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Solution and Application of Two Inverse Kinematics Subproblems *

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***Abstract* - This paper presents two new inverse kinematics subproblems. Subproblem ①: Rotating about two parallel axes to a given point and Subproblem ②: Rotating sequentially about two parallel axes to a point at the same distance to two given points. Subproblem② can be regarded as the extension of subproblem ①. The solutions to that kind of subproblem are derived. The inverse kinematics problem of some kinds of mechanic chains with two parallel rotate axes can be converted to solve those two subproblems and others known. Based on the RRSR chain, steps of inverse kinematics solution of this kind of chain are listed, and the problems of the multi-solution are analyzed and the geometric meanings of inverse kinematics are explained. Through an instance with exact parameters, the feasibility of the method is also testified.**

Index Terms - Screw, exponential product, inverse kinematics subproblems, RRSR chains

I. Introduction

A new approach for robotic kinematic analysis appeared which integrates screw theory and exponential product in[1]~[5]. The approach unifies translation and rotation, describes motion in the base coordinate, does not require coordinate transformation, and has the advantage of avoiding singularity in the kinematics solution. It has been utilized in the kinematics, dynamics and control of robots, and has been proved to be an effective way. Existing exponential product method is mainly applied to the kinematics analyses of open chain mechanisms. The geometrical algorithm for the inverse kinematics problem can be constructed by exponential formula for forward kinematics. The algorithm has been presented by Paden B in his works [1][2]. To solve the inverse kinematics problem, the first step is to tackle inverse kinematics subproblems, and the next step is to decompose the inverse kinematics problem into several known subproblems, which have the characteristics of patent geometrical meaning and numerical stability. The subproblems which have already been solved are: (1) rotating about one axis; (2) rotating about two intersecting axes consequently; (3) rotating to a given distance; and (4) rotating two different surface beeline consequently [5]. These subproblems can be used to solve a series of inverse kinematics problems of robotic subchains. However, this method does not have universality and special subproblems should be proposed to special chain.

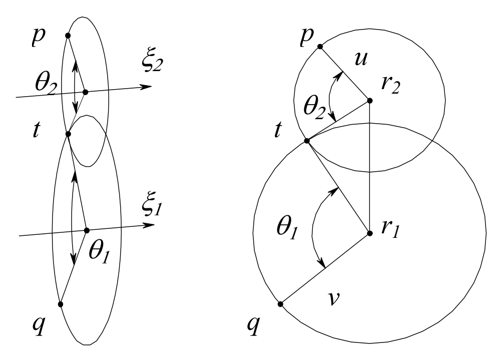
This paper presents two new inverse kinematics subproblems: Subproblem① rotating about two parallel axes to a given point; Subproblem② rotating sequentially about two parallel axes to a point at the same distance to two given points. Subproblem② can be regarded as the extension of subproblem①. The solution to that kind of subproblem is proposed later. The inverse kinematics problem of some kinds of mechanic links can be converted to solve those two subproblems and others already known. This will extend the scope of applicability of using subproblems to solve inverse kinematics.

II. Description and Solution of Subproblems ① and ②

A Solution procedure of the subproblem ①

As shown in fig.1, subproblem ① can be described as follows: assume $\xi_1$and ξ_2 as two parallel zero-pitch spinor (Their unit vectors are denoted by ω_1 and ω_2 respectively), and $p,q\in\Re^3$ are two points in the Cartesian space. Resolve θ_1 and θ_2, while p rotates around axis $\boldsymbol{\xi}_2$ to angle θ_2 and axis ξ_1 to θ_1 consequently, then coincides with the point q. In other words, resolve θ_1 and θ_2, satisfying: $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p=q$.

The solution procedure is as follows: the locus plane formed by point p rotating around the axis is denoted by Π. Because the axes of ξ_1 and ξ_2 are parallel, Π is perpendicular



(a) (b)

Fig.1 The diagrammatic sketch of subproblem ①. (a) The description of subproblem ① in space. (b) The description of subproblem ① in the projective plane Π.

