Contests for Status

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Abstract

We study the optimal design of organizations under the assumption that agents in a contest care about their relative position. A judicious definition of status categories can be used by a principal in order to influence the agents' performance. We first consider a pure status case where there are no tangible prizes. Our main results connect the optimal partition in status categories to properties of the distribution of ability among contestants. The top status category always contains a unique element. For distributions of abilities that have an increasing failure rate (IFR), a proliferation of status classes is optimal, while the optimal partition involves only two categories if the distribution of abilities is sufficiently concave. Moreover, for IFR distributions, a coarse partition with only two status categories achieves at least half of the output obtained in the optimal partition with a proliferation of classes. Finally, we modify the model to allow for status categories that are endogenously determined by monetary prizes of different sizes. If status is solely derived from monetary rewards, we show that the optimal partition in status classes contains only two categories.

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1 Introduction

One of the earliest designed society structures was that of Solon's (ca. 638 BC - 558 BC) *timokratia,* an oligarchy with a sliding scale of status determined by precisely defined ranges of measured output (fruit, grain, oil, etc.). Solon divided the entire population of Attica into four status classes,[1] and attached various, more or less tangible rights, to each class. Higher classes had more rights but were also expected to contribute more to the state.

The kings and queens of feudal states awarded titles of nobility such as *duke (or duchess), marquis, earl, count, viscount, baron, baronet* in return for special services to the crown. Initially there was a strong link between such titles and tangible assets, such as land and serfs, but this link weakened over time.[2]

Today's large corporations (such as large banks) have, besides a single *president*, several *executive vice presidents,* tens of *senior vice-presidents,* and several hundred "mere" *vice-presidents.* The New York Metropolitan Museum of Art offers eight different donor categories[3] for corporate members (such as "*Chairman's Circle*" for donations above \$100,000, "*Director's Circle*" for donations between \$60,000 and \$100,000, and so on) and 10 similar categories for private members.

The common denominator to the above examples is that agents care about social status, and that a self-interested principal is usually able to divert (or "manipulate") this concern to an avenue that is beneficial to himself/herself. The general importance of status concerns for explaining behavior has been long recognized by sociologists and economists.[4] Recent happiness research shows how wage rank affects workers' well-being,[5] and experimental studies pointed out that social status may play a role also in market exchanges.[6] Nevertheless, the literature focusing on the direct implications of status concerns for the design of societies and organizations is relatively thin.

[1]These were the Pentakosiomedimnoi, the Hippeis, the Zeugitai and the Thetes.

[2]Even today's citizens of the United Kingdom are eligible for more than 50 orders and decorations, awarded for special services to the "queen". These are structured in a strict precedence system, and play an important role in public life. The police currently investigates allegations that close associates of prime minister Blair facilitated the award of honors in exchange for large monetary contributions to the Labor party.

[3]See Amihai Glazer and Kai A. Konrad (1996) for some empirical evidence and a theoretical model that focuses on conspicuous giving.

[4]See Max Weber (1978), James S. Coleman (1990), Thorstein Veblen (1934), James S. Duesenberry (1949), Milton Friedman and Leonard J. Savage (1948), and Milton Friedman (1953) for some early contributions. Robert H. Frank (1985) offers an entertaining account of some of the issues.

[5]See Gorton Brown, Janathan Gardner, Andrew Oswald and Jian Qian (2004).

[6]See Sheryl Ball, Catherine Eckel, Philip J. Grossman and William Zame (2001).

William Goode (1979), a leading sociologist, offers a broad study of "prestige" as an instrument of social control. He notes that "individuals and groups give and withhold prestige and approval as a way of rewarding or punishing others."

In this paper we closely follow Goode's perspective, and we study the optimal design of organizations under the assumption that agents care about their relative position. We show how a judicious definition of the number and size of status classes based on performance rank can be used by a principal in order to maximize the agents' output in a contest situation. Our results offer both explanations for commonly observed phenomena (such as having a unique individual at the top) and suggestions for the design of the level structure in a hierarchy. As it will become clear below, major factors affecting the structure of the optimal partition in status categories are: 1) the distribution of abilities in the population, and 2) the relative weight of the monetary component in the determination of status. If outstanding talent is relatively rare or if differences in wealth are crucial for status perceptions, we find an optimal structure that distinguishes the top performer while lumping together everyone else, irrespective of their performance. This insight yields a novel potential explanation for the well-documented recent increase in the gap between CEO compensation and the compensation of other workers (or even other executives) within the firm. In contrast, if talent is relatively abundant and if status is not too tightly linked to wealth, we find an optimal structure where status categories proliferate and where relatively small differences in performance are rewarded with different status prizes. In those cases, status can serve as a potent substitute for money in order to drive performance.[7]

The tournament literature has shown how prizes based on rank-orders of performance can be effectively used to provide incentives (see Edward Lazear and Sherwin Rosen, 1981, Jerry Green and Nancy Stokey, 1983, and Barry Nalebuff and Joseph Stiglitz, 1983). Charles O'Reilly, Brian Main and Graef Crystal (1988) have emphasized the important role of status in executive compensation, and Donald Hambrick and Albert Cannela (1993) use relative standing as the main factor for explaining departures rates of executives of acquired firms. Michael Bognanno (2001) studies the empirical relation between the number of executive board members and the CEO's compensation in "corporate tournaments".

Benny Moldovanu and Aner Sela (2001, 2006) developed a convenient contest model that can easily accommodate several prizes of different size. Using their methodology, it is a natural step to

[7]For example, this seems to be the case in institutions devoted to scientific research and in many other not-for-profit organizations.

analyze the incentive effect of "status prizes," and the interplay between such prizes and tangible ones.

In our present model, several agents who are privately informed about their abilities engage in a contest, and are then partitioned into status categories (or classes) according to their performance. A status category consists of all contestants who have performances in a specified quantile, e.g., the top status class may consist of the individual with the highest output, the second class of individuals with the next three highest outputs, and so on... Each individual cares about the number of contestants in classes above and below him. We choose a convenient functional formulation that captures well the "zero-sum game" nature of concerns for relative position: if an individual gets higher (lower) status, one or more individuals must get lower (higher) status.

A designer (or principal) determines the number of status classes and their size in order to maximize total output. Since the contest equilibrium only depends on the structure of status classes, and not directly on the designer's goal, our type of analysis can, in principle, be performed for a variety of other goals.

