Proceedings of the 2007 IEEE
International Conference on Mechatronics and Automation
August 5 - 8, 2007, Harbin, China

A Novel Approach to Deriving the Unit-Homogeneous Jacobian Matrices of Mechanisms*

Minxiu Kong, Yong Zhang†, Zhijiang Du and Lining Sun
Robotics Institute
Harbin Institute of Technology
Harbin, Heilongjiang Province, P. R. China 150080
†zhangyong212@hit.edu.cn

***Abstract*— The condition number of a mechanism's Jacobian matrix can be considered as an indicator of its dexterity. However, the unit-inhomogeneity problem existing in the Jacobian matrix of some mechanisms with both translational and rotational degrees of freedom discourages the designers from using this index. Addressing this issue, this paper furthers the earlier work and presents a new general method for deriving the dimensionally homogeneous Jacobian matrix of mechanisms. The deriving process includes two steps: deriving the conventional Jacobian matrix, and then mapping the general velocity of the end effecter onto the linear velocities of three points on the top plate plane. A 6-DOF hybrid mechanism is used to demonstrate this deriving process, and its dexterity is initially analyzed.**

Index Terms*— *Jacobian matrix, unit-homogeneous Jacobian matrix, condition number, dexterity.

I. INTRODUCTION

The condition numbers of the Jacobian matrices of mechanisms have been usually utilized to measure their dexterity. Salibury and Craig [1] as the earliest used the condition number of the Jacobian matrix as an optimization criterion to obtain ideal dimensions for the fingers of the Stanford/JPL articulated hand. However, as first pointed out by Lipkin and Duffy [2], the condition number of a Jacobian matrix whose entries bear different physical units is of little practical significance. Thereafter quite a few papers have been published in succession to target this issue. And most of these approaches can be classified into three types as follows.

Gosselin [3] presented a new homogeneous Jacobian matrix that maps the actuator velocities onto the linear velocities of two or three points on the end-effector for serial planar or spatial mechanisms, respectively. Kim and Ryu [4] presented a new formulation of a dimensionally homogeneous Jacobian matrix for parallel manipulators with a planar mobile platform by using three end-effector points that are coplanar with the mobile platform joints. Pond and Carretero [5] recently furthered the earlier work to obtain a nonredundant normalized Jacobian matrix with independent entries.

Tandirci *et al*. [6] normalized the Jacobian matrix by dividing a "characteristic length (CL)" out of all translational elements in it. The CL that minimizes the condition number of the homogeneous Jacobian matrix was dubbed as the "natural length (NL)" by Ma and Angeles [7]. And Angeles [8] and Angeles *et al*. [9] used the NL as a design variable to obtain isotropic and nearly-isotropic architectures.

