ECO 2020 Tutorial 2

(3) 1) $(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$

If $\rho \to 1 \Rightarrow \alpha_1 x_1 + \alpha_2 x_2$ linear

If $\rho \to 0$

$$y = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$$

$$\log y = \frac{1}{\rho} \log(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)$$

$$\log y = \lim_{\rho \to 0} \frac{\log(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho}$$

$$= \lim_{\rho \to 0} \frac{\alpha_1 x_1^\rho \log x_1 + \alpha_2 x_2^\rho \log x_2}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho}$$

$$\log y = \alpha_1 \log x_1 + \alpha_2 \log x_2$$

$$= \log(x_1^{\alpha_1} x_2^{\alpha_2})$$

$$\Rightarrow y = x_1^{\alpha_1} x_2^{\alpha_2}$$

If $\rho \to -\infty$ Assume $x_1 \leq x_2$

$$\lim_{\rho \to \infty} x_1 \left(\alpha_1 + \alpha_2 \left(\frac{x_2}{x_1}\right)^\rho\right)^{\frac{1}{\rho}} = \lim_{\rho \to -\infty} x_1 \, \alpha^{\frac{1}{\rho}} = x_1$$

2) a) $$x_1(p, w) = \frac{w}{p_1 + p_2 \left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho - 1}}}$$

$$x_2(p, w) = \frac{w}{p_1 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho - 1}} + p_2}$$

$$v(p, w) = \left(p_1^{\frac{\rho}{\rho - 1}} + p_2^{\frac{\rho}{\rho - 1}}\right)^{\frac{1-\rho}{\rho}} w$$

b) i) HOD 0 $$x_1(\alpha p, \alpha w) = \frac{\alpha w}{\alpha p_1 + \alpha p_2 \left(\frac{\alpha p_2}{\alpha p_1}\right)^{\frac{1}{\rho - 1}}} = x_1(p, w)$$

(ii) $p_1 x_1(p, w) + p_2 x_2(p, w) = w$

iii) $u(x) = (x_1^{\rho} + x_2^{\rho})^{\frac{1}{\rho}}$ strictly quasiconcave

① concave $\rightarrow$ quasiconcave

② increasing transf on quasiconcave $\rightarrow$ quasiconcave
decreasing ———— $\rightarrow$ quasi convex

① $\rho = 1$, $u(x) = x_1 + x_2$ ✓

② $0 < \rho < 1$, $x_1^{\rho} + x_2^{\rho}$ strictly concave in x

$u(x) \rightarrow$ strictly quasi concave

~~③~~ $\rho > 1$, $x_1^{\rho} + x_2^{\rho}$ strictly convex in x

$u(x) \rightarrow$ strictly quasiconvex

④ $\rho < 0$, $x_1^{\rho} + x_2^{\rho}$ strictly convex

$u(x) \rightarrow$ strictly quasiconcave

$\therefore$ $\rho \leq 1 \rightarrow u(x)$ strictly quasiconcave

iv) $$\frac{\partial v(p,w)}{\partial w} = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} > 0$$

$$\frac{\partial v(p,w)}{\partial p_i} = w\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} \cdot p_i^{\frac{1}{\rho-1}} > 0 \quad \forall i$$

v) $\{(p, w) : v(p, w) \leq \bar{v}\}$ Convex for any $\bar{v}$

As $v(p, w)$ HOD 0

$\{(\frac{p}{w}, 1) : v(\frac{p}{w}, 1) \leq \bar{v}\}$

$\{p \in \mathbb{R}^2 : v(p, 1) \leq \bar{v}\}$ convex

① If $\rho = 1$, $v(p, w) = \max\{\frac{w}{p_1}, \frac{w}{p_2}\}$ quasi convex

② $0 < \rho < 1$, $p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ convex in p,

$v(p,w)$ quasiconvex in p.