We first analyze the "pure status" case where there are no other tangible prizes to motivate the contestants. We then extend our model to investigate a setting where the designer awards monetary prizes, and where status is purely derived from the differences in monetary compensation, i.e., having a higher monetary prize *per se* implies higher status.[8] These two models represent opposite extremes, and reality is often somewhere in the middle. In most cases, we think that individuals in organizations are, at least partly, motivated by status concerns, but that status is not solely derived from the monetary payoffs attached to various activities.[9]

Since status is a "zero-sum game", it seems, at first glance, that shifts in the allocation of status among agents should not affect total output. The missing factor in this argument is the heterogeneity in abilities. Since higher ability will be, in equilibrium, associated with higher performance, modifications of classes at different levels in the hierarchy may have quite different effects. In particular, because the expected benefit associated with a move upwards in the ranks (which is given by the expected increase in status minus the expected cost of producing an output that is sufficient for the upward move) depends on the bounds of the quantile defining the status class, a manipulation of these bounds affects behavior, and hence total output.

[8] See Arthur J. Robson (1992) for another model where status is defined by wealth.

[9] For example, Chaim Fershtmann and Yoram Weiss (1993) relate status to the length of the education necessary for a specific occupation (their motto is Adam Smith's nicely circular: "Honour makes a great part of the reward of all honourable professions").

Our results relate the structural features of the optimal partition in status categories to properties of the distribution of abilities in the society:

1) We show that, for any distribution of abilities, the top category in any optimal partition must contain a single agent.[10] This agrees well with the ubiquitous structure of many human (or animal) organizations and social structures, and brings to mind familiar roles such as "queen", "alpha-male", "CEO", etc....

2) Given a partition in status classes, adding a new element to an arbitrary class may, in fact, reduce output. But, we show that the adoption of a policy that resembles "hiring at the lowest level" (see George Baker, Michael Gibbs, and Bengt Holmstrom, 1994) always makes an increase in the number of (ex-ante symmetric) contestants beneficial to the principal.

3) We then identify the main factors leading either to a proliferation of status classes (where each individual is "in a class of his/her own") or to coarse partitions where it is optimal to have a wider range of performances bunched together in the same category. A proliferation of status classes is optimal if the distribution of abilities has an increasing failure (or hazard) rate. This finding points in the same direction as the well known empirical fact that job titles do proliferate, but only in organizations with a relatively professional work-force (see James N. Baron and William T. Bielby, 1986). In contrast, a coarse partition with only two status classes (where all individuals except one belong to the lower class) is optimal if the distribution of abilities is sufficiently concave.

4) If the distribution of abilities has an increasing failure rate, we show that the optimal partition in the class of partitions with only two status categories achieves at least half the performance of the overall optimal partition. Thus, whenever there are transaction costs attached to finer partitions, the coarsest possible non-trivial partition may be ultimately optimal.[11] This is related to an argument made by Preston McAfee (2002) in the context of "coarse matching" of two populations.

5) Finally, we introduce monetary prizes and consider status purely induced by these prizes. In order to add realism, we assume that the designer is budget constrained, and that agents can choose not to compete if the monetary prize is not enough to compensate them for a potential low status. In this framework, we show that the optimal structure is to have exactly two status classes: the top class consisting of the single most productive agent, while the lower class containing all other agents that get paid just enough to keep them in the contest. Since, as illustrated above,

[10] This is of course reminiscent of the optimal taxation literature, pioneered by Mirrlees (1971), which has a unique tax rate for the wealthiest individual.

[11] Think about the Econometric society, say, which has two status classes: members and fellows.

there are many real-life examples where status classes proliferate, our results suggest that in those situations status cannot be solely and entirely induced by monetary wealth.[12]. In contrast, the growing gap between CEO compensation and the compensation of other agents within the firm can be explained by an increase in the status value conferred by the monetary component.

Technically, our results are obtained by combining insights derived from the general analysis of contests with multiple prizes developed by Moldovanu and Sela (2001, 2006) with a novel application of statistical results about stochastic monotonicity properties of *normalized spacings* (i.e., differences) of order statistics (Richard Barlow and Frank Proschan, 1966). For large and interesting classes of distribution functions it is possible to say, for example, whether normalized spacings become stochastically more (less) compressed when we climb higher in the ability range, and we show that such features determine the structure of the optimal partition in status classes.

While many authors put "status" directly into the utility function,[13] the paper most closely related to ours is Pradeep Dubey and John Geanakoplos (2005). These authors study optimal grading of exams in situations where students care about relative ranks. We have borrowed from that paper the present specification of utility functions. Our determination of status categories based on relative effort rank corresponds to what Dubey and Geanakoplos call in their respective context "grading on a curve". There are many substantial differences between their model, technique and results and ours. For their main results, Dubey and Geanakoplos focus on absolute grading, assuming that there is complete information, that students are either homogenous or have discrete types, that effort choice is binary, and that the relation between effort and output is stochastic. Moreover, the designer's goal is to have all students choose the higher effort level out of the two possible ones. Their main finding is that status-conscious students may be better motivated to work hard by a professor who uses coarse grading (e.g., A,B,C,D rather than 100, 99,...). This should be contrasted with our main result about the optimality of the finest partition for a very large and ubiquitous class of distributions.

[12]On this topic see also Robert H. Frank (1999).

[13]Fershtman and Weiss (1993) construct a general equilibrium model where both status and wealth are determined endogenously. In Gary S. Becker, Kevin M. Murphy and Ivan Werning's (2005) model, status is bought in a market. They assume that there are at least as many status classes as individuals and that status is a complement to other consumption goods. Ed Hopkins and Tatiana Kornienko (2004) study the effect of an exogenous change of income distribution in a model where agents care about their rank in the distribution of consumption. Rick Harbaugh and Tatiana Kornienko (2001) draw a parallel between a decision model that assumes a concern for local status and prospect theory.

Another related paper is Rayo (2003). He analyzes the monopolistic design and pricing of positional goods that consumers use to signal their types. A main result is that a monopolist will restrict the variety of positional goods in order to extract surplus from consumers. In his model, a consumer's utility depends on the average type of consumers paying the same price. Thus (as in our model), utility from being in a certain class is manipulable by the designer. But, in Rayo's model, utility depends on the characteristics of consumers in the same class, whereas in our model utility depends on the number of agents in superior and inferior classes. Moreover, in Rayo's model there is a continuum of consumers who interact only indirectly (through the influence of perceptions on utility) - it is this feature which allows the usage of tools from the literature on monopolistic non-linear pricing. In contrast, we have here a finite number of agents who directly and strategically compete for a scarce resource (i.e., places in superior status classes) and therefore we need to use tools from the literature on contest design/statistics. In spite of these differences, several of Rayo's results resemble ours: the highest possible type should never be pooled with others; if a "virtual valuation function" is monotonic, full separation is optimal, whereas some pooling (which corresponds to coarseness in our model) is optimal if this condition is not satisfied.