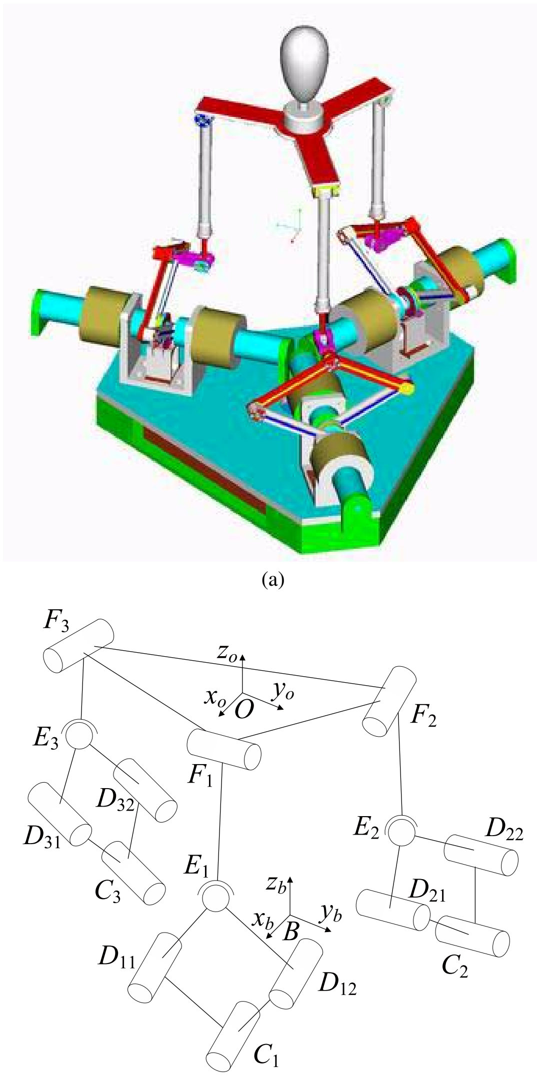
Stocco *et al*. [10], [11] presented a new design matrix normalizing technique to handle the problem of nonhomogeneous physical units and to provide a means of specifying a performance-based design goal. Then Stocco *et al*. [12] applied this technique to the optimal kinematic design of a haptic pen. Bowling and Khatib [13] developed the "dynamic capability equations (DCE)" to combine the analysis of the end-effector accelerations, velocities and forces. And one of its three major contributions is the treatment of the unit-inhomogeneity issue, namely, to map representations of translational and rotational quantities with different units onto representations of actuators with the same units.

The above three types of normalizing techniques are mathematically different and are all proved to be effective to some degree. However, it is difficult to apply the methods belonging to the first type to serial mechanisms and parallel mechanisms whose subchains are each actuated by more than one actuator. To address this issue, this paper presents a general method that can be used to obtain the homogeneous Jacobian matrix of various mechanisms. This method essentially comprises two steps. Firstly, the conventional Jacobian matrix is obtained by using the method of kinematic influence coefficients or other methods. Secondly, the general velocity of the top plate is mapped onto the translational velocities of three points which are coplanar with the top plate and centrosymmetric about the center of the top plate. Obviously, by multiplying the two mapping matrixes obtained separately through the last two steps, the normalized Jacobian matrix can be ultimately attained.

To demonstrate the deriving process, the rest of this paper presents the process of formulating the homogeneous Jacobian matrix of a three-subchain 6-DOF hybrid parallel

*This work is supported by the National High-Tech Research and Development Program (863 Program) of China under Contract Number 2002AA420100.

† Corresponding author. Tel: +86-451-86414462 ext 21; Fax: +86-451-86414174.



(a)

(b)

Fig. 1. Model of the mechanism. (a) CAD model. (b) Kinematic model.

mechanism [14] and the corresponding dexterity analysis of this mechanism.

II. Derivation of the Conventional Jacobian Matrix

This mechanism [14] illustrated in Fig. 1(a) is mainly composed of a base plate, three identical subchains and a top plate. Each subchain is composed of a pantograph mechanism driven by two servomotors fixed on the base plate, a RRR-type spherical joint, a connecting bar, and a revolute joint.

For the convenience of analysis, the RRR-type spherical joint is treated as a conventional ball-in-socket joint. Regarding one subchain, it is further assumed that the center of the spherical joint coincides with the top rotational joint of the pantograph mechanism, and that the radius of the base plate is reduced to the radius of the top plate. These assumptions are apparently rational and the reduced kinematic model is shown in Fig. 1(b).

TABLE I
Kinematic parameters

r_b	l_1	l_2	l_3	r_t
96 mm	90 mm	90 mm	180 mm	96 mm

Actually, one subchain of this hybrid mechanism can be seen as composed of two kinematic subchains. For example, there are two kinematic subchains in the first subchain, namely, $BC_1D_{11}E_1F_1O$ and $BC_1D_{12}E_1F_1O$.

The kinematic parameters include the radii of the base plate (denoted by r_b) and the top plate (r_t), the lengths of the bottom bars (l_1) and top bars (l_2) of the pantograph mechanisms, and the length of the connecting bars (l_3). And the assumed values of these parameters are shown in Table I and will be used for the dexterity analysis in Section IV.

