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Non-linear equity portfolio variance reduction under a mean–variance framework—A delta–gamma approach

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ABSTRACT

To examine the variance reduction from portfolios with both primary and derivative assets we develop a mean–variance Markovitz portfolio management problem. By invoking the delta–gamma approximation we reduce the problem to a well-posed quadratic programming problem. From a practitioner's perspective, the primary goal is to understand the benefits of adding derivative securities to portfolios of primary assets. Our numerical experiments quantify this variance reduction from sample equity portfolios to mixed portfolios (containing both equities and equity derivatives).
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1. Introduction

The main objective in portfolio management is the tradeoff between risk and return. Markovitz, [9,10] studied the problem of maximizing portfolio expected return for a given level of risk, or equivalently minimizing risk for a given expected return. One limitation of Markovitz's model, however, is that it considers only portfolios of primary assets. Mixed portfolios have been a topic of recent research from different perspectives with varying success. We list a few of the known results, and then describe our results in relation to the current research.

Recently, [11,1] looked at the optimal management of portfolios containing primary and derivative assets. In [11], the author introduced a technique for optimizing CVaR (Conditional Value at Risk) of a portfolio. The paper [1] observes that CVaR minimization for a portfolio of derivative securities is ill-posed. Furthermore, [1] has shown that this predicament can be overcome by including transaction costs.

[2,3,6] considered portfolio optimization with non-standard asset classes. In particular, [2] looked at the problem of maximizing expected exponential utility of terminal wealth under a continuous time model by trading a static position in derivative securities and a dynamic position in stocks. Separately, in a one period model, [3] analyzed the optimal investment and equilibrium pricing of primary and derivative instruments. Additionally, [6] has shown how to approximate dynamic positions in options by minimizing the mean-squared error.

To the best of our knowledge, this paper is the first work to consider the mean–variance Markovitz portfolio management problem in a one period model with derivative assets. For a portfolio containing many assets (primary and derivative) the estimation of the correlation matrix can be challenging. Practitioners often solve this difficulty by projecting portfolios onto a reduced set of factors. Projection methods motivate our approach to the mean–variance problem. However, if parametric approaches are used (we work in a multivariate normally distributed returns framework), this projection method creates another problem. Since projections are often non-linear, we must overcome non-linearities by the delta–gamma approximation.

The delta–gamma approximation is well known and often used in risk management and portfolio hedging. In industry practice this approximation works well for sufficiently small time intervals. By performing the delta–gamma approximation, the portfolio management problem with derivative assets is reduced to a quadratic program; however, the covariance matrix of the factors may not be positive definite. Since data are usually built from inconsistent datasets, this issue appears in some financial optimization problems. For example, for portfolios of stocks, the sample correlation matrix is just an approximate correlation, and hence need not be positive definite. This problem is addressed by [5,7]. These works focused on the extraction of a positive semi-definite variance–covariance matrix, obtained through the solution of a second-order conic mathematical programming problem. It is a way to convexify an a priori non convex problem. In [5,7], the smallest distortion of the original matrix which

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satisfies the desired properties (e.g., being a correlation matrix) is obtained by using Frobenius norm.

Our main motivation is to investigate the variance reduction portfolios achieved through the addition of derivative assets compared against straight equity portfolios. Options can be considered as a type of portfolio insurance. We postulate that by allowing investment in this asset class, one should be able to reduce the risk profile of optimal mean–variance portfolios (here risk is measured as variance). Therefore, it appears only important to study the risk reduction due to investing in options. We explore the size of this risk reduction, and how the risk reduction profile varies across different derivative structures.

To address these questions we implement several numerical experiments. One finding is that the largest variance reduction is obtained by adding options on one stock. It is interesting to point out that the optimal portfolio variance reduction is a unimodal function of annual returns (it increases for small values of annual returns, reaches its peak and then decreases for larger values of annual returns). The maximum variance reduction is ~85%–90%, and it occurs with a portfolio return per annum of ~12%–15%.

Our results can be applied to the problem of pricing and hedging in incomplete markets. For instance, we can consider instruments written on non-tradable factors (e.g., temperature), and they can be hedged with tradable instruments which are highly correlated (this procedure is called cross hedging). Take weather derivatives (e.g., HDD or CDD) as an example; energy prices are considered as the traded correlated instrument (in California a high correlation can be observed between temperature and energy prices). Perfect hedging is not possible in this paradigm. Minimizing the variance of the hedging error can be captured as a special case of mean–variance optimization problem for a portfolio of primary and derivative instruments. A survey paper on mean–variance hedging and mean–variance portfolio selection is explored in [12].