Postlewaite (1998) presents an excellent discussion on the advantages/disadvantages of the "direct" modeling approach versus the one where a concern for relative ranking is only implicit, or "instrumental" for other goals that are made explicit (see also Cole et al., 1992). In a nutshell, Postelwaite's argument against a direct approach is that, by adjusting utility functions at will, one can explain every phenomenon. For our purposes, the debate about the right way to model status concerns is only of secondary importance. Our main focus is on the optimal design of status classes (from an incentive point of view) given that agents care, for some direct or instrumental reason, about relative position. We view the assumed utility function as a simplification, and we ask the reader to judge the outcome by Hardy's dictum whereby good science must, at least, provide some "decent" distance between assumptions and results.

The rest of the paper is organized as follows: Section 2 presents the contest model with status concerns, and some useful facts about order statistics. In Section 3 we derive results that connect the form of the optimal partition in status categories to various properties of the distribution of ability in the population. We first show that, by always adding new entrants to the lowest status category, the designer can ensure that his payoff is monotonically increasing in the number of contestants. Thus, potential contestants need not be excluded from competing. We next show that the top status category in any optimal partition must contain a unique element. For distribution of abilities that have an increasing hazard rate, each status category in an optimal partition will contain a unique element — thus, in this case a proliferation of status classes is optimal. We also show that the optimal partition involves only two categories if the distribution of abilities is sufficiently concave. Finally, we study the properties of optimal partitions with only two status categories. In Section 4 we modify the model to allow for status categories that are endogenously determined by monetary prizes of different sizes. If status is solely derived from monetary rewards, we show that the optimal partition contains only two categories, with the top category being a singleton. Section 5 concludes. Several proofs and examples are relegated to an Appendix.

2 The Model

We consider a contest with n players where each player j makes an effort e_j. For simplicity, we postulate a deterministic relation between effort and output, and assume these to be equal. Efforts are submitted simultaneously. An effort e_j causes a cost denoted by e_j/a_j, where $a_j > 0$ is an ability parameter.

The ability (or *type*) of contestant j is private information to j. Abilities are drawn independently of each other from the interval $[0, 1]$ according to a distribution function F that is common knowledge. We assume that F has a continuous density $f = dF > 0$.

Contestants are ranked according to efforts. Let $\{(0, r_1], (r_1, r_2], ...(r_{i-1}, r_i], ..., (r_{k-1}, n]\ \}$ be a partition of the integers in the interval $(0, n]$ in $k \geq 1$ *status categories*, where $r_{i-1} < r_i$. Define also for convenience: $r_0 \equiv 0$ and $r_k \equiv n$. Given such a partition and the ordered list of efforts, contestants are divided into the k categories: a player is included in category i, if his effort is between the r_{i-1}-th and r_i-th highest ones.

Each player cares about the number of players in categories both below and above him, and we assume that the "pure status" prize of being in status category i is given by

$$v_i = r_{i-1} - (n - r_i).$$

Thus, a contestant is happier when he has more [less] people below [above] him. Note this formulation well captures the *zero-sum* nature of status: for any partition in status categories, the total value derived from status is given by :

$$\sum_{i=1}^{k}(r_i - r_{i-1})v_i = \sum_{i=1}^{k}(r_i - r_{i-1})(r_i + r_{i-1} - n) = 0$$

To summarize, the timing of the game is as follows: The designer chooses a partition $\{r_i\}_{i=0}^k$ and commits to it. Each contestant then gets privately informed about his/her ability. The contestants simultaneously choose effort level according to their ability types. Finally, agents are partitioned into different status categories according to their efforts and the chosen partition.

We assume that each player maximizes the value of the expected status prize minus the expected effort cost, and that the designer maximizes the value of expected total effort by adjusting the partition in status classes.

We use the following notation: 1) $A_{k,n}$ denotes k-th order statistic out of n independent variables independently distributed according to F (note that $A_{n,n}$ is the highest order statistic, and so on..); 2) $F_{k,n}$ denotes the distribution of $A_{k,n}$, and $f_{k,n}$ denotes its density; 3) $E(k,n)$ denotes the expected value of $A_{k,n}$. (Note that $E(n,n)$ is the expectation of the maximum, or highest order statistic, and so on..)

3 The Optimal Partition in Status Categories

This section contains our main results about the structure of the optimal partition in status categories. We focus on a symmetric equilibrium where all agents use the same, strictly monotonic equilibrium effort function β. In such an equilibrium, the output rank of player j will be the same as his ability rank among the n contestants.

Let $P_i(a)$ be the probability of a player with ability a being ranked in category i, i.e., her ability is between the r_i-th and r_{i-1}-th highest. These probabilities involve the order statistics of the distribution of abilities in the population. Applying the revelation principle, agent j with ability a chooses to behave as an agent with ability s to solve the following optimization problem:

$$\max_s \sum_{i=1}^{k} P_i\left(s\right)\left[r_{i-1}-\left(n-r_i\right)\right]-\frac{\beta\left(s\right)}{a}$$

In equilibrium, the above maximization problem must be solved by $s=a$. The calculation of equilibrium effort functions and total expected effort yields:

Theorem 1 *Assume that contestants are partitioned in k status categories according to the family $\{r_i\}_{i=0}^k$. Then, total expected effort in a symmetric equilibrium is given by*

$$E_{total}^{(k)}=\sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n)$$

Proof. See Appendix. ■

Given the above result, we can now formulate the designer's problem: she needs to determine the number of contestants (m) and status categories (k), and the size of each category ($r_i, i = 1, .., k-1$). Explicitly, we obtain the following discrete optimization problem:

$$\max_{m,k,\{r_i\}_{i=0}^{k}} [\sum_{i=1}^{k-1} (r_{i+1} - r_{i-1})(m - r_i)E(r_i, m)]$$

$$\begin{aligned} \text{subject to} \quad &: \\ \text{i) } 2 &\leq m \leq n \\ \text{ii) } 2 &\leq k \leq m \\ \text{iii) } 0 &= r_0 < r_1 < \; ... \; < r_{k-1} < r_k = m \end{aligned}$$

3.1 The Optimal Number of Contestants

In many relevant situations, the number of agents will be exogenously determined by various economic considerations within the group, and can therefore be considered fixed for our purposes. But, it is also of interest to understand whether the designer has incentives to restrict entry that directly stem from the status considerations.[14] We determine here the optimal number of contestants by analyzing the effect of changing the number of contestants (i.e., by entry or hiring) on total expected effort. Given the zero-sum nature of status, the answer is not clear-cut, and it depends on the designer's reaction to entry (i.e., on how the size and number of status categories change). The following example illustrates the possibility that a wrong post-entry adjustment policy may cause total effort to actually go down.