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to the axes of ξ_1 and ξ_2, and the intersecting points with them are denoted by r_1 and r_2 respectively. The point where p rotates about ξ_2 θ_2 is denoted by t. Define the vectors:

$$\mu = p - r_2;\ \nu = q - r_1;\quad \kappa_1 = t - r_1,$$

$$\kappa_2 = t - r_2;\ \gamma_{12} = r_1 - r_2;\ \gamma_{21} = r_2 - r_1 \tag{1}$$

Utilizing exponent product formula to derive the following relational expressions:

$$e^{\hat{\xi}_2\theta_2}\mu = \kappa_2;\quad e^{-\hat{\xi}_1\theta_1}\nu = \kappa_1;\ \|\mu\| = \|\kappa_2\|;\quad \|\mu_1\| = \|\nu_1\| \tag{2}$$

The separation angle between the vectors u and r_{12} is denoted by θ_{02}, which can be expressed as follow:

$$\theta_{02} = \mathrm{atan2}\left[\omega^T(u \times r_{12}), u^T r_{12}\right]. \tag{3}$$

In the triangle constructed by the points t, r_1 and r_2, the included angle between the vectors z_1 and r is denoted by ϕ. Then ϕ can be computed by the law of cosines:

$$\|u\|^2 + \|r_{12}\|^2 - 2\|u\|\|r_{12}\|\cos\phi = \|v\|^2 \tag{4}$$

So θ_2 can be derived as:

$$\theta_2 = \theta_{02} \pm \mathrm{acos}\left(\frac{\|u\|^2 + \|r_{12}\|^2 - \|v\|^2}{2\|u\|\|r_{12}\|}\right) \tag{5}$$

Then θ_{01} and θ_1 can be derived in the same manner:

$$\theta_{01} = \mathrm{atan2}\left[\omega^T(v \times r_{21}), v^T r_{21}\right] \tag{6}$$

$$\theta_1 = \theta_{01} \pm \mathrm{acos}\left(\frac{\|v\|^2 + \|r_{21}\|^2 - \|u\|^2}{2\|v\|\|r_{21}\|}\right) \tag{7}$$

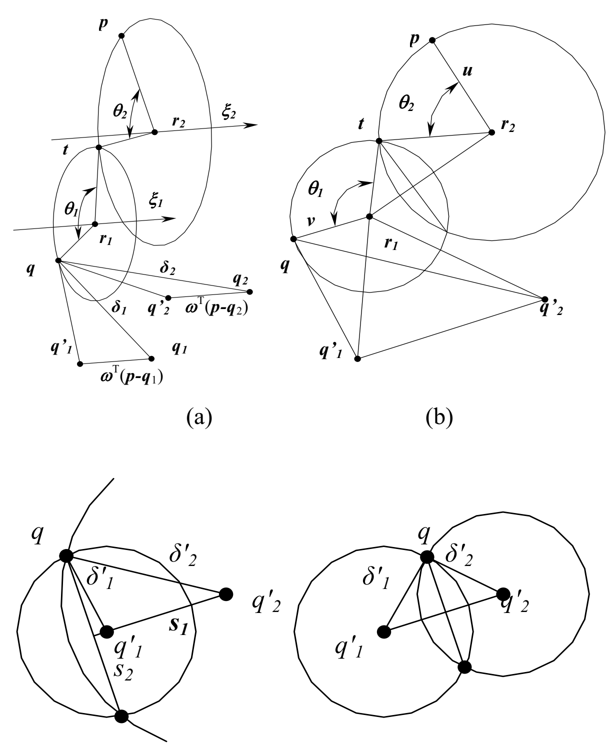

Fig.2 The diagrammatic sketch of subproblem ②. (a)(b)The spatial and planar description of subproblem② (c)(d) The graphic approach of obtaining the point q

B Solution procedure of subproblem ②

In fig.2, subproblem ② can be described as follows: solve θ_1 and θ_2, p rotates around axis ξ_2 to angle θ_2, rotates ξ_1 about θ_1, arriving at the point q with the distance to q_1 being δ_1, the distance to q_1 being δ_2. In other words, compute θ_1 and θ_2, satisfying:

$$\left\|e^{\hat{\xi}_1\theta_1}e^{\xi_2\theta_2}p - q_1\right\| = \delta_1;\ \left\|e^{\hat{\xi}_1\theta_1}e^{\xi_2\theta_2}p - q_2\right\| = \delta_2. \tag{8}$$

