 5 *$A(q_1,0)$ is strictly increasing (decreasing, respectively) if $a(\pi)$ is strictly increasing (decreasing, respectively). Moreover, $A(q_1,0)$ has the following shape:*

- *It is first strictly concave and then strictly convex in Case (d-scav).*
- *It is strictly convex in Case (d-wcvx).*
- *It is strictly concave in Case (i-wcav).*
- *It is first strictly convex and then strictly concave in Case (i-scvx).*

Using Lemma 5, we can quickly derive the concave envelope $A^{\#}(q_1,0)$ for Case (d-wcvx) and Case (i-wcav). In Case (d-wcvx), $A(q_1,0)$ is strictly convex and thus $A^{\#}(q_1,0)=q_1A(1,0)+(1-q_1)A(0,0)$. In Case (i-wcav), $A(q_1,0)$ is strictly concave and thus $A^{\#}(q_1,0)=A(q_1,0)$.

For the remaining cases, (d-scav) and (i-scvx), deriving the concave envelope $A^{\#}(q_1,0)$ requires a bit more work. Figure 4 illustrates the shape of $A(q_1,0)$ in these two cases. In Case (d-scav), $A(q_1,0)$ is initially concave and then convex (left panel), whereas in Case (i-scvx), it is initially convex and then concave (right panel). To derive the concave envelope $A^{\#}(q_1,0)$, we need to identify the tangent points l_1 and l_2.[10]

To define l_1 in Case (d-scav), we first define an auxiliary function Φ as

$$\Phi\left(l\right)\equiv\left(1-l\right)^{2}a\left(\pi^{*}\right)-2l\left(2-l\right)\left(\pi^{*}\right)^{2}\int_{l\pi^{*}}^{\pi^{*}}\frac{a\left(\pi\right)}{\pi^{3}}d\pi+\frac{2\left(1-l\right)a\left(l\pi^{*}\right)}{l}. \tag{9}$$

[10] At first glance, l_1 and l_2 may appear to be mirror images of each other because finding the concave envelope in Case (d-scav) is similar to finding the convex envelope in Case (i-scvx), and vice versa. However, they are not. The nested segment $(q_1,0)$ is symmetric when $q_1=0$, is asymmetric when $q_1>0$, and its level of asymmetry increases as q_1 increases from 0 to 1. Therefore, the concave envelope of $A(q_1,0)$ differs from (the mirror image of) its convex envelope, especially in the location of their respective tangent points. As shown in Lemma 6, l_2 is always interior with $l_2\in(0,1)$, but l_1 often takes the value of 0.

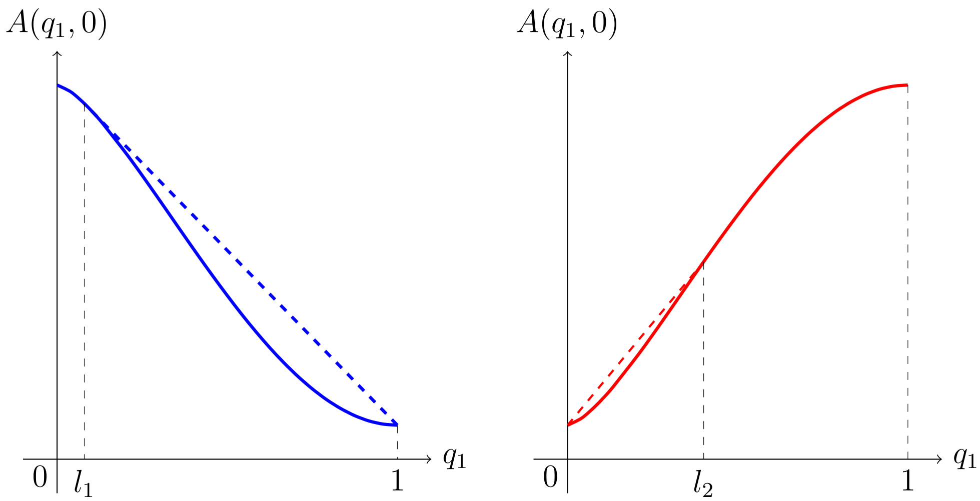

Figure 4: Definitions of l_1 and l_2

We can then use Φ to implicitly define l_1 as

$$l_1 = \inf \{l \in [0,1) : \Phi(l) \geq 0\}, \tag{10}$$

where we exclude the value of 1 because $\Phi(1) = 0$. For Case (i-scvx), we implicitly define l_2 as

$$l_2 = \sup\{l \in (0,1] : A(l,0) - 2a(l\pi^*) + a(0) \geq 0\}, \tag{11}$$

where we exclude the value of 0 because $A(0,0) - 2a(0) + a(0) = 0$. It is straightforward to verify that $A^{\#}(q_1, 0)$ is indeed tangent to $A(q_1, 0)$ at $q_1 = l_1$ if l_1 is interior in Case (d-wcav) and at $q_1 = l_2$ if l_2 is interior in Case (i-wcvx):

$$\frac{\partial A(q_1,0)}{\partial q_1}|_{q_1=l_1} = \frac{A(1,0) - A(l_1,0)}{1-l_1} \quad \text{and} \quad \frac{\partial A(q_1,0)}{\partial q_1}|_{q_1=l_2} = \frac{A(l_2,0) - A(0,0)}{l_2}.$$

It is important to note that both l_1 and l_2 depend only on the shape of $a(\pi)$; in particular, they are independent of the prior market configuration, γ_1 and γ_2.

The following lemma characterizes the concave envelope for nested segments. In particular, it provides a necessary and sufficient condition for $l_1 = 0$. We defer our discussion of this condition to Section 4.2 where we derive optimal segmentation for consumer surplus. In contrast, l_2 is always interior, as shown in the lemma.

Lemma 6 *The concave envelope $A^{\#}(q_1, 0)$ is given as follows.*

- *Case (d-scav): If $a(\pi^*) - a(0) - 2\pi^* a'(0) \geq 0$, then $l_1 = 0$ and*

$$A^{\#}(q_1, 0) = (1 - q_1)A(0,0) + q_1 A(1,0).$$

If $a(\pi^*) - a(0) - 2\pi^* a'(0) < 0$, *then* $l_1 \in (0,1)$ *and*

$$A^{\#}(q_1, 0) = \begin{cases} A(q_1, 0) & \text{if } \ q_1 \leq l_1 \\ \frac{1-q_1}{1-l_1} A(l_1, 0) + \frac{q_1 - l_1}{1-l_1} A(1, 0) & \text{if } \ q_1 > l_1 \end{cases} \tag{12}$$

- *Case (d-wcvx):* $A^{\#}(q_1, 0) = q_1 A(1, 0) + (1 - q_1) A(0, 0)$.
- *Case (i-wcav):* $A^{\#}(q_1, 0) = A(q_1, 0)$.
- *Case (i-scvx): We have* $l_2 \in (0,1)$ *and*

$$A^{\#}(q_1, 0) = \begin{cases} A(q_1, 0) & \text{if } \ q_1 \geq l_2 \\ \frac{q_1}{l_2} A(l_2, 0) + \frac{l_2 - q_1}{l_2} A(0, 0) & \text{if } \ q_1 < l_2 \end{cases} \tag{13}$$

Proposition 1, together with Lemmas 4 and 6, provides a procedure for solving the unified problem (OPT). If $a(\pi)$ is strictly increasing, we apply Lemma 6 to derive the concave envelope $A^{\#}(\gamma_1 + \gamma_2, 0)$ which is then combined with the preliminary solution (OPT-I) to obtain the final solution to Problem (OPT). If $a(\pi)$ is strictly decreasing, we apply Lemmas 4 and 6 to derive the concave envelopes for symmetric and nested segments and then solve the single variable maximization problem (OPT-D). The next section will demonstrate how this procedure can be adapted to solve our six optimization problems.

4 Best and Worst Market Segmentation

In this section, we will study the best and worst market segmentation for producer surplus, consumer surplus, and social surplus, respectively. We will begin by defining three types of market segmentation that will play an important role in the subsequent analysis.

Definition 2 *Fix a prior market* (γ_1, γ_2). *A market segmentation is called "nested segmentation" if it decomposes* (γ_1, γ_2) *into two nested segments as follows:*

$$\frac{\gamma_1}{\gamma_1 + \gamma_2} (\gamma_1 + \gamma_2, 0) + \frac{\gamma_2}{\gamma_1 + \gamma_2} (0, \gamma_1 + \gamma_2) \, . \tag{14}$$

A market segmentation is called "perfect segmentation" if it perfectly separates the three types of consumers in the prior market (γ_1, γ_2):

$$\gamma_1 (1, 0) + \gamma_2 (0, 1) + (1 - \gamma_1 - \gamma_2)(0, 0). \tag{15}$$

A market segmentation is called "field-leveling segmentation" if it decomposes (γ_1, γ_2) *into a symmetric segment and a nested segment as follows:*

$$(1-\gamma_1+\gamma_2)\left(\frac{\gamma_2}{1-\gamma_1+\gamma_2}, \frac{\gamma_2}{1-\gamma_1+\gamma_2}\right) + (\gamma_1-\gamma_2)(1,0). \tag{16}$$

The definitions of nested segmentation and perfect segmentation are straightforward. We will briefly comment on the definition of field-leveling segmentation. In the prior market (γ_1, γ_2), firm 1 has an advantage since it has more captive consumers than firm 2. To level the playing field, this segmentation removes a fraction $(\gamma_1 - \gamma_2)$ of firm 1's captive consumers to form the L-nested segment $(1, 0)$, while keeping all remaining consumers to form the symmetric segment. This symmetric segment is "maximal" in the sense that, after accounting for its size, it contains the largest fraction of the prior market.

To derive the best and worst segmentation for different welfare measures, we appropriately choose $a(\pi)$ to apply the solution to Problem (OPT) obtained in Section 3. While we sidestep the issue of uniqueness in solving Problem (OPT), we will address uniqueness here whenever possible.

4.1 P-Max and P-Min segmentation

For producer surplus, we set $a(\pi) = \pi$ and hence $A(q_1, q_2) = P(q_1, q_2)$ in Problem (OPT) with

$$P(q_1, q_2) = \frac{(2-q_1-q_2)\,q_1}{1-q_2}\pi^*. \tag{17}$$

Since $a(\pi) = \pi$ is linearly increasing, the P-Max problem belongs to Case (i-wcav), and the solution to Problem (OPT) is a P-Max segmentation. By Lemma 6, $A^{\#}(q_1, 0) = A(q_1, 0)$. By (OPT-I) in Proposition 1, nested segmentation (14) is a P-Max segmentation.

The following proposition shows that nested segmentation is in fact uniquely optimal for firms and consistently detrimental to consumers compared to uniform pricing.

Proposition 2 *Nested segmentation uniquely maximizes producer surplus. Moreover, it yields a lower consumer surplus than uniform pricing for any prior market.*

We provide intuition underlying Proposition 2. For the first part, consider a market segment (q_1, q_2) with $q_1 \geq q_2$. Suppose that we keep the total share of captive consumers $\ell = q_1 + q_2$ fixed and increase the fraction of firm 1's captive consumers q_1. As q_1 increases, the segment becomes more asymmetric, and firm 1's incentive to offer lower profit (i.e., cut price) to attract contested consumers decreases. Since profit (or price) offers are strategic complements, the equilibrium profit distribution shifts upward. Formally, we can use equation (17) to write the total profit of segment (q_1, q_2) as

$$P(q_1, q_2) = \frac{(2-\ell)q_1}{1-\ell+q_1}\pi^*,$$

which increases in q_1 for fixed ℓ. Therefore, for a fixed ℓ, nested segments $(\ell, 0)$ and $(0, \ell)$ exhibit maximal asymmetry and thus generate maximal profit.

For the second part, define $\ell = \gamma_1 + \gamma_2$ as the total share of captive consumers in the prior market. Consumer surplus is $C(\ell, 0)$ under nested segmentation and $C(\gamma_1, \gamma_2) = C(\gamma_1, \ell - \gamma_1)$ under uniform pricing. The second part claims that $C(\ell, 0) < C(\gamma_1, \ell - \gamma_1)$. To understand this inequality, note that, as we increase γ_1 while keeping ℓ fixed, two effects occur. First, as previously argued, when the market becomes more asymmetric, the equilibrium profit distribution shifts upward, which tends to lower consumer surplus. Second, the support of the equilibrium profit distribution, $[\frac{\gamma_1}{1+\gamma_1-\ell}\pi^*, \pi^*]$, shrinks and the variability of profit may go down, which tends to benefit consumers who are risk averse regarding offered profit π. It turns out that the first effect always dominates and consumers are always worse off.

The second part has antitrust implications. It suggests that if data brokers or third-party platforms are allowed to freely choose information structures for the product markets through public information provision, consumers are likely worse off compared to when no information is provided. To protect consumers, antitrust authorities may need to intervene, for example, by banning price discrimination or the sale of personal information.

Several remarks are in order regarding how Proposition 2 connects to and differs from the existing literature. First, Bergemann, Brooks and Morris (2020) and Albrecht (2020) have shown that nested segmentation is P-Max in the context of unit demand. It is not surprising that this result extends to our setting of downward sloping demand because there is one-to-one mapping between the price and the profit. By viewing firms as competing in profits rather than in prices, the analysis of equilibrium profits is identical to that with unit demand. Hence, the first part is known in the literature, but the second part is new.