The conventional Jacobian matrix is obtained here by the method of kinematic influence coefficients [15], [16]. The general velocity of the top plate is denoted by

$$\mathbf{V}_O = \begin{bmatrix} \mathbf{v}_O \\ \boldsymbol{\omega}_O \end{bmatrix} = \begin{bmatrix} v_{ox} & v_{oy} & v_{oz} & \omega_{ox} & \omega_{oy} & \omega_{oz} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^6 \tag{1}$$

where $\mathbf{v}_O$ and $\boldsymbol{\omega}_O$, respectively, denote the linear velocity along the three axes of the base coordinate system and the angular velocity about the three axes of the base coordinate system.

The angular velocities of the six joints of one of the six kinematic subchain are denoted by $\dot{\mathbf{q}}_i \in \mathbb{R}^6$, the general velocity of the top plate can be given by

$$\mathbf{V}_O = [\ G_O^{q_i}\ ]\,\dot{\mathbf{q}}_i \tag{2}$$

where

$$[\ G_O^{q_i}\ ] = \begin{bmatrix} \mathbf{S}_{i1} \times (\mathbf{P}_i - \mathbf{R}_{i1}) & \cdots & \mathbf{S}_{i2} \times (\mathbf{P}_i - \mathbf{R}_{i2}) \\ \mathbf{S}_{i1} & \cdots & \mathbf{S}_{i2} \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{3}$$

where, $\mathbf{S}_{ij}$ is a vector denoting axis of jth joint of the ith kinematic subchain, $\mathbf{P}_i$ is the position vector of the top plate under the coordinate system of the ith kinematic subchain, and $\mathbf{R}_{ij}$ denotes the position of the top plate under the coordinate system of the jth joint of the ith kinematic subchain.

Thereby,

$$\dot{\mathbf{q}}_i = [\ G_O^{q_i}\ ]^{-1}\,\mathbf{V}_O. \tag{4}$$

The first row of $[\ G_O^{q_i}\ ]^{-1}$ is denoted by $[\ G_O^{q_i}\ ]_{1:}^{-1}$. And the angular velocity of the actuated joints is denoted by $\dot{\boldsymbol{\theta}}$. We have

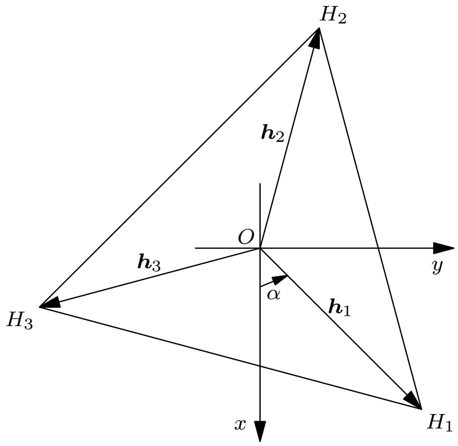


Fig. 2. Three points on the top plate plane.

$$\dot{\boldsymbol{\theta}} = \left[\ G_O^{\boldsymbol{\theta}}\ \right] \mathbf{V}_O \tag{5}$$

where

$$\left[\ G_O^{\boldsymbol{\theta}}\ \right] = \begin{bmatrix} \left[\ G_O^{q_1}\ \right]_{1:}^{-1} \\ \left[\ G_O^{q_2}\ \right]_{1:}^{-1} \\ \vdots \\ \left[\ G_O^{q_6}\ \right]_{1:}^{-1} \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{6}$$

which is the so-called conventional Jacobian matrix.

For further use, Eq. 6 is rewritten as

$$\mathbf{V}_O = \left[\ G_O^{\boldsymbol{\theta}}\ \right]^{-1} \dot{\boldsymbol{\theta}}. \tag{7}$$

III. Derivation of the Dimensionally Homogeneous Jacobian Matrix

After mapping the angular velocities of the servomotors onto the general velocity of the top plate, we now try to map the known general velocity to the linear velocity of three points which are on the plane of the top plate and centrosymmetric about the center of the top plate.