For geographic description, the seeked points are the intersecting points between two spheres with the center q_1 and q_2, the radii δ_1 and δ_2 and the locus plane of p. In fig.2, define points r_1 and r_2, and vectors r_{12}, r_{21}, u, v, z_1 and z_2. The projective points from the points q_1 and q_2 to the plane Π are denoted by q_1' and q_2'. Define the vectors:

$$\mu_1' = q_1 - q_1';\ \mu_2' = q_2 - q_2';\ \nu'_1 = q - q_1';$$

$$\nu'_2 = q - q_2';\ \varsigma_1 = q_1' - q_2';\ \tau_1 = \frac{\varsigma_1}{\|\varsigma_1\|};$$

$$\varsigma_2 = \omega_1 \times \tau_1;\ \tau_2 = \frac{\varsigma_2}{\|\varsigma_2\|}. \tag{9}$$

Therefore the problem of computing spatial points is converted to the problem of computing the points in the plane Π. The seeked points are the commonly intersecting points of three circulars whose centers are r_1, q_1'and q_2', whose radii are $\|v\|$, $\|v_1'\|$ and $\|v_2'\|$ respectively. From fig.2 both (c) and (d), we can get the following equations:

$$\mu_1' = \omega\omega^T(p - q_1);\ \mu_2' = \omega\omega^T(p - q_2)$$

$$\delta_1' = \|\nu_1'\| = \sqrt{\delta_1^2 - \|\mu_1'\|^2}$$

$$\delta_2' = \|\nu_2'\| = \sqrt{\delta_2^2 - \|\mu_2'\|^2} \tag{10}$$

Using the cosine rule, in the triangle with the vertexes q, q_1' and q_2', the angle at q_1' is

$$\theta_{q'} = \pm\arccos\left(\frac{\delta_1'^2 + \|\varsigma_1\|^2 - \delta_2'^2}{2\delta_1'\|\varsigma_1\|}\right) \tag{11}$$

Assumes $\nu_1' = \alpha\tau_1 + \beta\tau_2$, by vector relationship:

$$\alpha = -\delta_1'\cos(\theta_{q'});\ \beta = -\delta_1'\sin(\theta_{q'}) \tag{12}$$

After obtaining v_1', q can be obtained by equation (9). Therefore the problem can be solved by the solution procedure of subproblem ①.

III. THE APPLICATIONS OF SUBPROBLEMS

By now, there are some mechanic chains with two vicinal parallel axes in application (As shown in fig.3). In this section, an inverse kinematics of a single chain of the parallel robot described in lecture 3 will be given as an instance to show how to solve inverse kinematics by subproblems proposed above. The chain composed of R-R-S-R, where R stands for the rotative joint, S stands for the spherical joint and the axes of the first two rotative joints are parallel. The

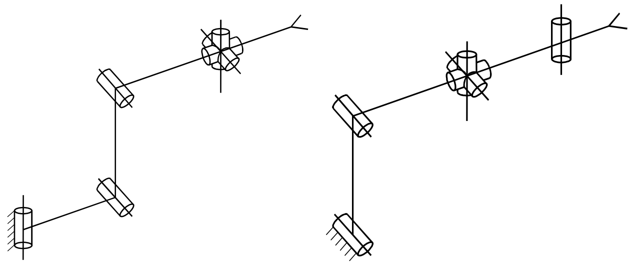

Fig.3 The mechanisms whose partial inverse kinematics can be solved by subproblem ① ② (a) the mechanic chain likes Motoman serial robot whose inverse kinematics relies on subproblem ①. (b) RRSR chain whose inverse kinematics relies on subproblem ②.

whole links have a 6-dof at the endeffector.

A. solution application

Inverse kinematics computes the active joint angles given the position and orientation of the endeffector. The inverse kinematics of subchains shown in fig.3 can be converted to the combination of subproblem① and ② and other subproblems.