Second, unlike the analysis of producer surplus, which is identical in both demand settings, the analysis of consumer welfare and social welfare differs. In the case of unit demand, consumers are always served in every possible segmentation, making total social surplus constant. This implies that what is best for firms must be worst for consumers and vice versa. However, in the case of downward-sloping demand, consumers are risk-averse to profit variation, and total welfare varies across different segmentations. The reformulation of firms choosing profits rather than prices does not simplify the analysis of consumer surplus. As a result, the analysis of price discrimination's effects on consumer welfare and social welfare is more nuanced in the setting of downward sloping demand, as we will demonstrate later in this section.

Third, we show that the P-Max segmentation unambiguously reduces consumer welfare relative to uniform pricing (i.e., no segmentation). In contrast, most of the literature on third-degree price discrimination takes market segmentation as *exogenously* given and finds the welfare consequences of price discrimination generally ambiguous. For example, Schmalensee (1981) and Varian (1985) show that the effect of monopolistic price discrimination on social welfare, relative to uniform pricing, depends on whether overall output increases.[11] In a symmetric duopoly model, Holmes (1989) shows that the effects of price discrimination on output and profit depend on cross-price elasticities and concavities of demand functions in the two submarkets.[12]

Finally, in the same model framework as here, Armstrong and Vickers (2019) find that, the effect of price discrimination (i.e., perfect segmentation) on consumer welfare relative to uniform pricing (i.e., no segmentation) is ambiguous and depends on the degree of asymmetry between firms. From the above proposition, we know that perfect segmentation does not maximize producer surplus, so the second part of the proposition does not apply. In fact, as shown in the next proposition, perfect segmentation minimizes producer surplus.

Proposition 3 *Perfect segmentation minimizes producer surplus.*

This result is intuitive because a firm's profit cannot fall below the level it earns from serving its own captive consumers, and perfect segmentation ensures that both firms achieve precisely that. The P-Min segmentation is not unique, as both $P(q,q)$ and $P(q,1-q)$ are linear in q. In Figure 3, we can mix or split in various ways along the 45-degree line and the line connecting $(0,1)$ and $(1,0)$, without changing the objective value of (OPT). In particular, field-leveling segmentation defined in (16) also minimizes producer surplus.

[11] See also Aguirre, Cowan and Vickers (2010) and Cowan (2012). Aguirre, Cowan and Vickers (2010) show that the effect of price discrimination depends on the relative curvature of the direct or inverse demand functions in the two submarkets. Cowan (2012) shows that consumer surplus may rise with discrimination if the ratio of pass-through to the elasticity at the uniform price is higher in the high-elasticity submarket.

[12] See also Corts (1998) who shows that if firms disagree over which submarkets are strong or weak, then price discrimination may lower profit and increase consumer surplus.

4.2 C-Max and C-Min segmentation

To find the C-Max segmentation, let $a(\pi) = v(\pi)$ and thus $a(\pi)$ is strictly decreasing and strictly concave. Therefore, the C-Max problem falls under Case (d-scav). Depending on whether the tangent point l_1 is interior, we can divide Case (d-scav) into two subcases: $l_1 = 0$ and $l_1 > 0$.

Consider first the simpler subcase where $l_1 = 0$ and thus $A(q_1, 0)$ is convex. By Lemma 6, $l_1 = 0$ if and only if

$$v(\pi^*) - v(0) \geq 2\pi^* v'(0). \tag{18}$$

Since $v(\pi^*) - v(0) \leq \pi^* v'(0)$ by concavity, condition (18) is satisfied if $v(\pi)$ is not too concave. For example, inequality (18) holds for all indirect utility functions generated by a linear demand.

Remark 2 *A sufficient condition for inequality (18) to hold is that $D'(p)p$ is decreasing for $p \in [0, p^*]$. To see this, let $p(\pi) \in (0, p^*)$ be the price that generates profit π under demand $D(p)$. Let $\overline{p}$ be the choke price such that $D(\overline{p})$=0. Then $v(\pi) = \int_{p(\pi)}^{\overline{p}} D(t)dt$, which implies that*

$$v'(\pi) = -D(p(\pi))p'(\pi) = \frac{-D(p(\pi))}{D(p(\pi)) + D'(p(\pi))p(\pi)}.$$

Hence, $v'(0) = -1$. Using $\pi^* = p^* D(p^*)$, we deduce that, with decreasing $D'(p)p$,

$$\begin{aligned}
v(\pi^*) - v(0) - 2\pi^* v'(0) &= \int_{p^*}^{\overline{p}} D(t)dt - \int_0^{\overline{p}} D(t)dt + 2\pi^* \\
&= D(p^*)p^* + \int_0^{p^*} D'(t)tdt \\
&= -D'(p^*)(p^*)^2 + \int_0^{p^*} D'(t)tdt \\
&= \int_0^{p^*} [D'(t)t - D'(p^*)(p^*)]dt \\
&\geq 0,
\end{aligned}$$

where the second equality follows from integration by parts and the third follows from the first-order condition for p^*. This sufficient condition implies Assumption 1, but the reverse is not true.

By Proposition 1, the C-Max problem is equivalent to Problem (OPT-D) with

$A(q_1, q_2) = C(q_1, q_2)$:

$$\max_{m_S \in [2\gamma_2, 1-\gamma_1+\gamma_2]} m_S C^{\#}\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1 - m_S) C^{\#}\left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0\right) \tag{19}$$

By Lemma 4, $C(q, q)$ is concave and $C^{\#}(q, q) = C(q, q)$. By Lemma 6, $l_1 = 0$ and $C^{\#}(q_1, 0) = q_1 C(1, 0) + (1 - q_1) C(0, 0)$. The objective function in (19) becomes

$$m_S C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1 - m_S - (\gamma_1 - \gamma_2)) C(0, 0) + (\gamma_1 - \gamma_2) C(1, 0).$$

Since $C(q, q)$ is concave, we have

$$\frac{m_S}{1 - \gamma_1 + \gamma_2} C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + \frac{1 - \gamma_1 + \gamma_2 - m_S}{1 - \gamma_1 + \gamma_2} C(0, 0) \leq C\left(\frac{\gamma_2}{1 - \gamma_1 + \gamma_2}, \frac{\gamma_2}{1 - \gamma_1 + \gamma_2}\right).$$

Therefore, the above objective is bounded above by

$$(1 - \gamma_1 + \gamma_2) C\left(\frac{\gamma_2}{1 - \gamma_1 + \gamma_2}, \frac{\gamma_2}{1 - \gamma_1 + \gamma_2}\right) + (\gamma_1 - \gamma_2) C(1, 0).$$

Moreover, this upper bound is attained by $m_S = 1 - \gamma_1 + \gamma_2$. This implies that field-leveling segmentation is a C-Max segmentation. In fact, as stated in the following proposition, it uniquely maximizes consumer surplus.

Proposition 4 *Suppose $v(\pi^*) - v(0) \geq 2\pi^* v'(0)$. Field-leveling segmentation uniquely maximizes consumer surplus.*

Intuitively, firms are more competitive in a more symmetric segment. Any further division of a symmetric segment can only increase the variability of the profit distribution, thereby lowering the expected consumer surplus, because by Lemma 4, $C(q, q)$ is concave. Therefore, the symmetric segment must be maximal in the C-Max segmentation.

Why is condition (18) necessary? Field-leveling segmentation randomly assigns a consumer captive to firm 1 to either the symmetric segment or the L-nested segment $(1, 0)$. Such consumers attain the lowest payoff in segment $(1, 0)$ since firm 1 will offer π^* for sure, and they attain the highest possible payoff in the maximal symmetric segment. That is, nested segmentation creates a maximal payoff disparity for such consumers. Therefore, if consumers are highly risk averse so that condition (18) fails, then field-leveling segmentation may not be a C-Max segmentation.

Next, consider the subcase where condition (18) fails. By Lemma 6, the tangent point l_1 is interior and $A^{\#}(q_1, 0)$ is given by (12). The C-Max problem is again reduced to Problem (OPT-D) which is a single-variable maximization problem, but its solution is less straightforward because $A^{\#}(q_1, 0)$ takes a more complex form. Rather than presenting the full solution here (which can be found in an online appendix), we focus on the case where the C-Max segmentation also increases producer surplus relative to uniform pricing. That is, the C-Max segmentation is a Pareto improvement over uniform pricing.[13]

Proposition 5 *Suppose that* $v(\pi^*) - v(0) < 2\pi^* v'(0)$ *and* $\rho(\gamma_1, \gamma_2) < l_1^C$ *where* $l_1^C \in (0,1)$ *is implicitly defined by* $\Phi(l_1^C) = 0$. *Then the following modified field-leveling segmentation*

$$m_S^* \left(\frac{\gamma_2}{m_S^*}, \frac{\gamma_2}{m_S^*} \right) + (1 - m_S^*) \left(\frac{\gamma_1 - \gamma_2}{1 - m_S^*}, 0 \right) \tag{20}$$

maximizes consumer surplus, where m_S^* *is interior and is implicitly determined by the first-order condition of Problem (OPT-D). Furthermore, the C-Max segmentation yields a strictly higher producer surplus than uniform pricing.*

Recall by Lemma 6 that the tangent point $l_1 = 0$ if condition (18) holds and $l_1 \in (0,1)$ otherwise. When l_1 is interior, it depends on the function $a(\pi)$. Here we write l_1^C to indicate that it is tailored for the C-Max problem. For the same reason, below we use l_2^C to denote the tangent point l_2 for the C-Min problem. Figure 5 provides a geometric illustration of the field-leveling segmentation (thick solid line) and its modification (thick dotted line) for the prior market (γ_1, γ_2).

Since a symmetric segment remains beneficial to consumers, the C-Max segmentation should still include a symmetric segment and an L-nested segment. In the optimal solution, $m_S^* < 1 - \gamma_1 + \gamma_2$ and thus the captive-to-reach ratios are ordered:

$$\rho\left(\frac{\gamma_1 - \gamma_2}{1 - m_S^*}, 0 \right) < \rho(1, 0) \quad \text{and} \quad \rho\left(\frac{\gamma_2}{1 - \gamma_1 + \gamma_2}, \frac{\gamma_2}{1 - \gamma_1 + \gamma_2} \right) < \rho\left(\frac{\gamma_2}{m_S^*}, \frac{\gamma_2}{m_S^*} \right).$$

Note that if $m_S^* = 1 - \gamma_1 + \gamma_2$, the modified field-leveling segmentation (20) is reduced to the standard field-leveling segmentation (16). Therefore, compared to field-leveling segmentation, the modified segmentation gives firm 1 a weaker incentive to cut price in the symmetric segment and a stronger incentive in the nested segment. Consequently, consumers receive a lower payoff in the symmetric segment but a higher payoff in the nested segment. When condition (18) fails, consumers are highly risk averse, so the reduced volatility in the profit distribution improves consumer welfare.

[13] The condition $\rho(\gamma_1, \gamma_2) < l_1^C$ in the following proposition is necessary for Pareto improvement. As we shown in the online appendix, if $\rho(\gamma_1, \gamma_2) = l_1^C$, uniform pricing (i.e., no segmentation) is consumer-optimal, and any price discrimination that strictly increases firms' profit must hurt consumers.

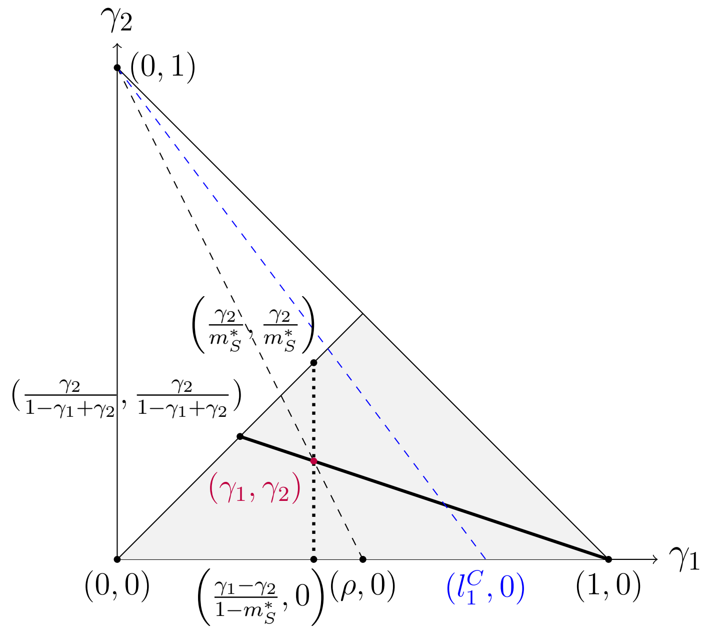

Figure 5: Geometry of field-leveling segmentation and its modification

To understand the argument for Pareto improvement, we first note that by decomposition (8) and Lemma 2, the producer surplus under uniform pricing is obtained by setting $m_S = \gamma_2(1 + \gamma_1 - \gamma_2)/\gamma_1$ in the segmentation (20). Next, we can use (17) to compute the producer surplus generated by the symmetric segment

$$m_S P\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) = 2\gamma_2\pi^*$$

and by the nested segment in the segmentation (20)

$$(1 - m_S)\, P\left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0\right) = \left(2 - \frac{\gamma_1 - \gamma_2}{1 - m_S}\right)(\gamma_1 - \gamma_2)\pi^*.$$

The former is constant in m_S, but the latter strictly decreases in m_S. Since $m_S^* < \gamma_2(1+\gamma_1-\gamma_2)/\gamma_1$, the C-Max segmentation strictly increases producer surplus compared to uniform pricing.[14]

We now turn to the C-Min problem. We set $a(\pi) = -v(\pi)$. Then $a(\pi)$ is strictly increasing and strictly convex, and thus the C-Min problem belongs to Case (i-scvx).