Another possible application of our results is the hedging of long maturity instruments with short maturity ones. As is well known, the market for long maturity instruments is illiquid, thus the issuers use (static) hedging portfolios of the more liquid short maturity instruments. The interested reader can Ref. [4].

This paper is organized as follows. In Section 2 we present the model; Section 3 introduces the delta–gamma approximation; Section 4 presents the reduction to quadratic programs. Numerical experiments are provided in Section 5; and Section 6 concludes our work.

2. The model

Portfolio returns are derived from the return of individual positions; however, in practice, it is not advisable to model the positions individually due to the latent correlation structure. If we have m instruments in our portfolio, we would need m separate volatilities, plus data on $\frac{m(m-1)}{2}$ correlations, so in total $\frac{m(m+1)}{2}$ pieces of information. For large m this may be difficult.

The resolution is to map m instruments onto a reduced set of risk factors, n. The mapping can be non-linear (e.g., BS (Black–Scholes formula) for options). Let us assume that the factors are represented by a stochastic vector process $S = (S_1, S_2, \dots, S_n)$, which at all times $t \in (0, \infty)$ is assumed to be of the form

$$S(t) = \Sigma W_t \tag{2.1}$$

where Σ is the variance–covariance matrix, which we take to be positive definite (the methodology proposed in [5,7] can be applied when the positive definite assumption fails), and W_t is a standard Brownian motion on a canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The portfolio value at time t, denoted by $V(S, t)$, is of the form

$$V(S,t) = \sum_{k=1}^{m} x_k(t) V_k(S,t), \tag{2.2}$$

where $V_k(S, t)$, $k = 1, \dots, m$, represents the value of the individual instruments (mapped onto the risk factors), and $x_k(t)$, $k = 1, \dots, m$, stands for the number of shares of instrument k held in the portfolio at time t. We choose the portfolio mix $x_k(t)$, $k = 1, \dots, m$, such that the portfolio return, ΔV, over time interval $[t, t + \Delta t]$,

$$\Delta V = V(S + \Delta S, t + \Delta t) - V(S, t), \tag{2.3}$$

is optimized as described below. It turns out to be more convenient to work with the vector of actual proportions of wealth invested in the different assets, thus, at time $t \in (0, \infty)$, we introduce portfolio weights $w_k(t)$, $k = 1, \dots, m$, by

$$w_k(t) = \frac{x_k(t)}{V(S,t)}, \quad k = 1, \dots, m. \tag{2.4}$$

In the following, we posit the Markowitz mean–variance type problem: given some exogenous benchmark return, $r_e(t)$, at time t an investor wants to choose among all portfolios having the same return, $r_e(t)$, the one with minimal variance, $\mathrm{Var}(\Delta V)$:

$$\text{(P1)} \ \min_{w} \mathrm{Var}(\Delta V)$$
$$\text{s.t. } \mathrm{E}(\Delta V) = r_e(t),$$
$$\sum_{k=1}^{m} w_k(t) V_k(S,t) = 1.$$

Another possible portfolio management problem is to choose the portfolio with minimal variance:

$$\text{(P2)} \ \min_{w} \mathrm{Var}(\Delta V)$$
$$\text{s.t. } \sum_{k=1}^{m} w_k(t) V_k(S,t) = 1.$$

There are some difficulties in solving (P1) and (P2). First, we may not be able to determine the moments of ΔV since ΔV non-linearly depends on changes in the underlying factors. Moreover, it is not obvious what distribution ΔV would follow—even if we perfectly learnt the pdf of ΔS. If we only required the moments of ΔV, the situation would not improve since the integration of moments might be intractable. One way out of this predicament is to use the delta–gamma approximation.

3. Delta–gamma approximation

The delta–gamma approximation states that a portfolio change during a short time period resulting from the change of underlying factors can be approximated by some second order polynomial function, the coefficients of which are given by the portfolio's sensitivities, such as the portfolio delta, gamma and theta. This approximation is an important tool in risk management and hedging; for instance, to hedge a portfolio of derivatives with respect to the underlying's change, the delta–gamma approximation is employed to match sensitivities of the portfolio with those of the hedging instruments.