③ $\rho < 0$, $1 p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ concave in p

$v(p,w)$ quasiconvex in p

vi) obvious

c) linear utility $u(x) = x_1 + x_2$

$$x(p,w) = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } p_1 < p_2 \\ \left(x_1, \frac{w - p_1 x_1}{p_2}\right) & \text{if } p_1 = p_2 \\ \left(0, \frac{w}{p_2}\right) & \text{if } p_1 > p_2 \end{cases}$$

$$v(p,w) = \max\left\{\frac{w}{p_1}, \frac{w}{p_2}\right\}$$

If $p_1 < p_2$, $\lim_{\rho \to 1^-} x_1(p,w) = \lim_{\rho \to 1^-} \frac{w}{p_1 + p_2\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}}} = \frac{w}{p_1}$

If $p_1 = p_2$ $\lim_{\rho \to 1^-} x_1(p,w) = \lim_{\rho \to 1^-} \frac{w}{2p_1} \in \left[0, \frac{w}{p_1}\right]$

If $p_1 < p_2$ $\lim_{\rho \to 1^-} \frac{w}{p_1}\left(\frac{1}{p_1} + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} = \frac{w}{p_1}$

If $p_1 = p_2$ $v(p,w) = 2^{\frac{1-\rho}{\rho}} p_1^{-1} w \to \frac{w}{p_1}$

$u(x_1, x_2) = \min\{x_1, x_2\}$

$$x(p,w) = \left(\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}\right)$$

$$v(p,w) = \frac{w}{p_1+p_2}$$

$$\lim_{\rho \to -\infty} x_1(p,w) = \frac{w}{p_1 + p_2\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}}} = \frac{w}{p_1+p_2}$$

$$\lim_{\rho \to -\infty} v(p,w) = \lim_{\rho \to -\infty} w\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} = \frac{w}{p_1+p_2}$$

$$\frac{x_1(p,w)}{x_2(p,w)} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

$$\varepsilon_{12}(p,w) = \frac{1}{1-\rho}$$

$$\lim_{\rho \to 1^-} \varepsilon_{12}(p,w) \to \infty$$

$$\lim_{\rho \to -\infty} \text{———} \to 0$$

$$\lim_{\rho \to 0} \text{———} \to 1$$

similar for h_2

$$h_1(p,u) = u\left(\frac{\alpha_2}{p_2}\right)^{\frac{1}{1-\rho}} \left(\alpha_1^{\frac{1}{1-\rho}} p_1^{\frac{\rho}{\rho-1}} + \alpha_2^{\frac{1}{1-\rho}} p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}$$

$$e(p,u) = u\left(p_1\left(\frac{\alpha_2}{p_1}\right)^{\frac{1}{1-\rho}} + p_2\left(\frac{\alpha_1}{p_1}\right)^{\frac{1}{1-\rho}} \left(\alpha_1^{\frac{1}{1-\rho}} p_1^{\frac{\rho}{\rho-1}} + \alpha_2^{\frac{1}{1-\rho}} p_2^{\frac{\rho}{\rho-1}}\right)\right)$$

$$\frac{\partial h_i(p,u)}{\partial p_i} = u\left(\frac{\alpha_j}{p_j}\right)^{\frac{1}{1-\rho}} \left(\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}} + \alpha_j^{\frac{1}{1-\rho}} p_j^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}-1} \frac{1}{1-\rho}\left(\frac{\alpha_i}{p_i}\right)^{\frac{1}{1-\rho}}$$

$j = i$

$\forall\ i = 1,2,\ j \neq i$

$$\frac{\partial h_i(p,u)}{\partial p_i} = u\left(\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{\rho}{\rho-1}} + \alpha_j\, p_j^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}-1} \frac{1}{1-\rho}\left(\frac{\alpha_i \alpha_j}{p_i p_j}\right)^{\frac{1}{1-\rho}}$$

$$\left(1 - \frac{p_i}{p_j} - \left(\frac{\alpha_j p_i}{\alpha_i p_j}\right)\right)^{\frac{1}{1-\rho}}$$

④ $V(p_x, p_y, w) = \max_{x,y} h(x) g(y)$ s.t. $p_x x + p_y y \le w$

$\Longleftrightarrow \max_{w_x} V_h(p_x, w_x) V_g(p_y, w - w_x)$

Cannot explain these intuitively, to show $A = \max_x B(x)$

need :