[14] For a numerical example, consider the demand function $D(p) = (p^{1.1} + 1)^{-1}$. We can verify that Assumption 1 holds, but condition (18) fails. Further calculations yield $p^* = 8.1113$, $\pi^* = 0.7374$, and $l_1^C = 0.2107$. Let $(\gamma_1, \gamma_2) = (0.1, 0.05)$ so that $\rho(\gamma_1, \gamma_2) < l_1^C$. The C-Max segmentation is given by $0.503(0.0994, 0.0994) + 0.497(0.1006, 0)$, which generates strictly higher consumer surplus and producer surplus than uniform pricing.

By Proposition 1, the objective value is given by

$$\frac{\gamma_1}{\gamma_1+\gamma_2}C^{\#}(\gamma_1+\gamma_2,0)+\frac{\gamma_2}{\gamma_1+\gamma_2}C^{\#}(0,\gamma_1+\gamma_2)$$

By Lemma 6, there is an interior $l_2^C \in (0,1)$ such that

$$C^{\#}(\gamma_1+\gamma_2,0)=\begin{cases} C(\gamma_1+\gamma_2,0) & \text{if } \gamma_1+\gamma_2 \geq l_2^C \\ \frac{\gamma_1+\gamma_2}{l_2^C}C(l_2^C,0)+\left(1-\frac{\gamma_1+\gamma_2}{l_2^C}\right)C(0,0) & \text{if } \gamma_1+\gamma_2 < l_2^C \end{cases}$$

Therefore, if $\gamma_1+\gamma_2 \geq l_2^C$ consumer surplus is minimized by the "standard" nested segmentation; otherwise it is minimized by the following "modified" nested segmentation:

$$\frac{\gamma_1}{l_2^C}\left(l_2^C,0\right)+\frac{\gamma_2}{l_2^C}\left(0,l_2^C\right)+\left(1-\frac{\gamma_1+\gamma_2}{l_2^C}\right)(0,0). \tag{21}$$

Figure 6 provides a geometric presentation of the nested segmentation for a prior market (γ_1', γ_2') with $\gamma_1'+\gamma_2' \geq l_2^C$ (thick solid line) and the modified nested segmentation for a prior market (γ_1,γ_2) with $\gamma_1+\gamma_2 < l_2^C$ (thick dotted line).

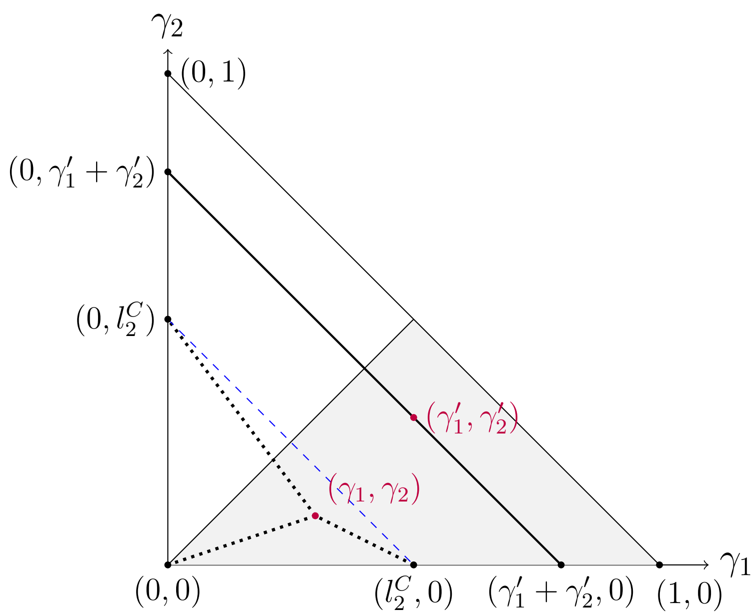

Figure 6: Geometry of nested segmentation and its modification

Proposition 6 *Let l_2^C be the tangent point defined in (11) with $a(\pi) = -v(\pi)$. If $\gamma_1+\gamma_2 \geq l_2^C$, nested segmentation uniquely minimizes consumer surplus; otherwise, the "modified" nested segmentation in (21) uniquely minimizes consumer surplus.*

As previously noted, a more asymmetric market leads to weaker competition between firms, resulting in lower consumer surplus. It is thus not surprising that a C-Min segmentation should include nested segments. Furthermore, for nested segments, consumer surplus can be further reduced by increasing the fraction of captive consumers. Therefore, if the total share of captive consumers is sufficiently large, i.e., $\gamma_1 + \gamma_2 \geq l_2^C$, then nested segmentation maximizes producer surplus and minimizes consumer surplus simultaneously. However, when the total share of captive consumers is too small, it is better to remove some contested consumers to form a third segment $(0, 0)$, thus increasing the fraction of captive consumers who remain in the nested segments. Under the modified nested segmentation, consumers gain from having the new segment $(0, 0)$, but this gain is outweighed by the loss in the two nested segments, $(l_2^C, 0)$ and $(0, l_2^C)$, due to less intense competition between firms.

4.3 S-Max and S-Min segmentation

To find the segmentation that maximizes social surplus, we set $a(\pi) = s(\pi) = \pi + v(\pi)$ and apply Proposition 1. By the concavity of $v(\pi)$, $s(\pi)$ is strictly concave. Therefore, if $s(\pi)$ is strictly increasing, the S-Max problem is similar to the P-Max problem and belongs to Case (i-wcav); if $s(\pi)$ is strictly decreasing, the S-Max problem is similar to the C-Max problem and falls in Case (d-scav). The proof for the first (second, respectively) part of the following proposition is similar to the proof for Proposition 2 (Proposition 4, respectively), and therefore we omit it.

Proposition 7 *If $s(\pi)$ is strictly increasing, then nested segmentation uniquely maximizes social surplus. If $s(\pi)$ is strictly decreasing and $v(\pi^*) - v(0) - 2\pi^* v'(0) - \pi^* \geq 0$, then field-leveling segmentation uniquely maximizes social surplus.*

To find the S-Min segmentation, we set $a(\pi) = -s(\pi) = -\pi - v(\pi)$ and thus $a(v)$ is strictly convex. If $s(\pi)$ is strictly increasing, the S-Min problem is similar to the P-Min problem and belongs to Case (d-wcvx). If $s(\pi)$ is strictly decreasing, the S-Min problem is similar to the C-Min problem and belongs to Case (i-scvx). Let l_2^S be the tangent point l_2 defined in (11) with $a(\pi) = -\pi - v(\pi)$. The following proposition is then immediate. Its proof is similar to the proof for Proposition 3 and Proposition 6, and thus is omitted.

Proposition 8 *If $s(\pi)$ is strictly increasing, then perfect segmentation minimizes social surplus. If $s(\pi)$ is strictly decreasing and $\gamma_1 + \gamma_2 \geq l_2^S$, then nested segmentation uniquely minimizes social surplus. If $s(\pi)$ is strictly decreasing and $\gamma_1 + \gamma_2 < l_2^S$, the* *following "modified" nested segmentation*

$$\frac{\gamma_1}{l_2^S}\left(l_2^S,0\right)+\frac{\gamma_2}{l_2^S}\left(0,l_2^S\right)+\left(1-\frac{\gamma_1+\gamma_2}{l_2^S}\right)(0,0)$$

uniquely minimizes social surplus.

If $s(\pi)$ is strictly increasing, perfect segmentation minimizes both social surplus and producer surplus.[15] Neither the S-Min segmentation nor the P-Min segmentation is unique. However, if $s(\pi)$ is strictly decreasing, the S-Min segmentation may not coincide with the C-Min segmentation. This discrepancy arises because l_2^S and l_2^C are defined for different $a(\pi)$ functions and thus different. One can show that $l_2^S \leq l_2^C$, so the S-Min segmentation and the C-Min segmentation coincide only if $\gamma_1+\gamma_2 \geq l_2^C$.

5 Concluding Remarks

We develop a unified information design approach to study the welfare effects of third-degree price discrimination in a duopoly market, considering all possible public market segmentations. Our findings show that firm-optimal market segmentation always harms consumers relative to uniform pricing, in contrast to the literature that assumes exogenous market segmentation and often finds ambiguous effects of price discrimination on consumer welfare. Additionally, we demonstrate that consumer-optimal segmentation can sometimes constitute a Pareto improvement over uniform pricing.

In a recent paper, Rhodes and Zhou (2024) extend the classic random-utility model by Perloff and Salop (1985) to a general oligopoly model with correlated product valuations and partial market coverage. Their model encompasses monopoly and the linear Hotelling model by Thisse and Vives (1998) as special cases. They find that the welfare effects of personalized pricing (first-degree discrimination) versus uniform pricing depend on market coverage. This raises the question of whether our information design exercise can be applied to their model and if our finding that firm-optimal price discrimination always harms consumers holds. The challenge lies in characterizing all possible equilibria for various market configurations, presenting an interesting direction for future research.

Throughout the paper, we focus on market segmentations based on publicly observable signals. In some cases, firms may not share the same consumer information. Allowing for multiple firms, Bergemann, Brooks and Morris (2021) identify an upper bound for the equilibrium distribution of prices and construct a private segmentation

[15] It is worth noting that field-leveling segmentation, which minimizes producer surplus, does not minimize social surplus even if social surplus is increasing in profit.

that achieves this bound for any symmetric prior. Their construction is as follows: if captive to firm 1, reveal s_1 to firm 1 and s_0 to firm 2; if captive to firm 2, reveal s_0 to firm 1 and s_1 to firm 2; if contested, reveal s_1 to firm 1 and s_2 to firm 2 with probability α, s_2 to firm 1 and s_1 to firm 2 with probability α, and s_2 to both with probability $1-2\alpha$. This private segmentation yields higher producer surplus than nested segmentation with a symmetric prior. However, this advantage disappears with a strongly asymmetric prior (e.g., $\gamma_1 = 0$ or $\gamma_2 = 0$). The optimal private segmentation for an asymmetric prior remains an open question.

6 Appendix: Proofs

Proof of Lemma 2. Take any segment (q_1, q_2) with $q_1 \geq q_2$ and decompose it into a symmetric segment and a nested segment as in (8). By writing $\rho = \rho(q_1, q_2)$, we note that

$$
\begin{aligned}
& \frac{q_2(1+q_1-q_2)}{q_1} A\left(\frac{q_1}{1+q_1-q_2}, \frac{q_1}{1+q_1-q_2}\right) + \frac{(q_1-q_2)(1-q_2)}{q_1} A\left(\frac{q_1}{1-q_2}, 0\right) \\
= \; & \frac{q_2(1+q_1-q_2)}{q_1} \frac{2\left(\frac{q_1}{1+q_1-q_2}\right)^2}{\left(1-\frac{2q_1}{1+q_1-q_2}\right)} (\pi^*)^2 \int_{\rho\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi \\
& + \frac{(q_1-q_2)(1-q_2)}{q_1} \left(\left(\frac{q_1}{1-q_2}\right)^2 a(\pi^*) + 2\left(\frac{q_1}{1-q_2}\right)^2 (\pi^*)^2 \int_{\rho\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi\right) \\
= \; & \frac{q_1(q_1-q_2)}{1-q_2} a(\pi^*) + \frac{2q_1^2(1-q_1)}{(1-q_2)(1-q_1-q_2)} (\pi^*)^2 \int_{\rho\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi \\
= \; & A(q_1, q_2).
\end{aligned}
$$

Therefore, any segment (q_1, q_2) with $q_1 \geq q_2$ in an optimal segmentation can be replaced by a symmetric segment and a nested segment without changing the objective value in Problem (OPT). The argument is symmetric for segment (q_1, q_2) with $q_1 < q_2$. Together, we conclude that Problem (OPT) has an optimal segmentation that consists only of symmetric and nested segments. ■

Proof of Lemma 3. Consider first the case that $a(\pi)$ is strictly increasing. We can decompose any symmetric segment (q, q) with $q \leq 1/2$ into two nested segments

$$(q, q) = \frac{1}{2}(2q, 0) + \frac{1}{2}(0, 2q).$$

Therefore, to show the optimality of $m_S = 0$, it is sufficient to show that

$$A(q,q) \leq A(2q,0) = \frac{1}{2}A(2q,0) + \frac{1}{2}A(0,2q).$$

It holds trivially with equality if $q = 1/2$. It remains to show that $A(q,q) \leq A(2q,0)$ for $q < 1/2$. Note that

$$\begin{aligned}
& A\left(2q,0\right) - A\left(q,q\right) \\
= \;& 4q^2 a\left(\pi^*\right) + 8q^2\left(\pi^*\right)^2 \int_{2q\pi^*}^{\pi^*} \frac{a\left(\pi\right)}{\pi^3} d\pi - \frac{2q^2}{1-2q}\left(\pi^*\right)^2 \int_{\frac{q}{1-q}\pi^*}^{\pi^*} \frac{a\left(\pi\right)}{\pi^3} d\pi \\
= \;& 4q^2 a\left(\pi^*\right) + \left(8q^2 - \frac{2q^2}{1-2q}\right)\left(\pi^*\right)^2 \int_{2q\pi^*}^{\pi^*} \frac{a\left(\pi\right)}{\pi^3} d\pi - \frac{2q^2}{1-2q}\left(\pi^*\right)^2 \int_{\frac{q}{1-q}\pi^*}^{2q\pi^*} \frac{a\left(\pi\right)}{\pi^3} d\pi,
\end{aligned}$$

where the last integration is well-defined since $\frac{q}{1-q} < 2q$. We consider two sub-cases.