As shown in Fig. 2, the three points on the top plate plane are denoted by H_i $(i = 1, 2, 3)$, and the vectors $\overrightarrow{OH_i}$ are denoted by $\mathbf{h}_i$.

Under the mobile coordinate system (O–$x_o y_o z_o$), the vectors $\mathbf{h}_i$ can be expressed as

$$\mathbf{h}_{i_O} = \begin{bmatrix} h\cos(\alpha + \frac{2}{3}\pi) & h\sin(\alpha + \frac{4}{3}\pi) & 0 \end{bmatrix}^{\mathrm{T}} \tag{8}$$

where h is the length of $\mathbf{h}_i$.

Then $\mathbf{h}_i$ under the base coordinate system (B–$x_o y_o z_o$) can be given by

$$\mathbf{h}_{i_B} = \mathcal{R}\mathbf{h}_{i_O} = \begin{bmatrix} h_{ix} & h_{iy} & h_{iz} \end{bmatrix}^{\mathrm{T}} \tag{9}$$

where $\mathcal{R} \in \mathbb{R}^{3\times 3}$ is the rotation transformation matrix of the top plate.

Thereby the velocity of H_i can be expressed as

$$\begin{aligned} \mathbf{v}_{H_i} &= \mathbf{v}_O + \boldsymbol{\omega}_O \times \mathbf{h}_{i_B} \\ &= \mathbf{v}_O - \hat{\mathbf{h}}_{i_B}\boldsymbol{\omega}_O \end{aligned} \tag{10}$$

where

$$\hat{\mathbf{h}}_{i_B} \overset{\text{def}}{=} \begin{bmatrix} 0 & -h_{iz} & h_{iy} \\ h_{iz} & 0 & -h_{ix} \\ -h_{iy} & h_{ix} & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}. \tag{11}$$

Rewriting the expression of $\mathbf{v}_{H_i}$ in matrix form gives

$$\mathbf{v}_{H_i} = \begin{bmatrix} \mathbf{I}_3 & -\hat{\mathbf{h}}_{i_B} \end{bmatrix} \mathbf{V}_O \tag{12}$$

where

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{13}$$

Writing the expressions of $\mathbf{v}_{H_1}$, $\mathbf{v}_{H_2}$ and $\mathbf{v}_{H_3}$ into an integrated matrix form gives

$$\mathbf{V}_H = \mathcal{J}_r \mathbf{V}_O \tag{14}$$

where

$$\mathbf{V}_H = \begin{bmatrix} \mathbf{v}_{H_1} \\ \mathbf{v}_{H_1} \\ \mathbf{v}_{H_1} \end{bmatrix} \in \mathbb{R}^9 \tag{15}$$

and

$$\mathcal{J}_r = \begin{bmatrix} \mathbf{I}_3 & -\hat{\mathbf{r}}_1 \\ \mathbf{I}_3 & -\hat{\mathbf{r}}_2 \\ \mathbf{I}_3 & -\hat{\mathbf{r}}_3 \end{bmatrix} \in \mathbb{R}^{9\times 6}. \tag{16}$$

Therefore, inserting Eq. (7) into Eq. (14) gives

$$\mathbf{V}_H = \mathcal{J}\dot{\boldsymbol{\theta}} \tag{17}$$

where

$$\mathcal{J} = \mathcal{J}_r \left[\ G_O^{\boldsymbol{\theta}}\ \right]^{-1} \in \mathbb{R}^{9\times 6} \tag{18}$$

directly maps $\dot{\boldsymbol{\theta}}$ onto $\mathbf{V}_H$ and is the normalized Jacobian matrix.