Mathematically speaking, this approximation is a second order Taylor expansion of the portfolio's change, ΔV, over the time interval $[t, t + \Delta t]$:

$$\Delta V \approx \delta V = \frac{\partial V}{\partial t} \Delta t + \delta^T \Delta S + \frac{1}{2} \Delta S^T \Gamma \Delta S, \tag{3.1}$$

where

$$\delta_i = \frac{\partial V}{\partial S_i}, \qquad \Gamma_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j}, \quad i = 1, \dots, n.$$

Since

$$V(S,t)=\sum_{k=1}^{m}x_k(t)V_k(S,t),$$

then

$$\delta_i=\frac{\partial V}{\partial S_i}=\sum_{k=1}^{m}x_k(t)\delta_i^k,\qquad \delta_i^k:=\frac{\partial V_k}{\partial S_i},\quad i=1,\dots,n,\ k=1,\dots,m, \tag{3.2}$$

$$\Gamma_{ij}=\frac{\partial^2 V}{\partial S_i\partial S_j}=\sum_{k=1}^{m}x_k(t)\Gamma_{ij}^k,\qquad \Gamma_{ij}^k:=\frac{\partial^2 V_k}{\partial S_i\partial S_j},\quad i=1,\dots,n,\ j=1,\dots,n,\ k=1,\dots,m. \tag{3.3}$$

It is well known that this approximation performs well for sufficiently small time intervals, Δt. At this point, we formulate the approximated versions of (P1) and (P2) as

(P3) $\min_w \mathrm{Var}(\delta V)$

s.t. $\mathrm{E}(\delta V)=r_e(t)$,

$\sum_{k=1}^{m}w_k(t)V_k(S,t)=1$,

and

(P4) $\min_w \mathrm{Var}(\delta V)$

s.t. $\sum_{k=1}^{m}w_k(t)V_k(S,t)=1$.

Our next step is to reduce (P3) and (P4) to quadratic programs and this is formulated in the following section.

4. Quadratic programs

In light of (2.1), $\Delta S\sim\mathcal{N}(0,\Sigma\sqrt{\Delta t})$. For computational convenience, we assume $\Delta t=1$. Next, we replace the vector of correlated normals, ΔS, with the vector of independent normals $Z\sim\mathcal{N}(0,I)$. This is done by setting

$$\Delta S=CZ\quad\text{with } CC^T=\Sigma.$$

In terms of Z, the quadratic approximation of ΔV becomes

$$\Delta V\approx\delta V=a+(C^T\delta)^TZ+\frac{1}{2}Z^T(C^T\Gamma C)Z,$$

with

$$a=\frac{\partial V}{\partial t}\Delta t. \tag{4.1}$$

At this point, it is convenient to choose the matrix C to diagonalize the quadratic term in the above expression. Let $\tilde{C}$ be a square matrix such that

$$\tilde{C}\tilde{C}^T=\Sigma \tag{4.2}$$

(e.g., the one given by the Cholesky factorization). The matrix $\frac{1}{2}\tilde{C}^T\Gamma\tilde{C}$ is symmetric, and thus admits the representation

$$\frac{1}{2}\tilde{C}^T\Gamma\tilde{C}=U\Lambda U^T, \tag{4.3}$$

where $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$, and U is an orthogonal matrix such that $UU^T=I$. Next, set $C=\tilde{C}U$ and observe that

$$CC^T=\tilde{C}UU^T\tilde{C}^T=\Sigma, \tag{4.4}$$

then we have

$$\frac{1}{2}C^T\Gamma C=\frac{1}{2}U^T(\tilde{C}^T\Gamma\tilde{C})U=U^T(U\Lambda U^T)U=\Lambda.$$

Thus, together with the fact that

$$b=C^T\delta, \tag{4.5}$$

we obtain

$$\Delta V\approx\delta V=a+b^TZ+Z^T\Lambda Z:=Y.$$

4.1. Moment generating function

In this subsection, we explore the moment generating function of Y and derive the mean and variance of Y. Since

$$Y=\sum_{i=1}^{n}(\lambda_iZ_i^2+b_iZ_i)+a \tag{4.6}$$

$$=\sum_{i=1}^{n}\lambda_i\left(Z_i+\frac{b_i}{2\lambda_i}\right)^2+a-\sum_{i=1}^{n}\frac{b_i^2}{4\lambda_i}, \tag{4.7}$$

it follows that the random variable Y has a student distribution, being – up to a constant – the sum of squared independent normally distributed random variables. It is well known that:

$$\mathrm{E}(\theta Y)=\exp(\eta(\theta)), \tag{4.8}$$

where

$$\eta(\theta)=a\theta+\sum_{j=1}^{n}\eta_j(\theta)=a\theta+\sum_{j=1}^{n}\frac{1}{2}\left(\frac{\theta^2b_j^2}{1-2\theta\lambda_j}-\log(1-2\theta\lambda_j)\right), \tag{4.9}$$

for all θ satisfying $\max_j\theta\lambda_j<\frac{1}{2}$. Direct computation leads to

$$\frac{d\left(e^{\eta(\theta)}\right)}{d\theta}=\exp(\eta(\theta))\frac{d\eta}{d\theta}=\exp(\eta(\theta))\left[a+\frac{1}{2}\sum_{j=1}^{n}\left(\frac{2\theta b_j^2(1-2\theta\lambda_j)-\theta^2(-2\lambda_j)}{(1-2\theta\lambda_j)^2}-\frac{-2\lambda_j}{1-2\theta\lambda_j}\right)\right]$$

and $\frac{d^2\left(e^{\eta(\theta)}\right)}{d\theta^2}$ as given in Box I.

Thus, the first and second moments of Y are:

$$\mathrm{E}(Y)=\left.\frac{d\left(e^{\eta(\theta)}\right)}{d\theta}\right|_{\theta=0}=a+\sum_{j=1}^{n}\lambda_j$$

and

$$\mathrm{E}(Y^2)=\left.\frac{d^2\left(e^{\eta(\theta)}\right)}{d\theta^2}\right|_{\theta=0}=\left(a+\sum_{j=1}^{n}\lambda_j\right)^2+\sum_{j=1}^{n}(b_j^2+2\lambda_j^2).$$

Hence,

$$\mathrm{Var}(Y)=\mathrm{E}(Y^2)-\mathrm{E}^2(Y)=\sum_{j=1}^{n}(b_j^2+2\lambda_j^2).$$

In order to ease the notation, we assume that $V(S,t)=1$, so the vector of shares x equals the vector of proportions w (also notice that for simplicity we dropped the t dependence of w). We would

$$\frac{d^2\left(e^{\eta(\theta)}\right)}{d\theta^2} = \exp(\eta(\theta))\left[a+\frac{1}{2}\sum_{j=1}^{n}\left(\frac{2\theta b_j^2(1-2\theta\lambda_j)-\theta^2(-2\lambda_j)}{(1-2\theta\lambda_j)^2}-\frac{-2\lambda_j}{1-2\theta\lambda_j}\right)\right]^2+\exp(\eta(\theta))\left[\frac{1}{2}\sum_{j=1}^{n}\left(2\lambda_j\frac{-2\lambda_j}{-(1-2\theta\lambda_j)^2}\right.\right.$$
$$\left.\left.+\ \frac{(2b_j^2-8\theta b_j^2\lambda_j+4\theta b_j^2\lambda_j)(1-2\theta\lambda_j)^2-(2\theta b_j^2(1-2\theta\lambda_j)+\theta^2 b_j^2 2\lambda_j)((-2\lambda_j)2(1-2\theta\lambda_j))}{(1-2\theta\lambda_j)^4}\right)\right].$$

Box I.

$$\Sigma = \begin{pmatrix} (1.144)10^{-4} & (5.026)10^{-5} & (4.687)10^{-4} & (2.285)10^{-5} & (7.010)10^{-5} \\ (5.026)10^{-5} & (1.638)10^{-4} & (5.973)10^{-5} & (2.849)10^{-5} & (6.050)10^{-5} \\ (4.687)10^{-5} & (5.973)10^{-5} & (1.69)10^{-4} & (3.158)10^{-5} & (6.023)10^{-5} \\ (2.285)10^{-5} & (2.849)10^{-5} & (3.158)10^{-5} & (1.276)10^{-4} & (2.961)10^{-5} \\ (7.010)10^{-5} & (6.050)10^{-5} & (6.023)10^{-5} & (2.961)10^{-5} & (2.433)10^{-4} \end{pmatrix},$$

Box II.

like to express the mean and variance of Y in terms of x. Combining (3.3), (4.2), (4.3) and trace properties, we have

$$\begin{aligned} \mathrm{E}(Y) &= a+\sum_{j=1}^{n}\lambda_j \\ &= a+\mathrm{tr}\left(U\Lambda U^T\right) \\ &= a+\frac{1}{2}\mathrm{tr}\left(\tilde{C}^T\Gamma\tilde{C}\right) \\ &= a+\frac{1}{2}\mathrm{tr}\left(\sum_{k=1}^{m}x_k\Gamma^k\Sigma\right) \\ &= a+x^Tp, \end{aligned} \tag{4.10}$$

where the vector p is defined as

$$p := \frac{1}{2}\left(\mathrm{tr}\left(\Gamma^1\Sigma\right), \mathrm{tr}\left(\Gamma^2\Sigma\right), \ldots, \mathrm{tr}\left(\Gamma^m\Sigma\right)\right)^T.$$