According the theory of screw and exponential product, the inverse kinematics of robot chains can be described by the spinor of the joints. The statement is as follow:

Given forward kinematics mapping g_{st}: $Q \rightarrow SE(3)$ and an expected pose $g_d \in SE(3)$, by solving $g_{st}(\theta) = g_d$, obtain $\theta \in Q$.

As show in Fig.4, to solve inverse kinematics, we constitute a base coordinate system ***B***(or ***S***), in which the spinor of the joints are ξ_1、ξ_2、ξ_3、ξ_4、ξ_5 and ξ_6. The pose g_d can be expressed by the coordinate transformation of reference coordinate system ***P*** fixed on the endeffector in the base coordinate system ***B***.

Assume the initial position and orientation as g_0, use screw to depict motion. Therefore g_d can be expressed with exponential product formula:

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_0 = g_d \tag{14}$$

Where, $\hat{\xi}_i$ is the spinor, and its coordinate is ξ_i. Transform equation (1), we get:

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} = g_d g^{-1}{}_0 e^{-\hat{\xi}_6\theta_6} \tag{15}$$

Select the spherical center p, so

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} p = g_d g^{-1}{}_0 e^{-\hat{\xi}_6\theta_6} p \tag{16}$$

Because the axes of ξ_3、ξ_4 and ξ_5 are intersected, so

$$e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} p = p \tag{17}$$

Equation (16) can be written as:

$$g_0 g^{-1}{}_d e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = e^{-\hat{\xi}_6\theta_6} p \tag{18}$$

Select two points q_1 and q_2(for calculating convenience, select q_2 at the intersecting point of ξ_5 and ξ_6. Subtract q_1 and q_2 from the both sides of equation (18) respectively, obtaining the equations (19) and (20):

$$g_0 g^{-1}{}_d e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p - q_1 = e^{-\hat{\xi}_6\theta_6}(p - q_1) \tag{19}$$

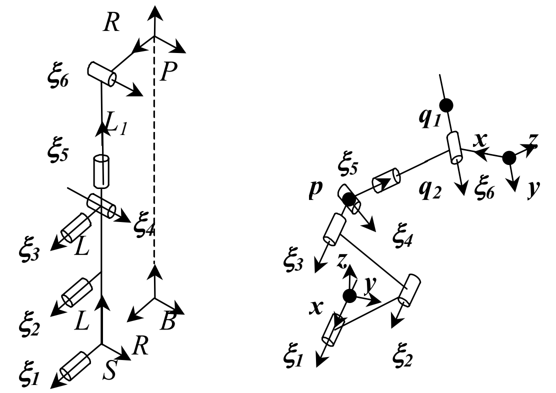

Fig.4 Two poses of RRSR chain in the solution of inverse kinematics. (a) The selected initial position and orientation. (b) The moved position and orientation

Rigid motion has the characteristic of maintaining the distance between two points. Calculate the norm of equations (21) and (22).

$$\left\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p - g_d g_0^{-1} q_1 \right\| = \left\| p - q_1 \right\| \tag{21}$$

$$\left\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p - g_d g_0^{-1} q_2 \right\| = \left\| p - q_2 \right\| \tag{22}$$

From equation (21) and (22), the calculation of θ_1 and θ_2 is converted to the inverse kinematics subproblem that the point p consequently rotates around two axis ξ_1 and ξ_2 to the point where the distances to $g_d g_0^{-1} q_1$ and $g_d g_0^{-1} q_2$ are $\|p - q_1\|$ and $\|p - q_2\|$ respectively. θ_1 and θ_2 can be obtained by utilizing the conclusion of subproblem ②. On the basis of that, by using equation (18), the calculation of θ_6 is converted to Panden-Kahan subproblem 1 (rotating around one axis). To calculate θ_3 and θ_4, select a point on the axis of ξ_5 randomly (for conveniently calculating, select the intersecting point q_2 between the axes ξ_5 and ξ_6). From equation (15), the equation of motion is given by:

$$e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} q_2 = e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g^{-1}{}_0 e^{-\hat{\xi}_6\theta_6} q_2 \tag{22}$$