IV. Dexterity Analysis

The dexterity of the mechanism can be measured by the condition number of the normalized Jacobian matrix, i.e.,

$$\kappa(\mathcal{J}) = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{19}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ respectively represent the maximum singular value and maximum singular value of the nonsquare normalized Jacobian matrix. And $\kappa(\mathcal{J})$ depends only on the pose of the mechanism when the architecture and kinematic parameters of the mechanism are already determined. A

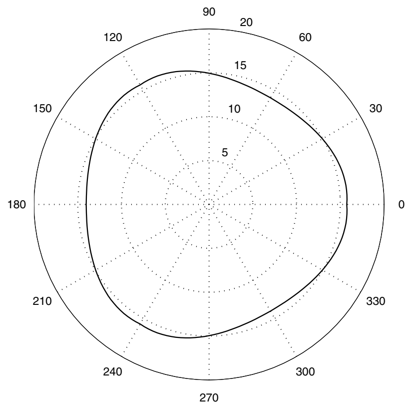


Fig. 3. Condition number representation when the top plate center moves on a circle.

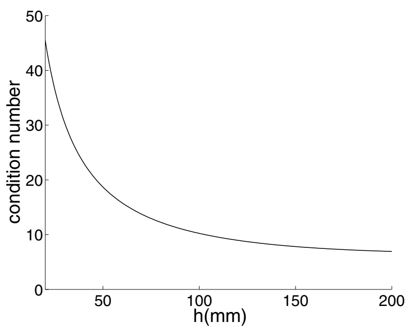


Fig. 4. Condition number varying with the radius h.

pose with smaller condition number corresponds to better dexterity.

With the top plate's orientation described by the Euler angles $(0\ \ 0\ \ 0)$, the center of the top plate moves on a circle whose center is at $(0\ 0\ 270\,\mathrm{mm})$ and radius is $60\,\mathrm{mm}$. The corresponding dexterity curve under the polar coordinate system is shown in Fig. 3, with the polar angle representing the position of the top plate and the polar radius representing the condition number. It can be noticed that polar angles $60°$, $180°$, and $300°$ corresponds to smaller condition numbers, which indicates that mechanism at these poses has better dexterity.

Although the elements of the homogeneous Jacobian matrix vary with the value of the angle α, the corresponding condition number is not affected.

However, the radius h could influence not only the elements of the Jacobian matrix but also the condition number. As far as this property is concerned, the first type of normalizing technique is similar to the second type, since the characteristic length has a similar effect on the Jacobian matrix.

When the center of the top plate is at $(0\ \ 0\ \ 270\,\mathrm{mm})$, the condition number of the normalized Jacobian matrix varies with the radius h, which is shown in Fig. 4. As h increases,

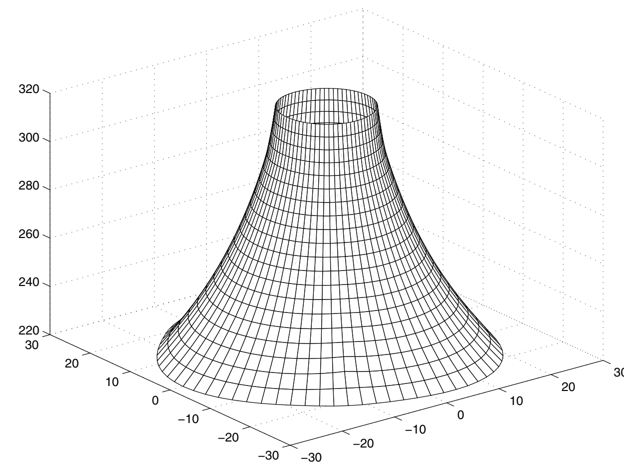


Fig. 5. Condition number representation when the top plate center moves on a cylinder.

the condition number decreases and its change rate decreases and tends towards 0. And when h is larger than 100 mm, the variation rate becomes moderate.

The center of the top plate moves on a cylinder whose radius is specified. And the three-dimensional (3-D) graphical representation of the corresponding condition numbers is obtained and shown in Fig. 5, with the z coordinate being the height of the top plate. It indicates that when the top plate moves higher, the dexterity of the mechanism improves.