As for the variance, recall that with b of (4.5), it follows that (see (3.2) and (4.4))

$$\sum_{j=1}^{n} b_j^2 = b^Tb = (C^T\delta)^T C^T\delta = \delta^T C^T C\delta = \frac{1}{2}x^T\hat{\Sigma}x, \tag{4.11}$$

where

$\hat{\Sigma} = 2M^T\Sigma M$ and

$M = (M_{ik}) = (\delta_i^k), \quad i = 1, \ldots, n,\ k = 1, \ldots, m.$

Obviously, $\hat{\Sigma}$ is positive semi-definite. In light of (4.4) and trace properties it can be shown that

$$\begin{aligned} \sum_{j=1}^{n}\lambda_j^2 &= \frac{1}{4}\mathrm{tr}\left((C^T\Gamma C)^T(C^T\Gamma C)\right) \\ &= \frac{1}{4}\mathrm{tr}\left(\Gamma CC^T\Gamma CC^T\right) \\ &= \frac{1}{4}\mathrm{tr}\left(\Gamma\Sigma\Gamma\Sigma\right) \\ &= \frac{1}{4}\mathrm{tr}\left(\left(\sum_{k=1}^{m}x_k\Gamma^k\Sigma\right)^2\right) \\ &= \frac{1}{4}\left(\sum_{k=1}^{m}x_k^2\mathrm{tr}\left((\Gamma^k\Sigma)^2\right)+2\sum_{i\neq k}x_ix_k\mathrm{tr}\left(\Gamma^i\Sigma\Gamma^k\Sigma\right)\right) \\ &= \frac{1}{4}x^TQx, \end{aligned}$$

where the matrix Q is defined by

$$Q_{ik} = \mathrm{tr}\left(\Gamma^i\Sigma\Gamma^k\Sigma\right), \quad i = 1, \ldots, m,\ k = 1, \ldots, m. \tag{4.12}$$

Therefore, we end up with

$$\mathrm{Var}(Y) = \sum_{j=1}^{n}(b_j^2+2\lambda_j^2) = \frac{1}{2}x^T(\hat{\Sigma}+Q)x. \tag{4.13}$$

Thus, from (4.10) and (4.13), the portfolio problem (P3) (recall that $x = w$) becomes

$$\text{(P5)}\ \min_x \frac{1}{2}x^T(\hat{\Sigma}+Q)x$$
$$\text{s.t. } a+x^Tp = r_e,$$
$$\sum_{k=1}^{m}V_k(t,S)x_k = 1,$$

and (P4) becomes

$$\text{(P6)}\ \min_x \frac{1}{2}x^T(\hat{\Sigma}+Q)x$$
$$\text{s.t. } \sum_{k=1}^{m}V_k(t,S)x_k = 1.$$