(i) If $0 < q \leq 3/8$, then $8q^2 - \frac{2q^2}{1-2q} \geq 0$. Since $a(\pi)$ is increasing, we have

$$\begin{aligned}
& A\left(2q,0\right) - A\left(q,q\right) \\
\geq \;& 4q^2 a\left(\pi^*\right) + \left(8q^2 - \frac{2q^2}{1-2q}\right)\left(\pi^*\right)^2 \int_{2q\pi^*}^{\pi^*} \frac{a\left(2q\pi^*\right)}{\pi^3} d\pi - \frac{2q^2}{1-2q}\left(\pi^*\right)^2 \int_{\frac{q}{1-q}\pi^*}^{2q\pi^*} \frac{a\left(2q\pi^*\right)}{\pi^3} d\pi \\
= \;& 4q^2 a\left(\pi^*\right) + a\left(2q\pi^*\right)\left(8q^2 - \frac{2q^2}{1-2q}\right)\left(\frac{1-4q^2}{8q^2}\right) - a\left(2q\pi^*\right)\frac{2q^2}{1-2q}\frac{(1-2q)(3-2q)}{8q^2} \\
= \;& 4q^2\left[a\left(\pi^*\right) - a\left(2q\pi^*\right)\right] \\
> \;& 0.
\end{aligned}$$

(ii) If $3/8 < q < 1/2$, then $8q^2 - \frac{2q^2}{1-2q} < 0$. Again it follows from the monotonicity of $a\left(\pi\right)$ that

$$\begin{aligned}
& A\left(2q,0\right) - A\left(q,q\right) \\
\geq \;& 4q^2 a\left(\pi^*\right) + \left(8q^2 - \frac{2q^2}{1-2q}\right)\left(\pi^*\right)^2 \int_{2q\pi^*}^{\pi^*} \frac{a\left(\pi^*\right)}{\pi^3} d\pi - \frac{2q^2}{1-2q}\left(\pi^*\right)^2 \int_{\frac{q}{1-q}\pi^*}^{2q\pi^*} \frac{a\left(2q\pi^*\right)}{\pi^3} d\pi \\
= \;& 4q^2 a\left(\pi^*\right) + a\left(\pi^*\right)\left(8q^2 - \frac{2q^2}{1-2q}\right)\left(\frac{1-4q^2}{8q^2}\right) - a\left(2q\pi^*\right)\frac{2q^2}{1-2q}\frac{(1-2q)(3-2q)}{8q^2} \\
= \;& \left(\frac{3}{4} - \frac{1}{2}q\right)\left[a\left(\pi^*\right) - a\left(2q\pi^*\right)\right] \\
> \;& 0.
\end{aligned}$$

In both sub-cases, we have $\frac{1}{2}A\left(2q,0\right) + \frac{1}{2}A\left(0,2q\right) > A\left(q,q\right)$. Therefore, if $a(\pi)$ is strictly increasing, we can safely exclude the symmetric segment from the optimal segmentation.

Next consider the case that $a(\pi)$ is strictly decreasing. Take any optimal segmentation m that includes at least one R-nested segment. Since the prior market (γ_1, γ_2) satisfies $\gamma_1 \geq \gamma_2$, the optimal segmentation m must also involve at least one L-nested segment. Note that, to find the concave closure of the function $A(q_1, 0)$ or $A(0, q_2)$, we need to randomize over at most two points. Therefore, it is without loss to assume that the segmentation m consists of at most two L-nested segments, $(y, 0)$, $(y', 0)$ with $y, y' \in (0, 1]$, and two R-nested segments, $(0, z)$, $(0, z')$ with $z, z' \in (0, 1]$. That is, we can write the segmentation m as

$$\begin{aligned}(\gamma_1, \gamma_2) \quad &= \quad \beta(y, 0) + \beta'(y', 0) + \delta(0, z) + \delta'(0, z') \\ &\quad + (1 - \beta - \beta' - \delta - \delta')\left(\frac{\gamma_1 - \beta y - \beta' y'}{1 - \beta - \beta' - \delta - \delta'}, \frac{\gamma_2 - \delta z - \delta' z'}{1 - \beta - \beta' - \delta - \delta'}\right)\end{aligned}$$

where $\beta, \beta', \delta, \delta'$ denote the weights assigned to the segments $(y, 0)$, $(y', 0)$, $(0, z)$, $(0, z')$, respectively, and the last segment is symmetric with

$$\gamma_1 - \gamma_2 = \beta y + \beta' y' - \delta z - \delta' z' \geq 0. \tag{22}$$

We will construct an alternative segmentation m' that excludes R-nested segments and weakly improves the objective value of Problem (OPT). A key step of our construction is to prove the following inequality:

$$A\left(\frac{yz}{y+z}, \frac{yz}{y+z}\right) - \frac{z}{y+z} A(y, 0) - \frac{y}{y+z} A(0, z) \geq 0. \tag{23}$$

To prove it, we first use the following observation

$$\frac{z}{y+z}\left(1 - y^2\right) + \frac{y}{y+z}\left(1 - z^2\right) + yz = 1$$

to write the left-hand side of the inequality as

$$
\begin{aligned}
& A\left(\frac{yz}{y+z}, \frac{yz}{y+z}\right) - \frac{z}{y+z}A(y,0) - \frac{y}{y+z}A(0,z) \\
= & \frac{z}{y+z}\left[\frac{2y^2z^2(1-y^2)}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi - 2y^2(\pi^*)^2\int_{y\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi\right] \\
& + \frac{y}{y+z}\left[\frac{2y^2z^2(1-z^2)}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi - 2z^2(\pi^*)^2\int_{z\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi\right] \\
& + yz\left[\frac{2y^2z^2}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi - a(\pi^*)\right]
\end{aligned}
$$

We now argue that the value of the terms in each big bracket is non-negative. Since

$$
\frac{2y^2z^2(1-y^2)}{(y+z)(y+z-2yz)} - 2y^2 = -\frac{2y^3(1-z)(y+2z-yz)}{(y+z)(y+z-2yz)} \leq 0
$$

the value of the terms in the first big bracket is non-negative:

$$
\begin{aligned}
& \frac{2y^2z^2(1-y^2)}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi - 2y^2(\pi^*)^2\int_{y\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi \\
= & \frac{2y^2z^2(1-y^2)}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{y\pi^*}\frac{a(\pi)}{\pi^3}d\pi \\
& - \frac{2y^3(1-z)(y+2z-yz)}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{y\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi \\
\geq & \frac{2y^2z^2(1-y^2)}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{y\pi^*}\frac{a(y\pi^*)}{\pi^3}d\pi \\
& - \frac{2y^3(1-z)(y+2z-yz)}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{y\pi^*}^{\pi^*}\frac{a(y\pi^*)}{\pi^3}d\pi \\
= & \frac{2y^2z^2(1-y^2)a(y\pi^*)}{(y+z)(y+z-2yz)}(\pi^*)^2\left(\frac{1}{2\left(\frac{yz}{y+z-yz}\pi^*\right)^2} - \frac{1}{2(y\pi^*)^2}\right) \\
& - \frac{2y^3(1-z)(y+2z-yz)a(y\pi^*)}{(y+z)(y+z-2yz)}(\pi^*)^2\left(\frac{1}{2(y\pi^*)^2} - \frac{1}{2(\pi^*)^2}\right) \\
= & 0
\end{aligned}
$$

The same algebra implies that the value of the terms in the second big bracket is also non-negative. Finally, the value of the last big bracket is non-negative because

$$\begin{aligned}
& \frac{2y^2z^2}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{\pi^*}\frac{a(\pi)}{\pi^3}d\pi - a(\pi^*) \\
\geq\ & \frac{2y^2z^2}{(y+z)(y+z-2yz)}(\pi^*)^2\int_{\frac{yz}{y+z-yz}\pi^*}^{\pi^*}\frac{a(\pi^*)}{\pi^3}d\pi - a(\pi^*) \\
=\ & \frac{2y^2z^2a(\pi^*)}{(y+z)(y+z-2yz)}(\pi^*)^2\left(\frac{1}{2\left(\frac{yz}{y+z-yz}\pi^*\right)^2}-\frac{1}{2(\pi^*)^2}\right) - a(\pi^*) \\
=\ & 0
\end{aligned}$$

This concludes the proof for inequality (23).

Inequality (23) says that we can weakly improve the objective value of Problem (OPT) by merging a portion of an L-segment $(y,0)$ with a portion of a R-nested segment $(0,z)$ to form a symmetric segment. In particular, inequality (23) implies

$$\left(\delta''+\frac{\delta''z}{y}\right)A\left(\frac{yz}{y+z},\frac{yz}{y+z}\right) \geq \frac{\delta''z}{y}A(y,0)+\delta''A(0,z) \tag{24}$$

$$\left(\delta'''+\frac{\delta'''z'}{y}\right)A\left(\frac{yz'}{y+z'},\frac{yz'}{y+z'}\right) \geq \frac{\delta'''z'}{y}A(y,0)+\delta'''A(0,z') \tag{25}$$

and

$$\begin{aligned}
& \left(\delta-\delta''+\frac{(\delta-\delta'')z}{y'}\right)A\left(\frac{y'z}{y'+z},\frac{y'z}{y'+z}\right) \\
\geq\ & \frac{(\delta-\delta'')z}{y'}A(y',0)+(\delta-\delta'')A(0,z) \qquad (26) \\
& \left(\delta'-\delta'''+\frac{(\delta'-\delta''')z}{y'}\right)A\left(\frac{y'z'}{y'+z'},\frac{y'z'}{y'+z'}\right) \\
\geq\ & \frac{(\delta'-\delta''')z'}{y'}A(y',0)+(\delta'-\delta''')A(0,z') \qquad (27)
\end{aligned}$$

where δ'' and δ''' are such that $\frac{\delta''z}{y}\leq\beta$, $\delta''\leq\delta$, $\delta'''\leq\delta'$, $\frac{\delta'''z'}{y}\leq\beta$, $\frac{(\delta-\delta'')z}{y'}\leq\beta'$ and $\frac{(\delta'-\delta''')z'}{y'}\leq\beta'$.

By (22), we can find $\delta''\in[0,\delta]$ and $\delta'''\in[0,\delta']$ such that

$$\delta''z+\delta'''z'\leq\beta y,(\delta-\delta'')z+(\delta'-\delta''')z'\leq\beta'y'.$$

Then consider an alternative segmentation m':

$$\begin{aligned}(\gamma_1,\gamma_2) &= \left(\beta-\frac{\delta''z}{y}-\frac{\delta'''z'}{y}\right)(y,0)+\left(\beta'-\frac{(\delta-\delta'')z}{y'}-\frac{(\delta'-\delta''')z'}{y'}\right)(y',0)\\ &+ \left(\delta''+\frac{\delta''z}{y}\right)\left(\frac{yz}{y+z},\frac{yz}{y+z}\right)+\left(\delta'''+\frac{\delta'''z'}{y}\right)\left(\frac{yz'}{y+z'},\frac{yz'}{y+z'}\right)\\ &+ \left(\delta-\delta''+\frac{(\delta-\delta'')z}{y'}\right)\left(\frac{y'z}{y'+z},\frac{y'z}{y'+z}\right)\\ &+ \left(\delta'-\delta'''+\frac{(\delta'-\delta''')z'}{y'}\right)\left(\frac{y'z'}{y'+z'},\frac{y'z'}{y'+z'}\right)\\ &+ (1-\beta-\beta'-\delta-\delta')\left(\frac{\gamma_1-\beta y-\beta'y'}{1-\beta-\beta'-\delta-\delta'},\frac{\gamma_2-\delta z-\delta'z'}{1-\beta-\beta'-\delta-\delta'}\right).\end{aligned}$$

Combining (25)-(27), we conclude that segmentation m' weakly improves upon the original segmentation m. Therefore, if $a(\pi)$ is decreasing, it is without loss to exclude R-nested segments from the optimal segmentation, i.e., $m_R = 0$. ■

Proof of Lemma 4. We focus on the case of weakly concave $a(\pi)$; the proof for the case of weakly convex $a(\pi)$ is symmetric. To show that $A(q,q)$ is weakly concave, we need to show that, for any $\lambda \in (0,1)$, any $0 < q_L < q_H < 1/2$ and $q = \lambda q_L + (1-\lambda)q_H$,

$$A(q,q) \geq \lambda A(q_L,q_L) + (1-\lambda)A(q_H,q_H).$$

Since by definition

$$A(q,q) = \int_{\rho\pi^*}^{\pi^*} a(\pi)dG(\pi;q,q),$$

it is sufficient to show that the distribution of the minimum of the profits in the two submarkets (q_L,q_L) and (q_H,q_H) is a mean-preserving spread of the minimum profit distribution in the single prior market (q,q).