Then the joystick rotates about the axis y_O by a specified angle γ, and rotates about the axis z_B by an angle β which ranges from 0 to 2π. This orientation can also be described by a modified set of Euler angles [17] [18], i.e., with the roll angle being 0 and the tilt angle being specified, the azimuth angle ranges from 0 to 2π. The advantages of this modified set of Euler angle were presented in [19]. With $\gamma = 5°$, the corresponding planar and 3-D graphical representations are respectively shown in Fig. 6 and Fig. 7. The 3-D representation also shows that the larger height of the top plate means better dexterity.

V. Conclusions

A general two-step method has been proposed which normalizes the conventional Jacobian matrix. Although a specific 6-DOF hybrid parallel mechanism was used to demonstrate this process, the deriving process can be applied consistently to serial and parallel mechanisms.

The graphical representations of the condition numbers the Jacobian matrix can be drawn and used to indicate that the performance of the mechanism at some poses is relatively better. Therefore they can be further used to write a guidebook of the mechanisms for the users. The normalized Jacobian matrix can also be further used to

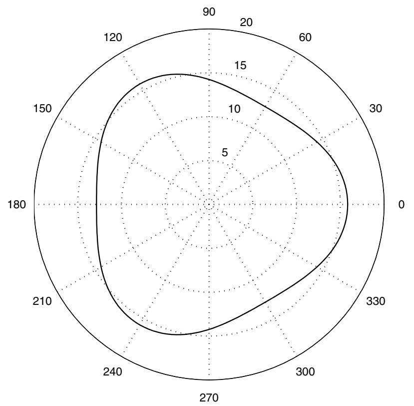


Fig. 6. Condition number representation when the top plate is at $(0\ 0\ 270\,\text{mm})$ and has a specified tilt angle $5°$.

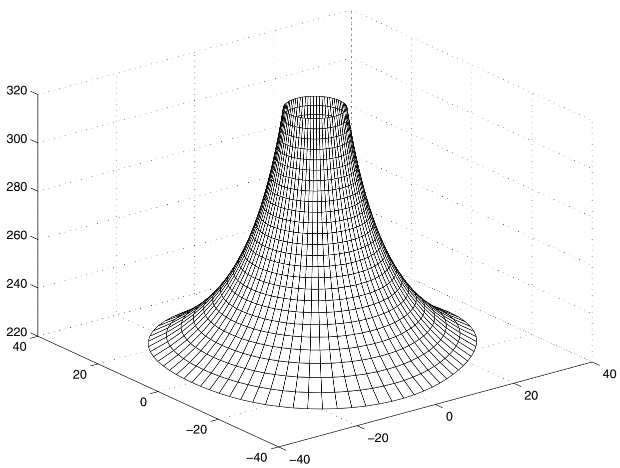


Fig. 7. Condition number representation when when the top plate is at $(0\ 0\ z)$ and has the specified tilt angle $5°$ and moves vertically.

optimize parameters or obtain dexterous workspace.

Acknowledgment

The authors wish to thank Liang for his advice concerning the CAD modeling of the mechanism.

References

[1] J. K. Salisbury and J. J. Craig, "Articulated hands: force control and kinematic issues," *The International Journal of Robotics Research*, vol. 1, no. 1, pp. 4–17, 1982.

[2] H. Lipkin and J. Duffy, "Hybrid twist and wrench control for a robotic manipulator," *ASME J. Mech. Trans. Automat. Des.*, vol. 110, pp. 138–144, June 1988.

[3] C. M. Gosselin, "Dexterity indices for planar and spatial robotic manipulators," in *Proc. IEEE Int. Conf. Robot. and Automat.*, Cincinnati, OH, May 1990, pp. 650–655.

[4] S-G. Kim and Jeha Ryu, "New dimensionally homogeneous jacobian matrix formulation by three end-effector points for optimal design of parallel manipulators," *IEEE Trans. Robot. Automat.*, vol. 19, no. 4, pp. 731–737, Aug. 2003.