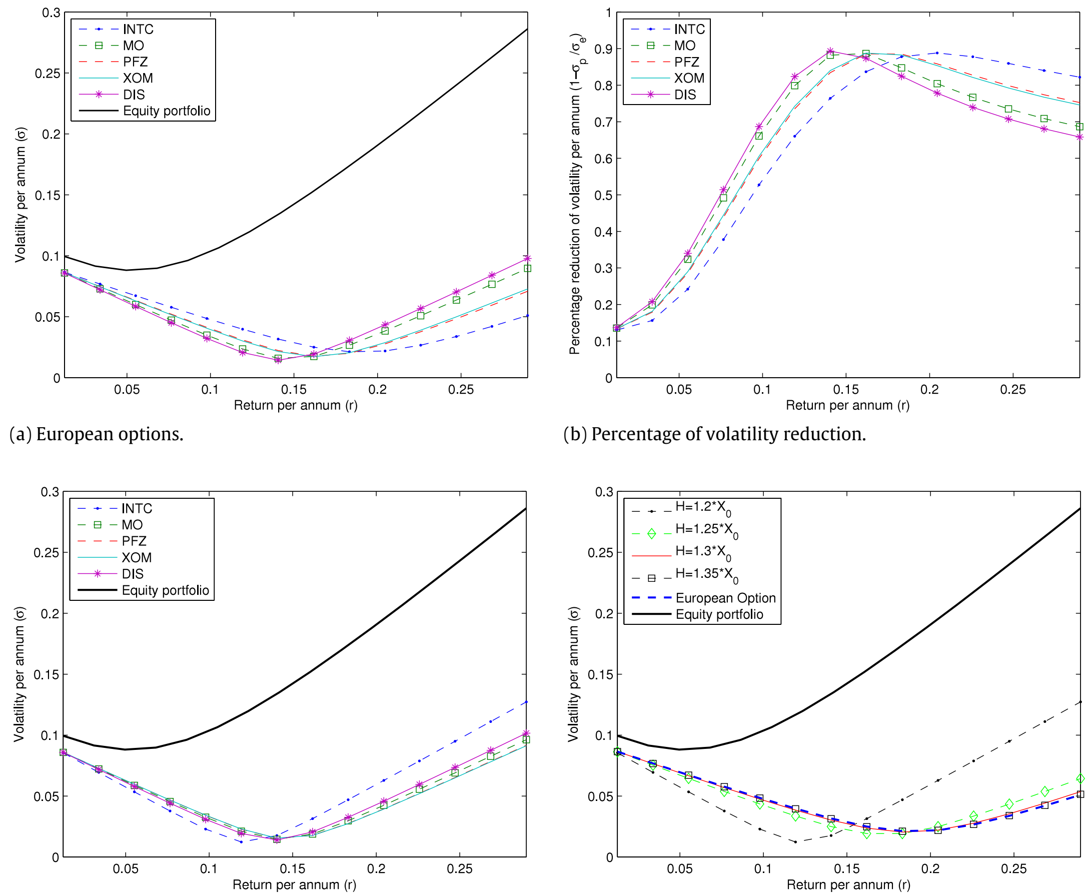
It turns out that (P5) has a similar form to the classical mean–variance portfolio problem: a quadratic objective function and linear constraints. Notice that the matrix $\hat{\Sigma}+Q$ is positive definite, since $\frac{1}{2}x^T(\hat{\Sigma}+Q)x = \mathrm{Var}(\delta V) > 0$. We wrap up our findings in the following theorem.

Theorem 4.1. (P3) *is equivalent to* (P5)*, which is a convex quadratic program, and thus solvable in polynomial time.*

5. Numerical study

In this section we examine the variance reduction achieved by adding derivative assets to equity portfolios through a numerical implementation of our earlier results. To illustrate these results, we derive optimal equity portfolios and portfolios containing both equities and equity derivatives. Following [8], we consider five major US stocks (INTC, MO, PFZ, XOM, DIS), and their daily returns between January 7th, 2002 and April 8th, 2005. Furthermore, we assume the log-returns are multivariate normal, with correlation matrix (of the five stocks derived by [8]) given in Box II: with mean vector of excess returns:

$$\mu = [0.010, 0.043, -0.045, 0.009, -0.017].$$



(a) European options.

(b) Percentage of volatility reduction.

(c) Barrier options.

(d) Various barriers.

Fig. 1. A base equity portfolio with the addition of a single option.

To implement our model we must transform between stocks and factors. The correspondence between stock, X_i, and factor, S_i, $i = 1, \dots, 5$, is:

$$X_i = \exp(S_i + \mu_i t).$$

To illustrate the variance reduction achieved by adding derivative assets to equity portfolios, we take two particular cases for (P1). In the first case, for some $1 < p < m$, we set

$$V_1(S,t) = X_1, \qquad V_2(S,t) = X_2, \dots, V_p(S,t) = X_p,$$
$$V_{p+1}(S,t) = V_{p+2}(S,t) = \cdots = V_m(S,t) = 0.$$

This is the base equity portfolio. After transformations from the delta–gamma approximation, and our results above, we recover the counterpart of (P1), (P5)—the quadratic programming problem. Let us call this optimal value of (P5) σ_e.

Our second case will be our base portfolio, as defined above, with the addition of derivative assets. This corresponds to

$$V_1(S,t) = X_1, V_2(S,t) = X_2, \dots, V_p(S,t) = X_p,$$
$$V_{p+1}(S,t), V_{p+2}(S,t), \dots, V_m(S,t)$$

are derivative pricing functionals.