Let $G(\pi;q) \equiv G(\pi;q,q)$ denote the probability that a consumer in market segment (q,q) is offered a minimum profit weakly lower than π. Then we must have

$$G(\pi;q) = \frac{(1-q)^2}{1-2q} - \frac{q^2}{1-2q}\left(\frac{\pi^*}{\pi}\right)^2.$$

$G(\pi;q)$ is strictly concave in q for all $\pi < \pi^*$ because

$$\frac{\partial^2 G(\pi;q)}{\partial q^2} = -\frac{2}{(1-2q)^3}\left(\left(\frac{\pi^*}{\pi}\right)^2 - 1\right) < 0.$$

Consider $q_L < q_H$, $\lambda \in (0,1)$ and $q = \lambda q_L + (1-\lambda) q_H$. Then for all $\pi \in \left[\frac{q_H}{1-q_H}\pi^*, \pi^*\right]$,

$$\overline{G}(\pi; q) \equiv \lambda G(\pi; q_L) + (1-\lambda) G(\pi; q_H) < G(\pi; q).$$

Since $q > q_L$, the support of $\overline{G}(\pi; q)$ contains the support of $G(\pi; q)$. Furthermore, for $\pi \in \left[\frac{q}{1-q}\pi^*, \frac{q_H}{1-q_H}\pi^*\right]$,

$$\frac{G'(\pi; q)}{\overline{G}'(\pi; q)} = \frac{1}{\lambda}\frac{q^2}{1-2q}\left(\frac{q_L^2}{1-2q_L}\right)^{-1} > 1$$

because function $f(x) = x^2/(1-2x)$ is strictly increasing and $q > q_L$. It follows that $G(\pi; q)$ crosses $\overline{G}(\pi; q)$ only once and from below. Finally, the two submarkets yield the same producer surplus of $2q\pi^*$ as the prior single symmetric market (q, q). Therefore, $\overline{G}(\pi; q)$ is a mean-preserving spread of $G(\pi; q)$.

Now if $a(\pi)$ is weakly concave, $A(q,q)$ is weakly concave and thus $A^{\#}(q,q) = A(q,q)$. If $a(\pi)$ is weakly convex, $A(q,q)$ is weakly convex and $A^{\#}(q,q)$ is the line segment linking $A(\frac{1}{2}, \frac{1}{2})$ and $A(0,0)$. That is, $A^{\#}(q,q) = 2qA(\frac{1}{2}, \frac{1}{2}) + (1-2q)A(0,0)$. ■

Proof of Lemma 5. We first prove the monotonicity of $A(q_1, 0)$. If $a(\pi)$ is strictly increasing, then $A(q_1, 0)$ is also strictly increasing because

$$\begin{aligned}
\frac{\partial A(q_1,0)}{\partial q_1} &= 2q_1 a(\pi^*) + 4q_1(\pi^*)^2 \int_{q_1\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 2\frac{a(q_1\pi^*)}{q_1} \\
&> 2q_1 a(\pi^*) + 4q_1(\pi^*)^2 \int_{q_1\pi^*}^{\pi^*} \frac{a(q_1\pi^*)}{\pi^3} d\pi - 2\frac{a(q_1\pi^*)}{q_1} \\
&= 2q_1 a(\pi^*) + 4q_1(\pi^*)^2 \frac{1-q_1^2}{2q_1^2(\pi^*)^2} a(q_1\pi^*) - 2\frac{a(q_1\pi^*)}{q_1} \\
&= 2q_1(a(\pi^*) - a(q_1\pi^*)) \\
&> 0.
\end{aligned}$$

The proof for the case with strictly decreasing $a(\pi)$ is symmetric.

Next, we prove the shape of $A(q_1, 0)$. Consider first Case (d-scav) where $a(\pi)$ is strictly decreasing and strictly concave. The second and third derivatives of $A(q_1, 0)$ are

$$\begin{aligned}
\frac{\partial^2 A(q_1,0)}{\partial q_1^2} &= 2a(\pi^*) + 4(\pi^*)^2 \int_{q_1\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - \frac{2}{q_1^2} a(q_1\pi^*) - \frac{2\pi^*}{q_1} a'(q_1\pi^*) \\
\frac{\partial^3 A(q_1,0)}{\partial q_1^3} &= -\frac{2(\pi^*)^2}{q_1} a''(q_1\pi^*)
\end{aligned}$$

It is immediate that

$$\frac{\partial^3 A(q_1,0)}{\partial q_1^3} > 0 \text{ and } \lim_{q_1\to 1} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) > 0.$$

To establish the claim that $A(q_1,0)$ is first strictly concave and then strictly convex, it is sufficient to show that $\lim_{q_1\to 0^+} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = -\infty$. Suppose by contradiction that $\lim_{q_1\to 0^+} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = y \in (-\infty,\infty)$. Then we can repeatedly apply the L'Hôpital's rule to rewrite y as

$$\begin{aligned}
& 2a(\pi^*) + \lim_{q_1\to 0^+} \frac{4(\pi^*)^2 q_1^2 \int_{q_1\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 2a(q_1\pi^*) - 2\pi^* q_1 a'(q_1\pi^*)}{q_1^2} \\
= \ & 2a(\pi^*) \\
& + \lim_{q_1\to 0^+} \frac{8(\pi^*)^2 q_1 \int_{q_1\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 4(\pi^*)^3 q_1^2 \frac{a(q_1\pi^*)}{(q_1\pi^*)^3} - 4\pi^* a'(q_1\pi^*) - 2(\pi^*)^2 q_1 a''(q_1\pi^*)}{2q_1} \\
= \ & 2a(\pi^*) + \lim_{q_1\to 0^+} \frac{4(\pi^*)^2 q_1^2 \int_{q_1\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - 2a(q_1\pi^*) - 2\pi^* q_1 a'(q_1\pi^*)}{q_1^2} - (\pi^*)^2 a''(0) \\
= \ & y - (\pi^*)^2 a''(0),
\end{aligned}$$

which leads to a contradiction. Therefore, $\lim_{q_1\to 0^+} \frac{\partial^2 A(q_1,0)}{\partial q_1^2}$ does not have a finite limit. But since $\frac{\partial^2 A(q_1,0)}{\partial q_1^2}$ is continuous and striclty increasing in q_1 for all $q_1 \in (0,1)$, we must have $\lim_{q_1\to 0^+} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = -\infty$.

In Case (d-wcvx), $a(\pi)$ is strictly decreasing and weakly convex, and

$$\frac{\partial^3 A(q_1,0)}{\partial q_1^3} \le 0 \text{ and } \lim_{q_1\to 1} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) > 0.$$

It follows that

$$\frac{\partial^2 A(q_1,0)}{\partial q_1^2} > 0 \text{ for all } q_1 \in [0,1],$$

and therefore $A(q_1,0)$ is strictly convex in q_1 for all $q_1 \in [0,1]$.

In Case (i-wcav), $a(\pi)$ is strictly increasing and weakly concave, and

$$\frac{\partial^3 A(q_1,0)}{\partial q_1^3} \ge 0 \text{ and } \lim_{q_1\to 1} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) < 0.$$

This implies that

$$\frac{\partial^2 A(q_1,0)}{\partial q_1^2} < 0 \text{ for all } q_1 \in [0,1],$$

which means that $A(q_1,0)$ is strictly concave.

Finally, for Case (i-scvx), since $a(\pi)$ is strictly increasing and strictly convex, it is immediate that

$$\frac{\partial^3 A(q_1,0)}{\partial q_1^3} < 0 \quad \text{and} \quad \lim_{q_1 \to 1} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) < 0.$$

As we have shown in the proof of Case (d-scav), $\frac{\partial^2 A(q_1,0)}{\partial q_1^2}$ does not have a finite limit as $q_1 \to 0^+$. We must have

$$\lim_{q_1 \to 0^+} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = +\infty.$$

Therefore, $A(q_1, 0)$ must be initially convex and then concave. ■

Proof of Lemma 6. For Case (d-scav), it remains to show that $l_1 = 0$, or equivalently $\Phi(0) \geq 0$, if and only if $a(\pi^*) - a(0) - 2\pi^* a'(0) \geq 0$. This is true because

$$\begin{aligned}
\Phi(0) &= a(\pi^*) - 2\lim_{l \to 0^+} \frac{(\pi^*)^2 \int_{l\pi^*}^{\pi^*} \frac{a(\pi)}{\pi^3} d\pi - \left(\frac{1-l}{l^2(2-l)}\right) a(l\pi^*)}{\frac{1}{l(2-l)}} \\
&= a(\pi^*) - 2\lim_{l \to 0^+} \frac{-\frac{a(l\pi^*)}{l^3} - \frac{1-l}{l^2(2-l)}\pi^* a'(l\pi^*) + \frac{\left(2l^2-5l+4\right)}{l^3(2-l)^2} a(l\pi^*)}{-\frac{2-2l}{l^2(2-l)^2}} \\
&= a(\pi^*) - 2\lim_{l \to 0^+} \frac{-\frac{1-l}{l^2(2-l)^2} a(l\pi^*) - \frac{1-l}{l^2(2-l)}\pi^* a'(l\pi^*)}{-\frac{2-2l}{l^2(2-l)^2}} \\
&= a(\pi^*) - \lim_{l \to 0^+} \left(a(l\pi^*) + (2-l)\pi^* a'(l\pi^*)\right) \\
&= a(\pi^*) - a(0) - 2\pi^* a'(0).
\end{aligned}$$

When l_1 is interior, the expression of concave envelope $A^{\#}(q_1, 0)$ directly follows from the definition of l_1 (see Figure 4).

The claims for Case (d-wcvx) and Case (i-wcav) directly follow from Lemma 5.

For Case (i-scvx), we first establish that l_2 is interior. In the proof of Lemma 5, we have shown that in Case (i-scvx)

$$\frac{\partial^3 A(q_1,0)}{\partial q_1^3} < 0 \quad \text{and} \quad \lim_{q_1 \to 1} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = -2\pi^* a'(\pi^*) < 0.$$

and that

$$\lim_{q_1 \to 0^+} \frac{\partial^2 A(q_1,0)}{\partial q_1^2} = +\infty.$$

Therefore, there must exist some $l^* \in (0,1)$ such that $\frac{\partial^2 A(q_1,0)}{\partial q_1^2} > 0$ if and only if $q_1 < l^*$. We can use the expressions for $A(q_1, 0)$ and $\partial A(q_1, 0)/\partial q_1$ to show that

$$\begin{aligned}
& A\left(l^*, 0\right) - 2a\left(l\pi^*\right) + a\left(0\right) \\
= \; & l^* \frac{\partial A(l_1^*, 0)}{\partial q_1} - A(l^*, 0) + A(0, 0) \\
= \; & l^* \frac{\partial A(l_1^*, 0)}{\partial q_1} - \int_0^{l^*} \frac{\partial A(q_1, 0)}{\partial q_1} dq_1 \\
= \; & \int_0^{l^*} \left(\frac{\partial A(l^*, 0)}{\partial q_1} - \frac{\partial A(q_1, 0)}{\partial q_1} \right) dq_1 \\
> \; & 0,
\end{aligned}$$

where the inequality follows from the convexity of $A(q_1, 0)$ for $q_1 < l^*$.

Since $A\left(l, 0\right) - 2a\left(l\pi^*\right) + a\left(0\right)$ is continuous in l and

$$A\left(1, 0\right) - 2a\left(\pi^*\right) + a\left(0\right) = -a\left(\pi^*\right) + a\left(0\right) < 0,$$

l_2 exists and $l_2 \in (0, 1)$. Then the expression of concave envelope $A^{\#}(q_1, 0)$ directly follows from the definition of l_2 (see Figure 4). ■

Proof of Proposition 2. For the first part, we only need to show uniqueness, which consists of three steps. First, a P-Max segmentation cannot contain any interior segment (q_1, q_2) with $0 < q_2 < \min\{q_1, 1 - q_1\}$ or $0 < q_1 < \min\{q_2, 1 - q_2\}$. To see this, take an interior segment (q_1, q_2) and decompose it into an L-nested segment and an R-nested segment as in (14). This decomposition strictly improves the objective in (OPT) because

$$\begin{aligned}
& \frac{q_1}{q_1 + q_2} P\left(q_1 + q_2, 0\right) + \frac{q_2}{q_1 + q_2} P\left(0, q_1 + q_2\right) - P\left(q_1, q_2\right) \\
= \; & \left(2 - q_1 - q_2\right)\left(q_1 + q_2\right)\pi^* - \frac{\left(2 - q_1 - q_2\right) q_1}{1 - q_2} \pi^* \\
= \; & \frac{\left(2 - q_1 - q_2\right)\left(1 - q_1 - q_2\right) q_2}{1 - q_2} \pi^* \\
> \; & 0,
\end{aligned}$$

where the first equality uses equation (17). It follows that a P-Max segmentation can use only symmetric segments, nested segments, and segments of form $(q_1, 1 - q_1)$.