Example 1 *Let $F(x) = x^{1/w}$, $w > 1$, and consider only partitions with two categories. Total effort is given by*

$$E_n = n(n-r)E(r,n) = n(n-r)\frac{n!\,(w+r-1)!}{(r-1)!\,(n+w)!}$$

where r is the division point. If we add an additional contestant to the higher category (that is, we do not change the value of r), we obtain for ω high enough:

$$E_{n+1} - E_n = \frac{(w+r-1)!n!}{(r-1)!\,(n+w)!}\left[\frac{(n+1)^2(n+1-r)}{(n+1+w)} - n(n-r)\right] < 0$$

That is, for sufficiently high w, total effort decreases in the number of players.

[14] Taylor (1995) and Fullerton and McAfee (1999) provide models of research tournaments where restricting entry may be beneficial for the designer.

We show below that a designer who optimally reacts to additional entry can always ensure that total effort increases. In particular, in the proof, we identify a very simple strategy (without the need of a complex re-optimization!) ensuring that total effort does not decrease: faced with more contestants, the designer can just increase the size of the lowest status category. For an intuition, consider for simplicity a partition with only two status categories. Then the number of "status prizes" is equal to the number of contestants in the top category, and each prize is worth n, the difference in payoffs between the high and low categories. If another agent is added, the value of each status prize becomes $n + 1$, independently of which status category is expanded. But, if the expansion is in the lower category, the number of status prizes remains fixed, while an expansion of the higher category also leads to an increase in the number of prizes. Such an increase has an adverse effect on the effort of high ability types, and this may offset the positive effect of having higher prizes. Thus, only by expanding the lower category, the designer increases the value of status prizes without simultaneously increasing their number.

Theorem 2 *Total effort in an optimal partition increases in the number of contestants.*

Proof. See Appendix. ■

3.2 The Optimal Partition

Given the above result, the designer has no incentives to restrict entry in the contest, and we thus assume below that all n potential contestants are included.[15]

Since the distribution of abilities determines the expected values of the various order statistics appearing in the designer's maximization problem, the optimal number of status categories and the optimal size of each category generally depend on this distribution. Our first main result identifies a robust and general feature that holds for any distribution:

Theorem 3 *In any optimal partition, the top status category contains a unique element.*

Proof. Suppose, by contradiction, that the k-th (top) category contains more than one element. Then, divide this category into two sub-categories, and denote by r_d the dividing point: $r_{k-1} < r_d < n$. Using the formula in Theorem 1, the difference in expected effort between the new and the

[15] See Section 4 where this result need not hold if the designer is budget constrained and if agents must be monetarily compensated for low status.

old partitions is given by:

$$
\begin{aligned}
E_{total}^{(k+1)} - E_{total}^{(k)} &= (n - r_{k-1})(n - r_d)E(r_d, n) - (n - r_{k-1})(n - r_d)E(r_{k-1}, n) \\
&= (n - r_{k-1})(n - r_d)\left[E(r_d, n) - E(r_{k-1}, n)\right] > 0
\end{aligned}
$$

The inequality follows since $A_{r_d,n}$ stochastically dominates $A_{r_{k-1},n}$. ■

Refining a top category that contains several elements does not affect the rewards going to agents outside that category. The reward and effort of those agents in the (new) second highest category is lower than before since these agents lose their top status. But, this loss is more than offset by the effort increase coming from the highest ability types whose status is increased by the refinement since they perceive more inferior agents after the change.

3.2.1 Optimal Fine Partitions

Our next main result identifies a condition on the distribution of abilities that allows us to extend the above logic to all status categories, thus exhibiting an optimal partition that is the finest possible. We use a statistic result about stochastic monotonicity of normalized differences (also called spacings) of order statistics. We first need to remind the reader some well-known concepts: The failure rate (or hazard rate) of a distribution F is defined by:

$$\lambda(a) = \frac{f(a)}{1 - F(a)}$$

A distribution function F has an *increasing failure rate* (IFR) if $\lambda(a)$ is increasing or, equivalently, if $\log(1 - F(a))$ is concave. Analogously, F has an *decreasing failure rate* (DFR) if $\lambda(a)$ is decreasing, or, equivalently, if $\log(1 - F(a))$ is convex.[16]

Armed with these concepts, we can now state:

Lemma 1 *(Barlow and Proschan, 1966) Assume that a distribution F with $F(0) = 0$ satisfies IFR (DFR). Then, $(n - i + 1)(A_{i,n} - A_{i-1,n})$ is stochastically decreasing (increasing) in i for a fixed n.*

In other words, up to a normalizing factor, the difference between the expected abilities of consecutively ranked contestants is higher at the bottom than at the top if the distribution is IFR, and the opposite holds for DFR distributions. An application of this result yields now:

[16] Most well known distributions belong to these important and much studied categories. The relationships between IFR, DFR, convexity and concavity of F are as follows: Convexity implies IFR , while DFR implies concavity. The only distribution that is both concave and convex is the uniform, while the only distribution that is both IFR and DFR is the exponential.

Theorem 4 *Assume that F, the distribution of abilities, has an increasing failure rate. Then, the optimal partition is the finest possible one: each status category contains a unique element.*

Proof. See Appendix. ■

The intuition behind the above result is analogous to one appearing in models of monopolistic quality/quantity discrimination: in "regular" settings, where marginal revenue is increasing in type (note that IFR is a sufficient condition for this to happen !), the optimal tariff allocates different qualities (here different status classes) to consumers with different types. In particular, lumping (or pooling) together several types cannot be optimal.

Splitting status class j in two sub-classes has two effects: there is a loss of expected effort stemming from the fact that several agents are now placed in the lower sub-class, and there is a gain from agents that are now placed in the higher sub-class (again, classes other than j are not affected by the split). The IFR condition ensures that the gain more than offsets the loss. For illustration purposes, assume that a category j has size two, and we refine it into two new categories, each with one element $r_j - r_d = r_d - r_{j-1} = 1$. This change is advantageous if the difference of expected efforts after and before the change is positive, i.e., if

$$\left\{ \begin{array}{c} (n - r_d + 1)\left[E\left(r_d, n\right) - E\left(r_{j-1}, n\right)\right] - (n - (r_d + 1) + 1)\left[E\left(r_d + 1, n\right) - E\left(r_d, n\right)\right] \\ +\left[E\left(r_d + 1, n\right) - E\left(r_d, n\right)\right] \geq 0 \end{array} \right\}$$

The first line is positive, because the normalized difference between the expected abilities of consecutively ranked contestants is higher at the bottom than at the top if the distribution is IFR, while the second line is positive because of usual stochastic dominance.[17]

3.2.2 Optimal Coarse Partitions

If the IFR condition (which represents, in fact, a convexity requirement with respect to the exponential distribution) is not satisfied, a coarse partition may be optimal. We now show that a very coarse partition with only two categories is optimal for sufficiently concave distributions. If there are only two categories, total effort is given by

$$E_{total}^{(2)} = n(n - r_1)E\left(r_1, n\right)$$

The intuition for the above expression is simple: this is a contest with $(n - r_1)$ equal prizes (for all those in the higher category), and each prize is worth here n (the difference in payoffs between