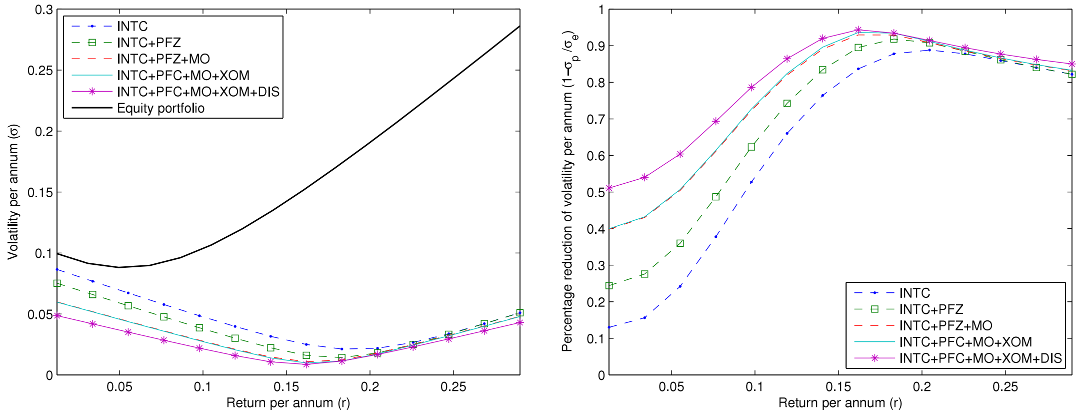
In our numerical experiments, we take the derivative pricing functional to be the BS functional on single assets from the base portfolio. For instance, given the portfolio of five equities listed above, write one European call option on one of the single assets. Here the BS functional is that for a European call written on a single equity of the underlying portfolio.

Again, after similar transformations we obtain (P5), the counterpart of (P1). Let us call this optimal value of (P5) σ_p. Our experiments, for this set of stocks, show that $\sigma_p < \sigma_e$ uniformly (i.e. under varying portfolio returns). In particular, we examine the quantity $(\sigma_e - \sigma_p)/\sigma_e$, the variance reduction as a percentage of the base equity portfolio's optimal variance.

We show our approach and results are robust by examining two different classes of equity derivatives. We first consider adding simple one month, at-the-money, European call options to equity portfolios. The proceeding analysis examines in detail the variance reduction achieved by this addition. The second class of equity derivatives we explore are barrier options. We have chosen barrier options since they are particularly amenable to this process: barrier prices and Greeks are readily available. Moreover, the asymptotic behavior of barrier options approaches their European counterparts, and provides additional insight into the variance reduction properties.

As a first step to exploring the variance reduction introduced through derivative instruments, we consider pure equity portfolios and portfolios with equities and one option. These European call options have spot/strike prices: $X_0 = [28.4, 60.01, 26.6, 65.53, 23.29]$.

We examine these portfolios in Fig. 1(a) where the bolded solid line indicates the efficient frontier of the base equity portfolio containing the five stocks mentioned above. Within the same figure, we also plot the efficient frontiers obtained by adding one European call option on one stock of the equity portfolio. As seen, the addition of this option provides significant variance reduction at any given expected return level. Fig. 1(b) plots the volatility reduction as a percentage from the base equity portfolio. More precisely, we plot $(\sigma_e - \sigma_p)/\sigma_e$, where σ_p and σ_e are the optimal volatilities from the modified portfolio (from derivative asset inclusion). The maximum variance reduction for the optimal



(a) European options.

(b) Percentage of volatility reduction.

Fig. 2. Equity portfolio + multiple options.

portfolio is approximately ∼85%–90%, and is achieved for annual portfolio returns of ∼12%–17%.

Under the same experimental setup, we add knock-out barrier options to our base equity portfolio. To test the robustness of the results the barrier, H, is set as $1.2 \times X_0$, where X_0 are the spot prices of the underlying stocks. From Fig. 1(c), the efficient frontier shows that similar variance reduction is achieved for barrier options.

In another experiment, we examine variance reduction achieved from different knock-out barrier levels, H. In Fig. 1(d), we compare the efficient frontiers of equity portfolios with European options on INTC, and equity portfolios with knock-out barrier options on INTC for different H. When $H = 1.2 \times X_0$, the variance from the barrier portfolio is quantitatively different from their European counterparts. However, as we increase the barrier, H, the efficient frontier tends to agree with its European counterpart. In particular, when $H = 1.35 \times X_0$, the efficient frontier from the European portfolio completely overlaps with the knock-out barriers. As H increases, the barrier becomes more unlikely to be realized, and gradually becomes indistinguishable from the corresponding European portfolios. Due to similar variance reduction properties between barrier and European options, we focus the remainder of our numerical analysis on portfolios containing European options.