[5] G. Pond and J. A. Carretero, "Formulating jacobian matrices for the dexterity analysis of parallel manipulators," *Mechanism and Machine Theory*, vol. 41, pp. 1505–1519, Dec. 2006.

[6] M. Tandirci, J. Angeles, and F. Ranjbaran, "The characteristic point and characteristic length of robotic manipulators," in *Proc. ASME 22nd Biennial Conf. Robotics, Spatial Mechanics, Mechanical Systems*, Scottsdale, AZ, Sept. 1992, pp. 203–208.

[7] O. Ma and J. Angeles, "Optimum architecture design of platform manipulators," in *Proc. IEEE Int. Conf. Adv. Robot.*, 1991, pp. 1130–1135.

[8] J. Angeles, "Kinematic isotropy in humans and machines," in *Proc. IFTOMM 9th World Congr. Theory Mach. Mech.*, Milan, Italy, Aug. 29–Sept. 2, 1995, vol. 1, pp. XLII–XLIX.

[9] J. Angeles, F. Ranjbaran, and R. V. Ratel, "On the design of the kinematic structure of seven-axes redundant manipulators for maximum conditioning," in *Proc. IEEE Int. Conf. Robot. and Automat.*, Nice, France, May 10–15, 1992, pp. 494–499.

[10] L. J. Stocco, S. E. Salcudean, and F. Sassani, "Matrix normalization for optimal kinematic design," in *Proc. IEEE Int. Conf. Robot. and Automat.*, Leuven, Belgium, May 16–21, 1998, pp. 1–6.

[11] L. J. Stocco, S. E. Salcudean, and F. Sassani, "On the use of scaling matrices for task-specific robot design," *IEEE Trans. Robot. Automat.*, vol. 15, no. 5, pp. 958–965, Oct. 1999.

[12] L. J. Stocco, S. E. Salcudean, and F. Sassani, "Optimal kinematic design of a haptic pen," *IEEE/ASME Trans. Mechatronics*, vol. 6, no. 3, pp. 210–220, Sept. 2001.

[13] A. Bowling and O. Khatib, "The dynamic capability equations: a new tool for analyzing robotic manipulator performance," *IEEE Trans. Robotics*, vol. 21, no. 1, pp. 115–123, Feb. 2005.

[14] J. Yoon and J. Ryu, "Design, fabrication, and evaluation of a new haptic device using a parallel mechanism," *IEEE/ASME Trans. Mechatronics*, vol. 6, no. 3, pp. 221–233, Sept. 2001.

[15] Z. Huang, "Modeling formulation of 6-DOF multi-loop parallel manipulators, part–1: kinematic influence coefficients," in *Proc. 4th IFTOMM Int. Symposium on Linkage and Computer Aided Design Methods*, Bucharest, Romania, 1985, vol. II–1, pp. 155–162.

[16] Z. Huang, "Modeling formulation of 6-DOF multi-loop parallel manipulators, part–2: kinematic influence coefficients," in *Proc. 4th IFTOMM Int. Symposium on Linkage and Computer Aided Design Methods*, Bucharest, Romania, 1985, vol. II–1, pp. 163–170.

[17] I. A. Bonev and J. Ryu, "Orientation workspace analysis of 6-DOF parallel manipulators," in *Proc. ASME Design Engineering Tech. Conf.*, Las Vegas, Nevada, USA, Sept. 1999, pp. 1–8.

[18] I. A. Bonev and J. Ryu, "A new approach to orientatin workspace analysis of 6-dof parallel manipulators," *Mechanism and Machine Theory*, vol. 36, pp. 15–28, Jan. 2001.

[19] I. A. Bonev and C. M. Gosselin, "Advantages of the modified euler angles in the design and control of pkms," in *Proc. 2002 Parallel Kinematic Machines Int. Conf.*, Chemnitz, Germany, Apr. 2002, pp. 171–188.