[17]The argument also indicates that the IFR condition is not necessary for class proliferation.

the high and low categories). By Theorem 3, when looking for optimal partitions, we can restrict attention to those where the top category consists of a unique element. In this case $r_1 = n-1$, and total effort is given by

$$E^{(2)}_{total} = nE\,(n-1, n)$$

In order to prove the result, we need to show that any other partition yields less effort if the distribution of abilities is sufficiently concave. The proof uses the following Lemma:

Lemma 2 *(Barlow and Porschan, 1966) Consider two distributions F and G such that $F(0) = G(0) = 0$, and such that $G^{-1}F$ is convex on the support of F.*[18] *Then $E_F(i,n)/E_G(i,n)$ is decreasing in i.*

Proposition 1 *Assume that the optimal partition of status categories under distribution F consists only of two categories, and consider another distribution G such that $G^{-1}F$ is convex on the support of F. Then the optimal partition under G also consists of two categories.*

Proof. See Appendix. ■

If we can show that there exists a distribution function for which the optimal partition consists indeed of two categories, then the above result immediately implies that the same will hold for **all** more concave distributions. The existence of such a distribution is established in the Appendix.

The intuition for the optimality of very coarse partitions for sufficiently concave distributions of ability is simple: most of the mass is then concentrated at the bottom and high ability individuals are rare. Thus, many "mediocre" types are motivated by a high reward (a unique high status prize) since they have a reasonable chance to get it. Moreover, the rare high ability individual lacks sufficient competition, and is therefore best motivated by a large reward.

3.3 How Good Are Partitions with Two Categories ?

In the above subsection we have identified conditions under which a partition with two categories are optimal. Here we take a somewhat different perspective that is not based on optimality: we show that, for the large and important class of IFR distributions (for which the optimal partition is the finest possible one), the designer can nevertheless achieve a substantial share of the optimal

[18] This means that G is more concave than F.

performance with a simple partition in two categories.[19] Thus, if very fine partitions are for some reason costly, a designer may find it optimal to choose the simplest non-trivial coarse partition. This seems to us a powerful argument in favor of coarse partitions.

Proposition 2 *Assume that F, the distribution of abilities, has an increasing failure rate. Then, the optimal partition in the class of partitions with only two status categories yields at least half the performance obtained by the overall optimal partition.*

Proof. Recall that in the IFR case, the overall optimal partition is the finest possible one, and hence has n status categories. Thus, total effort in the overall optimal partition is given by :

$$E_{total}^{(n)} = 2\sum_{i=1}^{n-1}(n-i)E(i,n)$$

Total effort in the optimal partition with only two categories is given by

$$E_{total}^{(2)} = n(n-i^*)E(i^*,n)$$

where $i^* \in \arg\max_i[n(n-i^*)E(i^*,n)]$. This immediately yields: $E_{total}^{(2)} > \frac{1}{2}E_{total}^{(n)}$. ■

The above approximation is rough, and the coarse partition with only two classes yields for "well-behaved" distributions a much higher percentage of the optimal performance. For example, $E_{total}^{(2)} \geq \frac{3}{4}E_{total}^{(n)}$ for a uniform distribution of abilities.

Our final result in this section gives further information about the optimal partition with two categories. Its proof is also based on Lemma 2 above.

Proposition 3 *Let r^* be the division point defining the optimal partition in two status categories, i.e. the optimal number of contestants in the lower class. If the distribution of abilities F is convex (concave) then $r^* \leq (\geq)\ n/2$.*

Proof. Suppose that r^* is the optimal division point. Then, total effort in the optimal partition is higher than in any other partition. In particular, it is higher than total effort in the partition

[19] We were not able to find a direct technical relation between our result and McAfee's (2002) paper on complete information matching of two continuum of populations. In McAfee's model the "optimal partition" is always (i.e., irrespective of distribution) the finest possible — assortative matching, whereas we get the optimality of the finest partition only under IFR. His result requires IFR on both distributions of abilities and on their survival functions, whereas we require IFR only on the distribution itself. Finally, his result holds for the partition with two categories where the cutoff is at the mean of each population, whereas our result holds for the optimal partition in the class of partitions with two categories.

where $r = n - r^*$. This yields:

$$\begin{aligned} n\left(n-r^*\right)E\left(r^*,n\right) &\geq n\left[n-\left(n-r^*\right)\right]E\left(n-r^*,n\right) \Leftrightarrow \\ \left(n-r^*\right)E\left(r^*,n\right) &\geq r^*E\left(n-r^*,n\right) \Leftrightarrow \\ \frac{E\left(r^*,n\right)}{r^*} &\geq \frac{E\left(n-r^*,n\right)}{n-r^*} \end{aligned}$$

By taking one of the distributions to be uniform in Lemma 2, we obtain that, for a fixed n, $E(i,n)/i$ is decreasing (increasing) in i if the distribution of abilities is convex (concave). Then, for a convex F, the last inequality above can hold only if $r^* \leq (n - r^*)$, which is equivalent to $r^* \leq n/2$. Analogously, if F is concave, it must be the case that $r^* \geq (n - r^*)$, which yields $r^* \geq n/2$. ■

A simple corollary is, of course, that exactly half of the agents should be in the low (high) category if abilities are uniformly distributed.

4 Status Derived from Monetary Prizes

So far, we focus on the pure effect of status in contests: there are no other real prizes to drive efforts. We now consider contests where status is being indirectly (and solely) induced by the rank of monetary prizes in the respective hierarchy. Higher effort leads to a (weakly) higher monetary prize, and, in addition, agents get positive utility proportional to the number of agents that have lower monetary prizes, and negative utility proportional to the number of agents that have higher monetary prizes.[20] In particular, we depart from the zero-sum world presented above.

A set of k monetary prizes $V_k \geq V_{k-1} \geq ... \geq V_1$ and a family of division points $\{r_i\}_{i=0}^k$ where $r_0 = 0$ and $r_k = n$ determines a partition with k categories: a contestant ranked in the top category k (i.e., a contestant whose effort is among the top $r_k - r_{k-1}$) receives a monetary prize of V_k, a contestant in the second highest category receives a prize of $V_{k-1} \leq V_k$, and so on till the lowest $V_1 \leq V_2 \leq ... \leq V_k$.

Thus, a player who is awarded the i-th highest monetary prize V_i perceives in fact a total prize (money + status) of :

$$v_i = V_i + r_{i-1} - (n - r_i).$$

In order to make the problem non-trivial, we add here two realistic assumptions: 1) The contest

[20] Dubey and Geanakoplos (2005) consider a status model where monetary prizes are awarded on the basis of absolute performance.

designer is financially constrained: the total amount of monetary prizes cannot exceed a given amount P. Otherwise, it is obvious that large enough monetary prizes can always swamp any status effects. 2) We impose an individual rationality constraint: the expected payoff of each contestant should be non-negative. Otherwise, contestants will leave without competing (the outside option being normalized to zero).