Next, we examine the variance reduction when options on two or more stocks are added to an equity portfolio. Fig. 2(a) shows the efficient frontiers when additional European options are included on top of the equity portfolios, and Fig. 2(b) represents the volatility reduction in percentage. The unimodality is consistent with our results on equity portfolios with one derivative. Intuitively, portfolios with additional options should lead to greater variance reduction (additional degrees of freedom), and our results as shown in Fig. 2 confirm this intuition. Efficient frontiers corresponding to more derivatives have a greater variance reduction than those with fewer derivatives. However, it is interesting to note that the marginal variance reduction appears to decrease as more options are added. For example, the extra variance reduction from options on both INTC+PFZ does not exceed more than 10% of the variance reduction achieved from options on INTC alone. In the portfolio where five options are added, the extra variance reduction (here we mean the variance reduction between considering a portfolio with one option and all five options added) is significant (around 50%). This occurs when the return is close to 0.

Another interesting observation from the numerical experiment is that the marginal variance reduction effect is more significant for small expected returns than larger expected returns: the differences in the percentages of volatility reduction in Fig. 2(b) are more visible for low returns. Consistent with our observations in the first step, in all the situations considered, the maximum variance reduction in the optimal portfolio is approximately ∼85%–90% and occurs for annual portfolio returns of ∼12%–18%.

6. Conclusion and future research

This paper considers a mean–variance analysis for portfolios of primary and derivative securities in a one period model. The delta–gamma approximation is employed in order to ensure tractability. Thus, the mean–variance optimization problem is reduced to a quadratic program which is well posed. Numerical experiments exhibit the optimal portfolio variance reduction obtained by adding options to an equity portfolio.

The approach we established so far can also be applied to hedging in incomplete markets. It is well known that in incomplete markets perfect hedging is not possible. One way to solve this problem is to consider quadratic hedging; that is, minimize the variance of the hedging error.

For example, let F be a payoff of the form $F = V_1(S_{t+\Delta t}, t+\Delta t)$, for some map V_1. We would like to hedge this payoff by some instruments which are of the form $V_k(S, t),\ k = 2, \ldots, l$ (with l possible less than n, whence the incompleteness). For simplicity, assume that in this market borrowing and lending of cash is done at zero interest rate (this can be easily achieved if one takes the zero coupon bonds as numeraire). Given the number of shares $(x_1, x_2, \ldots, x_l)$ in the hedging portfolio, the hedging error is

$$-\sum_{k=1}^{l+1} x_k \Delta V_k(S, t),$$

with $x_1 = -1$, and $\Delta V_{l+1}(S, t) = 1$. Therefore, the problem of minimizing the variance of hedging error is of the form (P2). The initial amount needed to finance the hedging portfolio is

$$x_{l+1} + V_1(S_t, t).$$

We leave further elaboration on this subject for future research.

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References

[1] S. Alexander, T.F. Coleman, Y. Li, Minimizing CVaR and VaR for a portfolio of derivatives, Journal of Banking & Finance 30 (2006) 583–605.
[2] I. Aytac, M. Jonsson, R. Sircar, Optimal investment with derivative securities, Finance and Stochastics 9 (2005) 585–595.

[3] P. Carr, D. Madan, Optimal positioning in derivative securities, Quantitative Finance 1 (2001) 19–37.
[4] P. Carr, L. Wu, Static hedging of standard options, 2004, Preprint.
[5] G. Cornuejols, R. Tutuncu, Optimization Methods in Finance, Cambridge University Press, 2007.
[6] M.B. Haugh, A.W. Lo, Asset allocation and derivatives, Quantitative Finance 1 (2001) 45–72.
[7] N.J. Higham, Computing the nearest correlation matrix—a problem from finance, IMA Journal of Numerical Analysis 22 (2002) 329–343.
[8] W. Hu, A. Kercheval, Portfolio optimization for student t and skewed t return, Quantitative Finance 10 (1) (2010) 91–105.
[9] H. Markowitz, Portfolio selection, The Journal of Finance 7 (1952) 77–91.
[10] H. Markowitz, Foundations of portfolio theory, Journal of Finance 46 (1991) 469–477.
[11] R.T. Rockafellar, S. Uryasev, Optimization of conditional value-at-risk, Journal of Risk 2 (2000) 21–41.
[12] M. Schweizer, Mean–variance hedging, in: Encyclopedia of Quantitative Finance, 2010, pp. 1177–1181.