It can also be rewritten as

$$e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} q_2 = e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g^{-1}{}_0 q_2 \tag{23}$$

θ_3 and θ_4 are calculated by using Paden-Kahan subproblem2 (consequently rotating around two parallel axes). Equation (23) can be written as:

$$e^{\hat{\xi}_5\theta_5} = e^{-\hat{\xi}_4\theta_4} e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g^{-1}{}_0 e^{-\hat{\xi}_6\theta_6} \tag{24}$$

Select a point not on ξ_5(p can be selected),

$$e^{\hat{\xi}_5\theta_5} p = e^{-\hat{\xi}_4\theta_4} e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_d g^{-1}{}_0 e^{-\hat{\xi}_6\theta_6} p \tag{25}$$

It can be converted to Padan-Kahan subproblem 1 to calculate θ_5.

B The analysis of the method

From the solution procedure of RRSR chain, it can be found that subproblem ② is the basis of the inverse kinematics of the whole inverse kinematics. For the parallel robot in lecture 3, after calculating θ_1 and θ_2 of each chain, the inverse kinematics of the robot is finished which will simplify the inverse kinematics of the mechanism.

The solution of subproblem② are composed of two steps: (1)to find the point q; (2)to calculate θ_1 and θ_2, using the conclusion of subproblem ①.

The point q in step (1) has three cases: the number of answers maybe one, two or zero, depending on the number of intersecting points between the circles whose centers are q_1' and q_2', radii are δ_1 and δ_2 respectively.

The step (2) which is also the solution of subproblem ① also has three cases: the solution number maybe one, two or zero, depending on the number of intersecting points between the circles whose centers are r_1 and r_2, radii are $\|\mu\|$ and $\|\nu\|$ respectively.

Therefore the number of answers of the subproblem ② maybe one, two, four or zero.

θ_6 is derived from equation (14), although θ_1 and θ_2 have four group of solution, θ_6 has only two cases because the left part of equation (18) is the solution of subproblem①. θ_6 is relied on θ_1 and θ_2. From equation (23), if the right part is fixed, θ_3 and θ_4 have two independent groups of solution at most. So θ_3 and θ_4 may have 8 solutions. From equation (25), θ_5 is relied on θ_1, θ_2, θ_3, θ_4 and θ_6. From the analysis above, we draw the conclusion than the chain has 8 independent groups of solution at most.

Besides, the calculation of θ_5 and θ_6 has a common characteristic: the number of solution depends on the number of the designated points in the sub problem rotating to the designated points. For instance, in equation (18), although θ_1 and θ_2 may have 4 groups of solutions, θ_6 has only two possible solutions. The reason is: through exponential product calculation, 4 groups of solution have two effects on points, thus achieving two different terminals. By utilizing that characteristic we can avoid repetitive calculation.

C Other Application and solution

Now, we look back to the mechanic chain similar to Motoman serial robot (Fig.3 (a)). For inverse kinematics of this chain, assumed g_0 is the initial position and orientation also, and then g_d can be expressed with equation (14). Solution to the first joint θ_1 is very simple, then solutions to the second and the third joints (θ_2 and θ_3) can be converted to subproblem ①. Select a point p on the wrist joint, after rotating around axis of joint 1, the new position q is obtained by the equation: $q = e^{\hat{\xi}_1\theta_1} p$. So the solution to θ_2 and θ_3 can be converted to the point q rotating axis of joint 2 and axis of joint 3 consequently and coinciding to the final point. According the solution procedure of subproblem ①, the results and number of solution are obtained by equations (3), (5), (6) and (7). The rest (θ_4, θ_5 and θ_6) can be done same as θ_3, θ_4 and θ_5 of the RRSR chain. Compared to the D-H method for solving θ_1, θ_2 and θ_3 in [6], the meaning of each solution shows clearly in this method.