Second, nested segmentation is uniquely optimal among segmentations that use only symmetric and nested segments, because the objective in Problem (OPT-S) is strictly concave. To see this, note that $m_S = 0$ when $a(\pi) = \pi$ and hence the objective of Problem (OPT-S) becomes

$$\begin{aligned} & m_L P^{\#}\left(\frac{\gamma_1}{m_L},0\right)+\left(1-m_L\right)P^{\#}\left(0,\frac{\gamma_2}{1-m_L}\right) \\ = \; & m_L\left(2-\frac{q_1}{m_L}\right)\frac{q_1}{m_L}\pi^*+\left(1-m_L\right)\left(2-\frac{q_2}{1-m_L}\right)\frac{q_2}{1-m_L}\pi^* \\ = \; & \left[\left(2-\frac{q_1}{m_L}\right)q_1+\left(2-\frac{q_2}{1-m_L}\right)q_2\right]\pi^* \end{aligned}$$

where the first equality follows from Lemma 6 that $P^{\#}(q_1,0)=P(q_1,0)=(2-q_1)q_1\pi^*$. It is easy to verify that the objective is strictly concave in m_L.

Third, a P-Max segmentation will never use a segment of the form $(q_1,1-q_1)$. We can decompose $(q_1,1-q_1)$ into $q_1(1,0)+(1-q_1)(0,1)$. By step two, this segment $(q_1,1-q_1)$ is strictly dominated by nested segmentation (14). This concludes the proof of uniqueness.

For the second part of the proposition, note that consumer surplus under nested segmentation is $C\left(\gamma_1+\gamma_2,0\right)$, so we need to show $C\left(\gamma_1,\gamma_2\right)\geq C\left(\gamma_1+\gamma_2,0\right)$. Define $\ell=\gamma_1+\gamma_2$ and rewrite

$$C\left(\gamma_1,\gamma_2\right)=C\left(\gamma_1,\ell-\gamma_1\right).$$

We prove that $C\left(\gamma_1,\ell-\gamma_1\right)$ is decreasing in γ_1 for fixed ℓ. That is, for a fixed total share of captive consumers, consumer surplus decreases as the distribution of captive consumers becomes more uneven between the two firms. Note that

$$C\left(\gamma_1,\ell-\gamma_1\right)=\frac{\gamma_1\left(2\gamma_1-\ell\right)}{1-\ell+\gamma_1}v\left(\pi^*\right)+\frac{2\gamma_1^2\left(1-\gamma_1\right)\left(\pi^*\right)^2}{\left(1-\ell\right)\left(1-\ell+\gamma_1\right)}\int_{\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*}^{\pi^*}\frac{v\left(\pi\right)}{\pi^3}d\pi.$$

The total derivative of $C\left(\gamma_1,\ell-\gamma_1\right)$ with respect to γ_1 is given by

$$\begin{aligned} & \frac{2\gamma_1^2+\left(4\gamma_1-\ell\right)\left(1-\ell\right)}{\left(1-\ell+\gamma_1\right)^2}v\left(\pi^*\right)-\frac{2\left(1-\gamma_1\right)}{\gamma_1}v\left(\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*\right) \\ + \; & \frac{2\left(2\gamma_1-3\gamma_1^2\right)\left(1-\ell+\gamma_1\right)-2\gamma_1^2\left(1-\gamma_1\right)}{\left(1-\ell+\gamma_1\right)^2\left(1-\ell\right)}\left(\pi^*\right)^2\int_{\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*}^{\pi^*}\frac{v\left(\pi\right)}{\pi^3}d\pi \end{aligned}$$

It is non-positive because we can rewrite it as

$$\begin{aligned}
& \frac{2\gamma_1-\ell}{1-\ell}\left[v\left(\pi^*\right)-v\left(\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*\right)\right] \\
& +\frac{\left(2\gamma_1-3\gamma_1^2\right)\left(1-\ell+\gamma_1\right)-\gamma_1^2\left(1-\gamma_1\right)}{\left(1-\ell+\gamma_1\right)^2\left(1-\ell\right)}\left(\pi^*\right)^2\int_{\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*}^{\pi^*}\frac{v'\left(\pi\right)}{\pi^2}d\pi \\
= & \frac{2\gamma_1-\ell}{1-\ell}\int_{\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*}^{\pi^*}v'\left(\pi\right)d\pi+\frac{\left(2\gamma_1-3\gamma_1^2\right)\left(1-\ell+\gamma_1\right)-\gamma_1^2\left(1-\gamma_1\right)}{\left(1-\ell+\gamma_1\right)^2\left(1-\ell\right)}\left(\pi^*\right)^2\int_{\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*}^{\pi^*}\frac{v'\left(\pi\right)}{\pi^2}d\pi \\
\leq & \left(\frac{2\gamma_1-\ell}{1-\ell}+\frac{\left(2\gamma_1-3\gamma_1^2\right)\left(1-\ell+\gamma_1\right)-\gamma_1^2\left(1-\gamma_1\right)}{\left(1-\ell\right)\gamma_1^2}\right)\left(\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*\right)^2\int_{\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*}^{\pi^*}\frac{v'(\pi)}{\pi^2}d\pi \\
= & \frac{2\gamma_1(1-\gamma_1)(\pi^*)^2}{(1-\ell+\gamma_1)^2}\int_{\frac{\gamma_1}{1-\ell+\gamma_1}\pi^*}^{\pi^*}\frac{v'(\pi)}{\pi^2}d\pi \\
\leq & \ 0,
\end{aligned}$$

where the first equality follows from integration by part, the first inequality follows because $1 \geq 2\gamma_1 \geq \ell$, $\pi \geq \frac{\gamma_1}{1-\ell+\gamma_1}\pi^*$ and $v'\left(\pi\right) \leq 0$, and the last inequality follows because $v'\left(\pi\right) \leq 0$. Therefore, $C\left(\gamma_1, \gamma_2\right) \geq C\left(\gamma_1+\gamma_2, 0\right)$. ■

Proof of Proposition 3. To minimize producer surplus, we take $a(\pi) = -\pi$ and thus $A(q_1, q_2) = -P(q_1, q_2)$. By Proposition 1, the P-minimization problem is equivalent to Problem (OPT-D):

$$\begin{aligned}
& \max_{m_S \in [2\gamma_2, 1-\gamma_1+\gamma_2]} m_S A^{\#}\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right)+\left(1-m_S\right)A^{\#}\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right) \\
= & \max_{m_S \in [2\gamma_2, 1-\gamma_1+\gamma_2]} 2\gamma_2 A\left(\frac{1}{2}, \frac{1}{2}\right)+\left(1-\gamma_1-\gamma_2\right)A(0,0)+\left(\gamma_1-\gamma_2\right)A\left(1,0\right) \\
= & -\gamma_2(P(1,0)+P(0,1))-\left(1-\gamma_1-\gamma_2\right)P(0,0)-(\gamma_1-\gamma_2)P(1,0) \\
= & -\gamma_1 P(1,0)-\gamma_2 P(0,1)-(1-\gamma_1-\gamma_2)P(0,0),
\end{aligned}$$

where the first equality follows because by Lemmas 4 and 6

$$\begin{aligned}
A^{\#}(q,q) &= 2qA(1/2,1/2)+(1-2q)A(0,0) \\
A^{\#}(q_1,0) &= q_1 A(1,0)+(1-q_1)A(0,0)
\end{aligned}$$

and the second equality follows because the objective is independent of m_S, $A(q_1, q_2) = -P(q_1, q_2)$, and $P(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}P(1,0) + \frac{1}{2}P(0,1)$. It is clear that perfect segmentation attains the optimum. ■

Proof of Proposition 4. It remains to show uniqueness. We proceeds with two steps. First, from the proof of Lemma 4, we can see that $C(q, q)$ is strictly concave because $v(\pi)$ is strictly concave. Therefore, field-leveling segmentation strictly dominates any other segmentation that consists of symmetric and L-nested segments.

Second, suppose by contradiction that there is another C-Max segmentation that assigns a positive weight to an interior segment (q_1, q_2). Without loss of generality, assume $q_2 < q_1$. Then by Lemma 2, we can replace it with a symmetric segment and a L-nested segment that generates the same consumer surplus. By step one, this can be strictly improved by applying field-leveling segmentation to (q_1, q_2). A contradiction.

Third, we can rule out R-nested segments in a C-Max segmentation, because whenever it includes an R-nested segment, given $\gamma_1 \geq \gamma_2$, it must also include an L-nested segment, and these two segments can be partly combined to form a symmetric segment and strictly improve consumer surplus.

Finally, suppose by contradiction that there is another C-Max segmentation that assigns a positive weight to a segment of the form $(q_1, 1 - q_1)$. Then we can replace it with a segmentation consisting of $(1, 0)$ and $(0, 1)$ that generates the same consumer surplus. By step 3, this can be strictly improved by applying the field-leveling segmentation. This completes the proof. ■

Proof of Proposition 5. The proof for the first part involves some tedious algebra, so we only provide a sketch here. The full details of the proof can be found in an online appendix. Since $v(\pi^*) - v(0) < 2\pi^* v'(0)$, the tangent point $l_1 = l_1^C$ is interior. We can use Case (d-scav) of Lemma 6 to rewrite the objective of Problem (OPT-D) as

$$\begin{cases} m_S C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1 - m_S)\, C\left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0\right) & \text{if } \frac{\gamma_1 - \gamma_2}{1 - m_S} \leq l_1^C \\ m_S C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1 - m_S)\left(\frac{1 - \frac{\gamma_1 - \gamma_2}{1 - m_S}}{1 - l_1^C} C\left(l_1^C, 0\right) + \frac{\frac{\gamma_1 - \gamma_2}{1 - m_S} - l_1^C}{1 - l_1^C} C\left(1, 0\right)\right) & \text{if } \frac{\gamma_1 - \gamma_2}{1 - m_S} > l_1^C \end{cases}$$

First, one can verify that the objective is continuous and weakly concave in m_S, so the optimal m_S^* is either corner with $m_S^* \in \{2\gamma_2, 1 - \gamma_1 + \gamma_2\}$ or interior with m_S^* implicitly determined by the first-order condition.

Second, by assumption $\frac{\gamma_1}{1-\gamma_2} \leq l_1^C$, so the solution m_S^* belongs to the case of $\frac{\gamma_1 - \gamma_2}{1 - m_S} > l_1^C$ only if $m_S^* \geq \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$. One can check that the objective in the case of $\frac{\gamma_1 - \gamma_2}{1 - m_S} > l_1^C$ is strictly decreasing in m_S at $m_S = \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$. Therefore, $m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$ and the case of $\frac{\gamma_1 - \gamma_2}{1 - m_S} > l_1^C$ is ruled out.

Finally, we can also verify that the objective is strictly increasing in m_S at $m_S = 2\gamma_2$ and thus $m_S^* > 2\gamma_2$. Since $l_1^C \in (0, 1)$, it follows from $\frac{\gamma_1 - \gamma_2}{1 - m_S} \leq l_1^C$ that $m_S < 1 - \gamma_1 + \gamma_2$. Therefore, we must have $m_S^* \in (2\gamma_2, 1 - \gamma_1 + \gamma_2)$ and it is implicitly determined by the first-order condition.

For the second part, recall from Lemma 2 that any prior market (γ_1, γ_2) with $\gamma_1 \geq \gamma_2$ can be decomposed into a symmetric segment and an L-nested segment:

$$\frac{(1+\gamma_1-\gamma_2)\,\gamma_2}{\gamma_1}\left(\frac{\gamma_1}{1+\gamma_1-\gamma_2}, \frac{\gamma_1}{1+\gamma_1-\gamma_2}\right) + \frac{(\gamma_1-\gamma_2)(1-\gamma_2)}{\gamma_1}\left(\frac{\gamma_1}{1-\gamma_2}, 0\right)$$

without changing the aggregate producer surplus that is generated. Now consider the following segmentation:

$$m_S\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1-m_S)\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right).$$

If $m_S = \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$, this segmentation coincides with the above decomposition and thus generates the same producer surplus as under uniform pricing. The producer surplus of this segmentation is

$$m_S P\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1-m_S)\,P\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right) = 2\gamma_2\pi^* + \left(2 - \frac{\gamma_1-\gamma_2}{1-m_S}\right)(\gamma_1-\gamma_2)\pi^*,$$

which is strictly decreasing in m_S. As we have shown earlier that $m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$, the producer surplus under the C-Max segmentation must be strictly higher than under uniform pricing. ■

Proof of Proposition 6. It remains to show uniqueness. By Lemma 6 and the definition of l_2^C, any other segmentation that consists of L-nested and R-nested segments generates a strictly higher consumer surplus. Using an argument similar to the one used to prove uniqueness in Proposition 4, we can show that a C-Min segmentation cannot contain an interior segment, a symmetric segment, or a segment of the form $(q_1, 1-q_1)$. ■

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Online Appendix: Not for Publication

This online appendix contains omitted results of consumer-optimal segmentations when $v(\pi^*) - v(0) - 2\pi^* v'(0) < 0$. Proposition A1 characterizes a consumer-optimal segmentation and subsumes the first part of Proposition 5 in the paper as a subcase. Although we do not have uniqueness in general, all these optimal segmentations yield the same producer surplus, and therefore we can compare them with uniform pricing in terms of producer surplus in Proposition A2. Proposition A2 shows that there is a Pareto improvement if and only if the captive-to-reach ratio is not too high and is a stronger statement than the second part of Proposition 5 in the paper.