By calculations similar to those performed for the case of pure status concerns, total effort in a symmetric equilibrium is given by

$$E_{total}^{(k)} = \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i)$$

Therefore, the designer's problem is as follows:

$$\begin{aligned} \max_{k,\{r_i\}_{i=1}^k,\{V_i\}_{i=1}^k} E_{total}^{(k)} &= \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\ \text{subject to} &: \ (1) \ \ 1 \leq k \leq n \\ &: \ (2) \ \ \sum_{i=1}^{k}(r_i - r_{i-1})V_i = P \\ &: \ (3) \ \ V_1 \geq n - r_1 \\ &: \ (4) \ \ V_k \geq V_{k-1} \geq \ \ldots \ \geq V_1 \end{aligned}$$

Note that constraint (3) guarantees that the expected payoff of the lowest type, who does not make any effort, is non-negative. By a standard monotonicity argument, all other types will have positive expected payoffs.

Theorem 5 *If $P > n$, (i.e., if the available budget is as least as large as the number of contestants), the optimal solution to the designer's problem has the following structure: The designer induces a partition with two status categories such that the contestant with the highest effort receives a monetary prize $V_2 = P - (n-1)$,while all other contestants receive a monetary prize $V_1 = 1$. If $P \leq n$, it is optimal to restrict entry to the contest.*

Proof. See Appendix. ■

The intuition behind the optimality of the above described partition is as follows: Take a partition with two categories and a singleton in the top category, and refine it, for example, by dividing the low category in new "middle" and "low" categories. Then, the agents in the new low category perceive a decline in status, and this decline must be compensated by a higher monetary prize (in order to satisfy their individual rationality constraint). Since status is derived from monetary prizes, the agents in the new middle category must obtain a monetary prize that is at least as large as that of the agents in the new low category. Thus, via the budget constraint, the monetary prize of the agent in the top category must go down — this decline necessarily induces a decline in the effort of high ability types. Since the strongest effect of prizes is on high ability agents, the potential increase in effort of middle ability agents is not enough to compensate for the decline at the top. This insight is related to the optimality of a unique "first" prize in Moldovanu and Sela's (2001) contest model with linear cost functions and purely monetary prizes. That optimality naturally translates here into a partition in two status classes, with a singleton in the top category.

Our previous result suggests that an upward shift in the relative weight of the monetary part in determining status will lead to a larger gap between CEO compensation and the compensation of the other agents in a firm.[21] Frydman (2005) documents the relatively recent dramatic increase in this gap in the US, and offers an explanation based on a shift in the importance of general versus firm-specific skills.

5 Conclusion

We have studied a contest model where heterogeneous agents who care about relative standing are ranked according to output, and are then partitioned into status categories. Our main results describe the structure of the optimal partition into status classes from the point of view of a designer who maximizes total output. The model explains ubiquitous phenomena such as a top status class that contains a unique individual, and the proliferation of status classes in organizations where high-skilled individuals are not rare. We also studied the interplay between pure status and monetary prizes.

As already mentioned in the introduction, in most real-life situations status is only partly determined by measurable differences in monetary compensation. Social, cultural and other economic considerations that may be connected to a concern for relative position in a future interaction are also important determinants. Modeling a specific situation requires a simple combination of the two variants displayed here, and the corresponding results will be driven by the relative strengths of the monetary versus the less tangible parts.

Finally, note that, in principle, an analysis analogous to ours is possible for other agents'

[21] For example, the average ratio of highest to fifth highest compensation in US firms jumped from about 2.8 in the middle of 20th century to 6.1 at the beginning of the 21th century. The increase in the ratio of CEO compensation to average compensation in the firm is much more dramatic.

utility functions, or for other designer's goals. In particular, for given, fixed utility functions, the equilibrium analysis is not affected by the designer's goal which can be modified according to the desired application. Thus, our model offers a convenient framework for the study of the various implications of concerns for social status on organizational design.

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7 Appendix

A few useful facts about order statistics:

It is well-known that:

$$\begin{aligned} F_{k,n}(s) &= \sum_{j=k}^{n} \binom{n}{j} F(s)^j [1-F(s)]^{n-j} \\ f_{k,n}(s) &= \frac{n!}{(k-1)!(n-k)!} F(s)^{k-1} [1-F(s)]^{n-k} f(s) \end{aligned}$$

Let $F_i^n(s)$, $i = 1, 2, ...n$ denote the probability that a player's type s ranks exactly i-th highest among n random variables distributed according to F. Then

$$F_i^n(s) = \frac{(n-1)!}{(i-1)!(n-i)!} [F(s)]^{i-1} [1-F(s)]^{n-i}$$

Defining $F_{n,n-1} \equiv 0$, and $F_{0,n-1} \equiv 1$, it is immediate that the relation between $F_{i,n}(s)$ and $F_i^n(s)$ is

$$F_i^n(s) = F_{i-1,n-1}(s) - F_{i,n-1}(s)$$

Finally, let $P_i(s)$ be the probability of a player with type s being ranked in category i, i.e., her type is between the r_i-th and r_{i-1}-th highest. Then:

$$P_i(s) = \sum_{j=1}^{r_i - r_{i-1}} F^n_{r_{i-1}+j}(s) = F_{r_{i-1},n-1}(s) - F_{r_i,n-1}(s)$$

Proof of Theorem 1:

Proof. Let a partition with k categories be given by $\{ (0, r_1], (r_1, r_2], ...(r_{i-1}, r_i], ..., (r_{k-1}, n] \}$. Assuming a symmetric equilibrium in strictly increasing strategies,[22] the optimization problem of a player with ability a is

$$\max_s \left\{ \begin{array}{c} [1 - F_{r_1,n-1}(s)][-(n-r_1)] \\ + \sum_{i=2}^{k-1} \left[F_{r_{i-1},n-1}(s) - F_{r_i,n-1}(s) \right] [r_{i-1} - (n - r_i)] \\ + F_{r_{k-1},n-1}(s) r_{k-1} - \frac{\beta(s)}{a} \end{array} \right\}$$

[22] It can be shown that there is a unique symmetric equilibrium.

where the first term is the utility of being in the lowest category, the second term is the utility of being in categories 2 till $(k-1)$, the third term is the utility of being in the highest category, and the last term is the disutility of exerting effort $\beta(s)$.