IV. CALCULATION INSTANCE

Now an instance will be given to testify the results above mentioned. According the Fig.4, the distance from the original point of *B-xyz* to the link L is 96mm, the length of link L is 90mm, the length of the upper link L_1 is 180mm, the position and orientation of the endeffector P is given as {10, 20, 260, 10°,0°,0°}, referring to the base coordinate system *B-xyz*. Then, we can get:

$$g_d = T(x, y, z)\, R_z(\text{roll})\, R_y(\text{pitch})\, R_x(\text{yaw})$$

Where T denotes translation transformation, R denotes rotation transformation, and subscripts are the coordinate axes rotated around. Refer to fig.5 (a), we can get the followings:

$$g_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 360 \\ 0 & 0 & 0 & 1 \end{bmatrix};\; g_d = \begin{bmatrix} 0.9254 & 0.0180 & 0.3785 & 10.0 \\ 0.1632 & 0.8826 & -0.4410 & 20.0 \\ -0.3420 & 0.4698 & 0.8138 & 100 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we can obtain p, q_1 and q_2, they are described by Cartesian coordinate referring to the base system as follows:

$p(96,0,180)$, $q_1(96,0,360)$, $q_2(96,10,360)$.

From the position of p, θ_{02} can be obtained: θ_{02}=3.1416.

The included angle between the vectors μ and γ_{12} is π. According to the equation, we can get the θ_1, θ_2 and θ_6 is the follows:

θ_1(°)	θ_2(°)	θ_6(°)
-24.4609	-146.6275	3.1847
-171.0884	146.6275	3.1847

When the position and orientation of the endeffector P is given as {-10, 0, 310, 0°, 20°, 0°}, we can get the θ_1, θ_2 and θ_6 is the follows:

θ_1(°)	θ_2(°)	θ_6(°)
-87.4973	174.9946	-25.0324
87.4973	-174.9946	-25.0324

V. CONCLUSIONS

The paper presents a new category of inverse kinematics subproblems, those are subproblem① and subproblem②. The solution using geometric method is derived. The method has good universality and directly geometric meaning and can be applied to the inverse kinematics of some mechanic subchains which have two vicinal parallel rotated axes. For RRSR mechanic chain, the inverse kinematics solution by applying the proposed subproblems and the other existing subproblems are also derived. Closed solution can be obtained, and we can draw the conclusion that the number of inverse kinematics solution of this type of chain is no more than 8. We also can give the exact number of solution from the judgment of some intersecting points during procedure of derivation. Compared with D-H method, the advantages of the method are simpler algorithm, less calculation cost and faster calculation. Screw and exponent product are combined to solve inverse kinematics problems, using only two coordinate systems(the base coordinate system and component coordinate system) to describe rigid motion with explicit geographic meaning and numerical stability.

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REFERENCES

[1] B. Paden, "Kinematics and Control Robot Manipulators," *Ph. D thesis*, Department of Electrical and Computer Sciences, University of California, Berkeley, 1986.

[2] B. Paden and S. S. Sastry, "Optimal kinematic design of 6R manipulators," in *International Journal of Robotics Research*, 1988, pp. 43-61.

[3] J. Yoon, J. Ryu, "Design, Fabrication, and Evaluation of a New haptic Device Using a Parallel Mechanism" in *IEEE/ASME Transactions on mechtronics,* 2001, pp. 521-533.

[4] Z. Y Chang, X. Q. Yang, J. Z. Tan, "Application of exponential product method in spatial movement analysis of mechanism," *Mechanism Design (in Chinese)*, Vol. 7, 2002, pp. 26-28

[5] J. Zhao, Y. B. Liu, H. G. Cai, "Solution for One Type of Inverse Kinematics Sub-problem in Screw Theory and Its Application, " in *Robot (in Chinese)*, .Vol. 27, 2005. pp. 163-167

[6] D. L. Pieper, "The Kinematics of Manipulators Under Computer Control, " in *Artificial Intelligence Project memo*, No.72, 1968, Computer science department, Stanford University, Palo Alto, Calif.

[7] C. S. G. Lee, M. Ziegler, "A Geometric Approach in Solving the Inverse Kinematics of PUMA Robots," in *IEEE Trans. Aerospace and Electronic Systems*, Vol. AES-20, no. 6, 1984, pp.695-760.