By Proposition 1, the C-Max problem is equivalent to Problem (OPT-D) with $A(q_1, q_2) = C(q_1, q_1)$:

$$\max_{m_S \in [2\gamma_2, 1-\gamma_1+\gamma_2]} m_S C^{\#}\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1 - m_S) C^{\#}\left(\frac{\gamma_1 - \gamma_2}{1 - m_S}, 0\right).$$

By Lemma 4, $C^{\#}(q, q) = C(q, q)$. By Lemma 6,

$$C^{\#}(q_1, 0) = \begin{cases} C(q_1, 0) & \text{if} \quad q_1 \leq l_1^C \\ \frac{1-q_1}{1-l_1^C} C\left(l_1^C, 0\right) + \frac{q_1 - l_1^C}{1-l_1^C} C(1, 0) & \text{if} \quad q_1 > l_1^C \end{cases}$$

where $l_1^C \in (0, 1)$ is implicitly defined by $\Phi\left(l_1^C\right) = 0$ with

$$\Phi(l) = (1-l)^2 v(\pi^*) - 2l(2-l)(\pi^*)^2 \int_{l\pi^*}^{\pi^*} \frac{v(\pi)}{\pi^3} d\pi + \frac{2(1-l)v(l\pi^*)}{l}.$$

By solving this problem, we can characterize an optimal segmentation in the following proposition. There are four cases and Proposition 5 belongs to Case (1).

Proposition A1 *If* $v(\pi^*) - v(0) - 2\pi^* v'(0) < 0$, *the following segmentation maximizes consumer surplus among all possible market segmentations:*

(1) If $\frac{\gamma_1}{1-\gamma_2} < l_1^C$,

$$m_S^* \left(\frac{\gamma_2}{m_S^*}, \frac{\gamma_2}{m_S^*}\right) + (1 - m_S^*) \left(\frac{\gamma_1 - \gamma_2}{1 - m_S^*}, 0\right)$$

where m_S^* *is implicitly determined by the first-order condition of Problem (OPT-D) such that* $2\gamma_2 < m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$;

(2) If $\frac{\gamma_1}{1-\gamma_2} = l_1^C$, (γ_1, γ_2);

(3) If $\frac{\gamma_1}{1-\gamma_2} > l_1^C$ and $\frac{\gamma_2(1+l_1^C)}{l_1^C} < 1-\gamma_1+\gamma_2$,

$$\frac{1-\gamma_1-\gamma_2 l_1^C}{1-l_1^C}\left(\frac{l_1^C(1-\gamma_1-\gamma_2)}{1-\gamma_1-\gamma_2 l_1^C}, \frac{\gamma_2(1-l_1^C)}{1-\gamma_1-\gamma_2 l_1^C}\right) + \frac{\gamma_1+\gamma_2 l_1^C - l_1^C}{1-l_1^C}(1,0);$$

(4) If $\frac{\gamma_1}{1-\gamma_2} > l_1^C$ and $\frac{\gamma_2(1+l_1^C)}{l_1^C} \geq 1-\gamma_1+\gamma_2$,

$$(1-\gamma_1+\gamma_2)\left(\frac{\gamma_2}{1-\gamma_1+\gamma_2}, \frac{\gamma_2}{1-\gamma_1+\gamma_2}\right) + (\gamma_1-\gamma_2)(1,0).$$

Proof. Take second-order derivatives with respect to m_S, we have

$$\frac{\partial^2 C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right)}{\partial\left(\frac{\gamma_2}{m_S}\right)^2}\frac{\gamma_2^2}{m_S^3} + \frac{\partial^2 C^{\#}\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right)}{\partial\left(\frac{\gamma_1-\gamma_2}{1-m_S}\right)^2}\frac{(\gamma_1-\gamma_2)^2}{(1-m_S)^3} < 0$$

because $C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right)$ is strictly concave and $C^{\#}\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right)$ is weakly concave. Thus m_S^* is either corner with $m_S^* \in \{2\gamma_2, 1-\gamma_1+\gamma_2\}$, or interior with m_S^* uniquely implicitly determined by the first-order condition. We will prove four cases separately.

Case (1): There are two steps. In Step 1, we will show that $m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$. For $m_S^* \leq \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$, because $\frac{\gamma_1}{1-\gamma_2} < l_1^C$, $\frac{\gamma_1-\gamma_2}{1-m_S} < l_1^C$ and therefore the objective function of Problem (OPT-D) is

$$m_S C\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1-m_S) C\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right). \quad (28)$$

Then the first-order derivative is

$$-\frac{2\gamma_2^2}{(m_S-2\gamma_2)^2}(\pi^*)^2\int_{\frac{\gamma_2}{m_S-\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi + \frac{2(m_S-\gamma_2)}{m_S-2\gamma_2}v\left(\frac{\gamma_2}{m_S-\gamma_2}\pi^*\right)$$
$$+\frac{(\gamma_1-\gamma_2)^2}{(1-m_S)^2}v(\pi^*) + \frac{2(\gamma_1-\gamma_2)^2}{(1-m_S)^2}(\pi^*)^2\int_{\frac{\gamma_1-\gamma_2}{1-m_S}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi - 2v\left(\frac{\gamma_1-\gamma_2}{1-m_S}\pi^*\right).$$

We will show that the first-order derivative is negative at $m_S^* = \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$. Substitute

$m_S = \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$ into the first-order derivative, we have

$$\begin{aligned}
& -2\frac{\gamma_1^2}{(\gamma_1+\gamma_2-1)^2}(\pi^*)^2\int_{\frac{\gamma_1}{1-\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi + \frac{2-2\gamma_2}{1-\gamma_1-\gamma_2}v\left(\frac{\gamma_1}{1-\gamma_2}\pi^*\right) \\
& +\frac{\gamma_1^2}{(\gamma_2-1)^2}v(\pi^*) + \frac{2\gamma_1^2}{(\gamma_2-1)^2}(\pi^*)^2\int_{\frac{\gamma_1}{1-\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi - 2v\left(\frac{\gamma_1}{1-\gamma_2}\pi^*\right) \\
= & \frac{\gamma_1^2}{(\gamma_2-1)^2}v(\pi^*) - \frac{2\gamma_1^3(2-\gamma_1-2\gamma_2)}{(\gamma_2-1)^2(\gamma_1+\gamma_2-1)^2}(\pi^*)^2\int_{\frac{\gamma_1}{1-\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi + \frac{2\gamma_1 v\left(\frac{\gamma_1}{1-\gamma_2}\pi^*\right)}{1-\gamma_1-\gamma_2} \\
= & \frac{\gamma_1^2}{(\gamma_1+\gamma_2-1)^2}\Phi\left(\frac{\gamma_1}{1-\gamma_2}\right) < 0
\end{aligned}$$

where the last equality follows from the definition of l_1^C and that $\frac{\gamma_1}{1-\gamma_2} < l_1^C$. Thus, $m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$ and $\frac{\gamma_1-\gamma_2}{1-m_S} < l_1^C$.

In Step 2, we will show that $m_S^* > 2\gamma_2$ by showing the first-order derivative is strictly positive at $m_S = 2\gamma_2$. Because $\frac{\gamma_1-\gamma_2}{1-m_S} < l_1^C$, the objective function of Problem (OPT-D) is (28). Then at $m_S = 2\gamma_2$, the first-order derivative is

$$\begin{aligned}
& \lim_{m_S\to 2\gamma_2}\left(-\frac{2\gamma_2^2(\pi^*)^2}{(m_S-2\gamma_2)^2}\int_{\frac{\gamma_2}{m_S-\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi + \frac{2m_S-2\gamma_2}{m_S-2\gamma_2}v\left(\frac{\gamma_2}{m_S-\gamma_2}\pi^*\right)\right) \\
& +\frac{(\gamma_1-\gamma_2)^2}{(1-m_S)^2}v(\pi^*) + \frac{2(\gamma_1-\gamma_2)^2}{(1-m_S)^2}(\pi^*)^2\int_{\frac{\gamma_1-\gamma_2}{1-m_S}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi - 2v\left(\frac{\gamma_1-\gamma_2}{1-m_S}\pi^*\right).
\end{aligned}$$

First, by L'Hospital's rule,

$$\begin{aligned}
& \lim_{m_S\to 2\gamma_2}\left(-\frac{2\gamma_2^2(\pi^*)^2}{(m_S-2\gamma_2)^2}\int_{\frac{\gamma_2}{m_S-\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi + \frac{2m_S-2\gamma_2}{m_S-2\gamma_2}v\left(\frac{\gamma_2}{m_S-\gamma_2}\pi^*\right)\right) \\
= & \lim_{m_S\to 2\gamma_2}\left(\frac{(2m_S-2\gamma_2)(m_S-2\gamma_2)v\left(\frac{\gamma_2}{m_S-\gamma_2}\pi^*\right) - 2\gamma_2^2(\pi^*)^2\int_{\frac{\gamma_2}{m_S-\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi}{(m_S-2\gamma_2)^2}\right) \\
= & \lim_{m_S\to 2\gamma_2}\left(\frac{2(m_S-2\gamma_2)v\left(\frac{\gamma_2}{m_S-\gamma_2}\pi^*\right) - \frac{2\gamma_2(m_S-2\gamma_2)}{m_S-\gamma_2}v'\left(\frac{\gamma_2}{m_S-\gamma_2}\pi^*\right)\pi^*}{2(m_S-2\gamma_2)}\right) \\
= & v(\pi^*) - v'(\pi^*)\pi^*. \qquad (29)
\end{aligned}$$

Second,

$$\begin{aligned}
&\frac{(\gamma_1-\gamma_2)^2}{(1-2\gamma_2)^2}v(\pi^*)+\frac{2(\gamma_1-\gamma_2)^2}{(1-2\gamma_2)^2}(\pi^*)^2\int_{\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*}^{\pi^*}\frac{v(\pi)}{\pi^3}d\pi-2v\left(\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*\right)\\
=\;&\frac{(\gamma_1-\gamma_2)^2}{(1-2\gamma_2)^2}(\pi^*)^2\int_{\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*}^{\pi^*}\frac{v'(\pi)}{\pi^2}d\pi-v\left(\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*\right). \qquad (30)
\end{aligned}$$

Combining (29) and (30), the first-order derivative at $m_S=2\gamma_2$ equals

$$\begin{aligned}
&\frac{(\gamma_1-\gamma_2)^2}{(1-2\gamma_2)^2}(\pi^*)^2\int_{\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*}^{\pi^*}\frac{v'(\pi)}{\pi^2}d\pi-v\left(\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*\right)+v(\pi^*)-v'(\pi^*)\pi^*\\
=\;&\frac{(\gamma_1-\gamma_2)^2}{(1-2\gamma_2)^2}(\pi^*)^2\int_{\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*}^{\pi^*}\frac{v'(\pi)}{\pi^2}d\pi+\int_{\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*}^{\pi^*}v'(\pi)\,d\pi-v'(\pi^*)\pi^*\\
\geq\;&\frac{(\gamma_1-\gamma_2)^2}{(1-2\gamma_2)^2}(\pi^*)^2\int_{\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*}^{\pi^*}\frac{v'(\pi^*)}{\pi^2}d\pi+\int_{\frac{\gamma_1-\gamma_2}{1-2\gamma_2}\pi^*}^{\pi^*}v'(\pi^*)\,d\pi-v'(\pi^*)\pi^*\\
=\;&-\frac{(\gamma_1-\gamma_2)^2}{(2\gamma_2-1)^2}v'(\pi^*)\pi^*>0
\end{aligned}$$

where the first inequality follows because $v(\pi)$ is strictly concave. Because $l_1^C<1$, $\frac{\gamma_1-\gamma_2}{1-m_S}<l_1^C$ implies that $m_S^*<1-\gamma_1+\gamma_2$. Therefore, we must have $m_S^*\in(2\gamma_2,1-\gamma_1+\gamma_2)$ and it is implicitly determined by the first-order condition.

Case (2): Similarly to Case (1), because $\frac{\gamma_1}{1-\gamma_2}=l_1^C$, the first-order derivative at $m_S=\frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$ equals

$$\frac{\gamma_1^2}{(\gamma_1+\gamma_2-1)^2}\Phi\left(\frac{\gamma_1}{1-\gamma_2}\right)=\frac{\gamma_1^2}{(\gamma_1+\gamma_2-1)^2}\Phi\left(l_1^C\right)=0.$$

Thus, the optimal solution $m_S^*=\frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$ and an optimal segmentation is

$$\frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}\left(\frac{l_1^C}{1+l_1^C},\frac{l_1^C}{1+l_1^C}\right)+\frac{(\gamma_1-\gamma_2)(1-\gamma_2)}{\gamma_1}\left(l_1^C,0\right).$$

By Lemma 2, this segmentation generates the same consumer surplus as the prior market and this completes the proof.