The solution of the resulting differential equation with boundary condition $\beta(0) = 0$ is

$$\beta(a) = \int_0^a x \left\{ f_{r_1,n-1}(x)(n-r_1) + \sum_{i=2}^{k-1} \left[f_{r_{i-1},n-1}(x) - f_{r_i,n-1}(x) \right] (r_{i-1} + r_i - n) + f_{r_{k-1},n-1}(x) r_{k-1} \right\} dx \tag{1}$$

Thus, total effort is given by:

$$E_{total} = n \int_0^1 \beta(a) f(a) da \tag{2}$$

The above integral can be calculated by inserting formula 1 in 2 and by integrating by parts the constituent terms, who all have the form $b \int_0^1 \left[\int_0^a x f_{r,n-1}(x) dx \right] f(a) da$ where b is a constant. Note that :

$$\begin{aligned}
& \int_0^1 \left[\int_0^a x f_{r,n-1}(x) dx \right] f(a) da \\
= & \left[F(a) \int_0^a x f_{r,n-1}(x) dx \right]_0^1 - \int_0^1 F(a) a f_{r,n-1}(a) da \\
= & \int_0^1 a \left[1 - F(a) \right] f_{r,n-1}(a) da \\
= & E(r, n-1) - \frac{r}{n} E(r+1, n) \\
= & \frac{n-r}{n} E(r, n)
\end{aligned}$$

The last equality follows by a well known identity among order statistics (see David and Nagaraja, 2003, page 44). Assembling all terms in equation 2, and recalling that $r_0 = 0$, and $r_k = n$ finally yields:

$$\begin{aligned}
E_{total}^{(k)} &= \left\{ \begin{array}{c} (n-r_1)^2 E(r_1, n) \\ + \sum_{i=2}^{k-1} (r_{i-1} + r_i - n) \left[(n - r_{i-1}) E(r_{i-1}, n) - (n - r_i) E(r_i, n) \right] \\ + r_{k-1}(n - r_{k-1}) E(r_{k-1}, n) \end{array} \right\} \\
&= \sum_{i=1}^{k-1} (r_{i+1} - r_{i-1})(n - r_i) E(r_i, n)
\end{aligned}$$

■

Proof of Theorem 2:

Proof. Consider a partition $\{r_i\}_{i=0}^k$ for a given number of contestants m. Total effort is given by

$$\begin{aligned} E_{total} &= \sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i,m) \\ &= r_2(m-r_1)E(r_1,m)+\sum_{i=2}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i,m) \end{aligned}$$

Assume now that a designer faced with $m+1$ contestants expands by one the size of the lowest status category: thus, consider the new partition $\{r_i'\}_{i=0}^k$ where $r_0'=0, r_1'=r_1+1, r_2'=r_2+1, ..., r_{k-1}'=r_{k-1}+1, r_k'=m+1$.

Total effort for this new partition is given by

$$\begin{aligned} E_{total}^{'} &= \sum_{i=1}^{k-1}(r_{i+1}'-r_{i-1}')(m+1-r_i')E(r_i',m+1) \\ &= (r_2+1)(m-r_1)E(r_1+1,m+1)+\sum_{i=2}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i+1,m+1) \end{aligned}$$

We obtain:

$$\begin{aligned} & E_{total}^{'}-E_{total} \\ = \;& (m-r_1)E(r_1+1,m+1)+\sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)[E(r_i+1,m+1)-E(r_i,m)]\geq 0 \end{aligned}$$

The last inequality holds since, for all i, m, $A_{i+1,m+1}$ stochastically dominates $A_{i,m}$.[23] The claim follows now by starting from an optimal partition for m contestants, and expanding the size of the lowest category as above. Further eventual optimization of the partition for $m+1$ contestants must weakly increase the total effort even further, thus yielding the wished result. ■

Proof of Theorem 4:

Proof. Suppose that, in an optimal partition with k categories, the j-th ($1 \leq j \leq k$) category contains more than one element. Divide the j-th category into two sub-categories and denote by r_d the dividing point, $r_{j-1} < r_d < r_j$. Letting $E(0,n) \equiv 0$, the difference in total effort between the

[23] See Shaked and Shanthikumar (1994) for more details.

new and the initial partition is given by:

$$E_{total}^{(k+1)} - E_{total}^{(k)} = \left\{ \begin{array}{c} (r_j - r_{j-1})(n - r_d)E(r_d, n) \\ -(r_j - r_d)(n - r_{j-1})E\left(r_{j-1}, n\right) \\ -(r_d - r_{j-1})(n - r_j)E(r_j, n) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (r_j - r_d)\left[(n - r_d)\, E\left(r_d, n\right) - (n - r_{j-1})\, E\left(r_{j-1}, n\right)\right] \\ -\left(r_d - r_{j-1}\right)\left[(n - r_j)\, E\left(r_j, n\right) - (n - r_d)\, E\left(r_d, n\right)\right] \end{array} \right\}$$

Let $t = r_j - r_{j-1}$, $r_d = r_{j-1} + 1$. Then,

$$E_{total}^{(k+1)} - E_{total}^{(k)}$$

$$= \left\{ \begin{array}{c} (t-1)\left[(n - r_d)\, E\left(r_d, n\right) - (n - (r_d - 1))\, E\left(r_{j-1}, n\right)\right] \\ -\left[(n - (r_d + t - 1))\, E\left(r_d + t - 1, n\right) - (n - r_d)\, E\left(r_d, n\right)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (t-1)\left[(n - r_d)\, E\left(r_d, n\right) - (n - (r_d - 1))\, E\left(r_d - 1, n\right)\right] \\ -\left[(n - (r_d + t - 1))\, E\left(r_d + t - 1, n\right) - (n - (r_d + t - 2))\, E\left(r_d + t - 2, n\right)\right] \\ -\left[(n - (r_d + t - 2))\, E\left(r_d + t - 2, n\right) - (n - (r_d + t - 3))\, E\left(r_d + t - 3, n\right)\right] \\ -.... \\ -\left[(n - (r_d + 1))\, E\left(r_d + 1, n\right) - (n - r_d)\, E\left(r_d, n\right)\right] \end{array} \right\}$$

Note that

$$\begin{aligned} & (n - r)\, E\left(r, n\right) - (n - (r - 1))\, E\left(r - 1, n\right) \\ = \;& (n - r + 1)\left[E\left(r, n\right) - E\left(r - 1, n\right)\right] - E\left(r, n\right) \end{aligned}$$

By Barlow and Proschan's Lemma about IFR distributions, and by the fact that $-E\left(r, n\right)$ is decreasing in r, it immediately follows that $\left[(n - r)\, E\left(r, n\right) - (n - (r - 1))\, E\left(r - 1, n\right)\right]$ is decreasing in r. Therefore $E_{total}^{(k+1)} - E_{total}^{(k)} > 0$. This contradicts the assumption that the initial partition was optimal. Therefore, each category in the optimal partition must contain a unique element. ■

Proof of Proposition 1:

Proof. By Theorem 3 we can restrict the argument to partitions for which the top status class contains a unique element. By Theorem 1, the total effort in a partition with k status categories is given by

$$\begin{aligned} E_{total}^{(k)} &= \sum_{i=1}^{k-1} (r_{i+1} - r_{i-1})(n - r_i)E(r_i, n) \\ &= \sum_{i=1}^{k-2} (r_{i+1} - r_{i-1})(n - r_i)E(r_i, n) + (n - r_{k-2})E(n-1, n) \end{aligned}$$

The optimal partition contains only two status classes iff $E^{(2)}_{total} \geq E^{(k)}_{total}$ for all $2 \leq k \leq n$. That is, the following claim must hold for all $2 \leq k \leq n$ and all admissible partition sequences $\{r_i\}_{i=1}^{k}$:

$$\begin{aligned} nE(n-1,n) &\geq \sum_{i=1}^{k-2}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n)+(n-r_{k-2})E(n-1,n) \\ &\Rightarrow\ r_{k-2}E(n-1,n) \geq \sum_{i=1}^{k-2}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n) \\ &\Rightarrow\ r_{k-2} \geq \sum_{i=1}^{k-2}(r_{i+1}-r_{i-1})(n-r_i)\frac{E(r_i,n)}{E(n-1,n)} \qquad (3) \end{aligned}$$

By Lemma 2 above, we know that $E_F(i,n)/E_G(i,n)$ is decreasing in i. This yields

$$\frac{E_F(r_i,n)}{E_G(r_i,n)} \geq \frac{E_F(n-1,n)}{E_G(n-1,n)}$$

which in turn implies

$$\frac{E_F(r_i,n)}{E_F(n-1,n)} \geq \frac{E_G(r_i,n)}{E_G(n-1,n)} \qquad (4)$$

Thus, if inequality (3) holds under F, it must also hold under G, and the desired result follows. ■

Existence of a distribution for which a partition with two categories is optimal:

Proof. By the proof of Proposition 1, it is sufficient to show that there exists a distribution function for which condition (3) is satisfied. Consider $F(x)=x^{\frac{1}{w}}$, $w>1$. Then

$$E(r,n)=\frac{n!(w+r-1)!}{(r-1)!(n+w)!}$$

and

$$\frac{E(r,n)}{E(n-1,n)}=\frac{(n-2)!}{(r-1)!}\frac{(w+r-1)!}{(w+n-2)!}$$

It can be easily verified that $\lim_{w\to\infty}\frac{(w+r-1)!}{(w+n-2)!}=0$. Therefore, for a sufficiently large w, condition (3) is satisfied, and the result follows. ■

Proof of Theorem 5

Proof. The designer's problem is:

$$\begin{aligned} \max_{k,\{r_i\}_{i=1}^{k},\{V_i\}_{i=1}^{k}} E^{(k)}_{total} &= \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n)+\sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\ \text{subject to} &: \ 1)\ \ 1 \leq k \leq n \\ &: \ 2)\ \ \sum_{i=1}^{k}(r_i-r_{i-1})V_i=P \\ &: \ 3)\ \ V_1 \geq n-r_1 \\ &: \ 4)\ \ V_k \geq V_{k-1} \geq .... \geq V_1 \end{aligned}$$

Assume first that a given partition with k status categories is fixed. We derive the optimal allocation of money prizes consistent with such a partition. Subsequently, we find the optimal partition.

Note that $\frac{dE_{total}^{(k)}}{dV_1} < 0$, and therefore $V_1 = n - r_1$. The maximization problem reduces to:

$$\begin{aligned} &\max_{\{V_i\}_{i=1}^{k}} \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\ &\text{subject to: } \sum_{i=1}^{k}(r_i - r_{i-1})V_i = P \\ : \quad & V_k \geq V_{k-1} \geq ....V_1 = n - r_1 \end{aligned}$$

Assuming that all the constraints $V_k \geq .... \geq V_1 = n - r_1$ are binding, the Lagrangian is

$$\begin{aligned} L \;\; = \;\; & \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + \sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) - \\ & \alpha_0(\sum_{i=1}^{k}(r_i-r_{i-1})V_i - P) + \sum_{i=1}^{k}\alpha_i(V_i-(n-r_1)) \end{aligned}$$

The first order conditions are

$$\frac{dL}{dV_i} = [(n-r_{i-1})E\left(r_{i-1},n\right) - (n-r_i)E\left(r_i,n\right)] - \alpha_0(r_i - r_{i-1}) - \alpha_i = 0, \quad i = 1, ..., k$$

The solution of this problem is:

$$\begin{aligned} V_{k-1} &= ... = V_1 = (n-r_1); \\ V_k &= \frac{P - r_{k-1}(n-r_1)}{n - r_{k-1}} \\ \alpha_0 &= E(r_{k-1}, n); \\ \alpha_i &= [(n-r_{i-1})E\left(r_{i-1},n\right) - (n-r_i)E\left(r_i,n\right)] - \alpha_0(r_i - r_{i-1}),\; i = 1,..,k \end{aligned}$$

Note that :

$$\begin{aligned} \alpha_i &= [(n-r_{i-1})E\left(r_{i-1},n\right) - (n-r_i)E\left(r_i,n\right)] - \alpha_0(r_i - r_{i-1}) \\ &< (r_i - r_{i-1})(E(r_i,n) - E(r_{k-1},n)) \leq 0 \end{aligned}$$

That is, our assumption that all the constraints $V_{k-1} \geq ... \geq V_1 = n - r_1$ are binding ($V_k \geq n - r_1$ is not binding) was correct. Now, at the optimal solution, total effort is given by

$$E_{total}^{(k)} = \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n) + E(r_{k-1},n)(P - n(n-r_1))$$

For a partition with $k = 2$ with division point r_1', the above formula yields:

$$E_{total}^{(2)} = PE(r_1', n)$$

which is maximized for $r_1' = n - 1$. Noting that $\sum_{i=1}^{k-1}(n - r_i)(r_{i+1} - r_{i-1}) = n(n - r_1)$, and that for any k, $r_{k-1} \leq n - 1$, we obtain that

$$\begin{aligned}
& E_{total}^{(2)} - E_{total}^{(k)} \\
= \;& PE(n-1, n) - \left(\sum_{i=1}^{k-1}(n - r_i)(r_{i+1} - r_{i-1})E(r_i, n) + E(r_{k-1}, n)(P - n(n - r_1)) \right) \\
= \;& P[E(n-1, n) - E(r_{k-1}, n)] - \sum_{i=1}^{k-1}(n - r_i)(r_{i+1} - r_{i-1})[E(r_i, n) - E(r_{k-1}, n)] \geq 0
\end{aligned}$$

Thus, a partition with two status categories where the top category contains a unique element is optimal. ■