Case (3): We will first show that the optimal solution to Problem (OPT-D) is $m_S^*=\frac{\gamma_2\left(1+l_1^C\right)}{l_1^C}$. At $m_S=\frac{\gamma_2\left(1+l_1^C\right)}{l_1^C}$, the objective function is

$$m_S C\left(\frac{\gamma_2}{m_S},\frac{\gamma_2}{m_S}\right)+(1-m_S)\left(\frac{1-\frac{\gamma_1-\gamma_2}{1-m_S}}{1-l_1^C}C\left(l_1^C,0\right)+\frac{\frac{\gamma_1-\gamma_2}{1-m_S}-l_1^C}{1-l_1^C}C(1,0)\right).$$

Substituting $m_S = \frac{\gamma_2\left(1+l_1^C\right)}{l_1^C}$ into the first-order derivative, we have

$$
\begin{aligned}
& -\frac{2\gamma_2^2}{\left(m_S-2\gamma_2\right)^2}\left(\pi^*\right)^2\int_{\frac{\gamma_2}{m_S-\gamma_2}\pi^*}^{\pi^*}\frac{v\left(\pi\right)}{\pi^3}d\pi+\frac{2\left(m_S-\gamma_2\right)}{m_S-2\gamma_2}v\left(\frac{\gamma_2}{m_S-\gamma_2}\pi^*\right) \\
& +\frac{l_1^C C\left(1,0\right)-C\left(l_1^C,0\right)}{1-l_1^C} \\
= & -2\frac{(l_1^C)^2}{\left(l_1^C-1\right)^2}\left(\pi^*\right)^2\int_{l_1^C\pi^*}^{\pi^*}\frac{v\left(\pi\right)}{\pi^3}d\pi+\frac{2}{1-l_1^C}v\left(l_1^C\pi^*\right)+\frac{l_1^C C\left(1,0\right)-C\left(l_1^C,0\right)}{1-l_1^C} \\
= & -2\frac{(l_1^C)^2}{\left(l_1^C-1\right)^2}\left(\pi^*\right)^2\int_{l_1^C\pi^*}^{\pi^*}\frac{v\left(\pi\right)}{\pi^3}d\pi+\frac{2}{1-l_1^C}v\left(l_1^C\pi^*\right) \\
& +(l_1^C)^2 v\left(\pi^*\right)+2(l_1^C)^2\left(\pi^*\right)^2\int_{l_1^C\pi^*}^{\pi^*}\frac{v\left(\pi\right)}{\pi^3}d\pi-2v\left(l_1^C\pi^*\right) \\
= & (l_1^C)^2 v\left(\pi^*\right)-2\frac{(l_1^C)^3\left(2-l_1^C\right)}{\left(l_1^C-1\right)^2}\left(\pi^*\right)^2\int_{l_1^C\pi^*}^{\pi^*}\frac{v\left(\pi\right)}{\pi^3}d\pi+\frac{2l_1^C}{1-l_1^C}v\left(l_1^C\pi^*\right) \\
= & \frac{(l_1^C)^2}{\left(1-l_1^C\right)^2}\Phi\left(l_1^C\right)=0.
\end{aligned}
$$

Because $\frac{\gamma_2(1+l_1^C)}{l_1^C} < 1-\gamma_1+\gamma_2$, the optimal solution is $m_S^* = \frac{\gamma_2(1+l_1^C)}{l_1^C}$ and an optimal segmentation is

$$
\begin{aligned}
& \frac{\gamma_2(1+l_1^C)}{l_1^C}\left(\frac{l_1^C}{1+l_1^C},\frac{l_1^C}{1+l_1^C}\right)+\frac{l_1^C-\gamma_2(1+l_1^C)-l_1^C\left(\gamma_1-\gamma_2\right)}{l_1^C\left(1-l_1^C\right)}\left(l_1^C,0\right) \\
& +\frac{\gamma_1-\gamma_2-l_1^C+\gamma_2(1+l_1^C)}{1-l_1^C}\left(1,0\right).
\end{aligned}
$$

By Lemma 2, we can replace the first two components in this segmentation with $\frac{1-\gamma_1-\gamma_2 l_1^C}{1-l_1^C}\left(\frac{l_1^C(1-\gamma_1-\gamma_2)}{1-\gamma_1-\gamma_2 l_1^C},\frac{\gamma_2(1-l_1^C)}{1-\gamma_1-\gamma_2 l_1^C}\right)$. This completes the proof.

Case (4): Similarly to case (3), the first-order derivative equals 0 at $m_S = \frac{\gamma_2\left(1+l_1^C\right)}{l_1^C}$. Because $\frac{\gamma_2(1+l_1^C)}{l_1^C} \geq 1-\gamma_1+\gamma_2$, for $m_S \in \left[2\gamma_2, 1-\gamma_1+\gamma_2\right)$, the first-order derivative is strictly positive. Thus, the optimal solution to Problem (OPT-D) is $m_S^* = 1-\gamma_1+\gamma_2$ and an optimal segmentation is as specified in the proposition. ■

We have explicit solution except for Case (1). In Case (1), because $\frac{\gamma_1}{1-\gamma_2} < l_1^C$ and $m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$, the symmetric segment in the optimal segmentation $(\frac{\gamma_2}{m_S^*},\frac{\gamma_2}{m_S^*})$ has a higher captive-to-reach ratio than the prior segment (which equals $\frac{\gamma_1}{1-\gamma_2}$) and the nested segment has a lower captive-to-reach ratio than the prior segment. Similarly, because $m_S^* < 1-\gamma_1+\gamma_2$, compared with the field-leveling segmentation, in the optimal segmentation, the symmetric segment has a higher captive-to-reach ratio and the nested segment has a lower captive-to-reach ratio.

We do not have uniqueness for Case (2) and (3) because we can replace any segment whose captive-to-reach ratio equals l_1^C with infinite combinations of segments that have the same captive-to-reach ratio. By Lemma 2, all these combinations yield the same producer surplus. Indeed, all consumer-optimal segmentations generate the same producer and consumer surplus. Thus, we can compare the producer surplus between an optimal segmentation and uniform pricing.

Proposition A2 *Suppose* $v(\pi^*) - v(0) - 2\pi^* v'(0) < 0$. *A consumer-optimal segmentation yields strictly higher producer surplus than uniform pricing if and only if* $\frac{\gamma_1}{1-\gamma_2} < l_1^C$.

Proof. By Lemma 2, we can replace a segment with a symmetric segment and a nested segment with the same captive-to-reach ratio without changing the consumer surplus. Thus, we have the following corollary of Proposition A1.

Corollary A1 *If* $v(\pi^*) - v(0) - 2\pi^* v'(0) < 0$, *the following segmentation maximizes cosumer surplus among all possible market segmentations:*

(1) If $\frac{\gamma_1}{1-\gamma_2} < l_1^C$,

$$m_S^* \left(\frac{\gamma_2}{m_S^*}, \frac{\gamma_2}{m_S^*} \right) + (1 - m_S^*) \left(\frac{\gamma_1 - \gamma_2}{1 - m_S^*}, 0 \right)$$

where m_S^* *is implicitly determined by the first-order condition of Problem (OPT-D) such that* $2\gamma_2 < m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$;

(2) If $\frac{\gamma_1}{1-\gamma_2} = l_1^C$,

$$\frac{\gamma_2 (1 + \gamma_1 - \gamma_2)}{\gamma_1} \left(\frac{l_1^C}{1 + l_1^C}, \frac{l_1^C}{1 + l_1^C} \right) + \frac{(\gamma_1 - \gamma_2)(1 - \gamma_2)}{\gamma_1} \left(l_1^C, 0 \right);$$

(3) If $\frac{\gamma_1}{1-\gamma_2} > l_1^C$, and $\frac{\gamma_2(1+l_1^C)}{l_1^C} < 1 - \gamma_1 + \gamma_2$,

$$\frac{\gamma_2(1 + l_1^C)}{l_1^C} \left(\frac{l_1^C}{1 + l_1^C}, \frac{l_1^C}{1 + l_1^C} \right) + \frac{l_1^C - \gamma_2(1 + l_1^C) - l_1^C (\gamma_1 - \gamma_2)}{l_1^C (1 - l_1^C)} \left(l_1^C, 0 \right)$$
$$+ \frac{\gamma_1 - \gamma_2 - l_1^C + \gamma_2(1 + l_1^C)}{1 - l_1^C} (1, 0);$$

(4) If $\frac{\gamma_1}{1-\gamma_2} > l_1^C$, and $\frac{\gamma_2(1+l_1^C)}{l_1^C} \geq 1 - \gamma_1 + \gamma_2$,

$$(1 - \gamma_1 + \gamma_2) \left(\frac{\gamma_2}{1 - \gamma_1 + \gamma_2}, \frac{\gamma_2}{1 - \gamma_1 + \gamma_2} \right) + (\gamma_1 - \gamma_2)(1, 0).$$

As is shown in the proof for Proposition A1, any other segmentations that consist of symmetric and L-nested segments than the one specified in Corollary A1 are suboptimal. Then as in Lemma 2, consumer-optimal segmentations specified in Corollary A1 yield strictly higher consumer surplus than uniform pricing if $\frac{\gamma_1}{1-\gamma_2} \neq l_1^C$ and otherwise the same consumer surplus. This also implies if a consumer-optimal segmentation contains an interior segment (q_1, q_2) with $0 < q_2 < \min\{1-q_1, q_1\}$ or $0 < q_1 < \min\{1-q_2, q_2\}$, $\frac{q_1}{1-q_2} = l_1^C$. Then again as in Lemma 2, we can replace it with $\frac{q_2(1+q_1-q_2)}{q_1}\left(\frac{l_1^C}{1+l_1^C}, \frac{l_1^C}{1+l_1^C}\right) + \frac{(q_1-q_2)(1-q_2)}{q_1}\left(l_1^C, 0\right)$ which yields the same producer and consumer surplus. Similarly, for captive segments (q_1, q_2), we can also replace it with $q_1(1,0) + q_2(0,1)$ which yields the same producer and consumer surplus. Thus it is sufficient to show that the consumer-optimal segmentation which consists of only symmetric and L-nested segments yields strictly higher producer surplus than uniform pricing if and only if $\frac{\gamma_1}{1-\gamma_2} < l_1^C$, which is equivalent to show that the consumer-optimal segmentation characterized in Corollary A1 yields strictly higher producer surplus than uniform pricing if and only if $\frac{\gamma_1}{1-\gamma_2} < l_1^C$.

If $\frac{\gamma_1}{1-\gamma_2} = l_1^C$, as in Lemma 2, the consumer-optimal segmentation in Corollary A1 yields the same producer surplus as uniform pricing. Then we consider the case of $\frac{\gamma_1}{1-\gamma_2} < l_1^C$ and $\frac{\gamma_1}{1-\gamma_2} > l_1^C$ separately. Note that in both cases, as in Lemma 2, the segmentation

$$\frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}\left(\frac{\gamma_1}{1+\gamma_1-\gamma_2}, \frac{\gamma_1}{1+\gamma_1-\gamma_2}\right) + \frac{(\gamma_1-\gamma_2)(1-\gamma_2)}{\gamma_1}\left(\frac{\gamma_1}{1-\gamma_2}, 0\right) \tag{31}$$

yields the same producer surplus as uniform pricing. It is thus sufficient to compare this segmentation with the segmentation characterized in Corollary A1.

In case 1, $\frac{\gamma_1}{1-\gamma_2} < l_1^C$, which corresponds to Case (1) in Corollary A1. Since $m_S^* < \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}$, it is sufficient to show that the following problem is strictly decreasing in m_S:

$$\max_{m_S \in \left[2\gamma_2, \frac{\gamma_2(1+\gamma_1-\gamma_2)}{\gamma_1}\right]} m_S P\left(\frac{\gamma_2}{m_S}, \frac{\gamma_2}{m_S}\right) + (1-m_S) P\left(\frac{\gamma_1-\gamma_2}{1-m_S}, 0\right).$$

Because $P(\gamma_1, \gamma_2) = \frac{(2-\gamma_1-\gamma_2)\gamma_1}{1-\gamma_2}\pi^*$, the objective function is

$$2\gamma_2\pi^* + \left(2 - \frac{\gamma_1-\gamma_2}{1-m_S}\right)(\gamma_1-\gamma_2)\pi^*$$

which is clearly strictly decreasing in m_S. This completes the proof.

In case 2, $\frac{\gamma_1}{1-\gamma_2} > l_1^C$. First, if $\frac{\gamma_2(1+l_1^C)}{l_1^C} < 1-\gamma_1+\gamma_2$, because $P(q,q)$ is linear in q, the consumer-optimal segmentation in Corollary A1 yields the same producer surplus as the segmentation

$$
\begin{aligned}
&\frac{\gamma_2\left(1+\gamma_1-\gamma_2\right)}{\gamma_1}\left(\frac{\gamma_1}{1+\gamma_1-\gamma_2},\frac{\gamma_1}{1+\gamma_1-\gamma_2}\right)\\
&+\left(\frac{\gamma_2(1+l_1^C)}{l_1^C}-\frac{\gamma_2\left(1+\gamma_1-\gamma_2\right)}{\gamma_1}\right)(0,0)\\
&+\frac{l_1^C-\gamma_2(1+l_1^C)-l_1^C\left(\gamma_1-\gamma_2\right)}{l_1^C\left(1-l_1^C\right)}\left(l_1^C,0\right)+\frac{\gamma_1-\gamma_2-l_1^C+\gamma_2(1+l_1^C)}{1-l_1^C}\left(1,0\right).
\end{aligned}
$$

Because $P\left(q_1,0\right)$ is strictly concave in q_1, we can increase the producer surplus by replacing the last three components of this segmentation with $\frac{(\gamma_1-\gamma_2)(1-\gamma_2)}{\gamma_1}\left(\frac{\gamma_1}{1-\gamma_2},0\right)$, which leads to the segmentation in (31). This completes the proof.

Second, similarly, if $\frac{\gamma_2(1+l_1^C)}{l_1^C}\geq 1-\gamma_1+\gamma_2$, the optimal segmentation in Case (4) of Corollary A1 yields a strictly lower producer surplus than the segmentation in (31). ■