① $\forall x$ $A \geq B(x)$

② $\exists x$ $A \leq B(x)$

$V(p_x, p_y, w) \le V_h(p_x, w_x) V_g(p_y, w - w_x)$

Let x^* and y^* be $(x^*, y^*) = \arg\max$

$p_x x^* = w_x \Rightarrow p_y y^* = w - w_x$

$x^* \in B(p_x, w_x) \Longrightarrow h(x^*) \le V_h(p_x, w_x)$

$y^* \in B(p_y, w - w_x) \Longrightarrow g(y^*) \le V_g(p_y, w - w_x)$

$V(p_x, p_y, w) = h(x^*) g(x^*) \le V_h(p_x, w_x) V_g(p_y, w - w_x)$

$V(p_x, p_y, w) \geq V_h(p_x, w_x) V_g(p_y, w - w_x)$

Let $x^* \in X(p_x, w_x)$ and $y^* \in X(p_y, w - w_x)$

$V_h(p_x, w_x) = h(x^*)$

$V_g(p_y, w - w_x) = g(y^*)$

$V(p_x, p_y, w) \geq h(x^*) g(y^*) = V_h(p_x, w_x) V_g(p_y, w - w_x)$

$e(p_x, p_y, w) = \min(e(p_x, w_x) + e(p_y, w - w_x))$

$e(p_x, p_y, u) = \min p_x x + p_y y$ s.t. $h(x) g(y) \geq u$

Let x^* and y^* $h(x^*) = u_x$ and $g(y^*) = \frac{u}{u_x}$

$p_x x^* \geq e(p_x, u_x)$

$p_y y^* \geq e(p_y, \frac{u}{u_x})$

$e(p_x, p_y, u) \geq e_h + e_g$

$$e(p_x, p_y, u) \leq e(p_x, u_x) + e(p_y, \frac{u}{u_x})$$

x^* and y^*

$$e(p_x, p_y, u) \leq p_x x^* + p_y y^*$$

$$e(p_x, u_x) \qquad e(p_y, \frac{u}{u_x})$$

$$u(x_0 \ldots x_L) = x_0^{1-\alpha} \left(\sum_{l=1}^{L} x_l^{\beta} \right)^{\frac{\alpha}{\beta}}$$

$$h: \quad \max_{x_0} x_0^{1-\alpha} \quad s.t. \quad p_0 x_0 \leq w_0 \implies x_0 = \frac{w_0}{p_0}$$

$$V_h = \left(\frac{w_0}{p_0}\right)^{1-\alpha}$$

$$g: \quad \max_{x_1 \ldots x_L} \left(\sum_{l=1}^{L} x_l^{\beta} \right)^{\frac{\alpha}{\beta}} \quad s.t. \quad \sum p_l x_l \leq w_x = w - w_0$$

$$\mathcal{L}: \left(\sum x_l^{\beta} \right)^{\alpha/\beta} - \lambda \left(\sum p_l x_l - w_l \right)$$

$$x_k = \frac{w_x}{\sum p_l^{\frac{\beta}{\beta-1}}} \cdot p^{\frac{1}{\beta-1}}$$

possible to have $x_l = 0$ for a subset of the goods.

$$V_g = w_x^{\alpha} \left(\sum p_l^{\frac{\beta}{\beta-1}} \right)^{\frac{\alpha}{\beta} - \alpha}$$

$$V(p_x, p_y, w) = \max_{w_0} \left(\frac{w_0}{p_0}\right)^{1-\alpha} (w - w_0)^{\alpha} \left(\sum p_l^{\frac{\beta}{\beta-1}} \right)^{\frac{\alpha}{\beta} - \alpha}$$

$$w_0 = (1-\alpha) w, \qquad w_x = w - w_0$$

$$h: \; e_h(p_0, u_0) = p_0 \cdot u_0^{\frac{1}{1-\alpha}}$$

$$g: \; e_g(p_x, u_x), \quad u_x = \frac{u}{u_0}$$

$$\min \sum p_l x_l \quad s.t \quad \sum x_l^{\beta} \geq (u_x)^{\beta/\alpha}$$

$$\mathcal{L} = \sum p_l x_l - \lambda \left(\sum x_l^{\beta} - u_x^{\beta/\alpha} \right)$$

$$X_l = \frac{u_x^{\frac{1}{\alpha}}}{\left(\sum p_l^{\frac{\beta}{\beta-1}}\right)^{\frac{1}{\beta}}} \cdot p_l^{\frac{1}{\beta-1}}$$

$$e(p_x, u_x) = u_x^{\frac{1}{\alpha}} \cdot \left(\sum p_l^{\frac{\beta}{\beta-1}}\right)^{1-\frac{1}{\beta}}$$

$$e(p_x, p_y, u) = \min_{u_0} \; p_0 \cdot u_0^{\frac{1}{1-\alpha}} + \left(\frac{u}{u_0}\right)^{\frac{1}{\alpha}} \left(\sum p_l^{\frac{\beta}{\beta-1}}\right)^{1-\frac{1}{\beta}}$$

$$u_0 = \left(\frac{\frac{1}{\alpha} u^{\frac{1}{\alpha}} \left(\sum p_l^{\frac{\beta}{\beta-1}}\right)^{1-\frac{1}{\beta}}}{\frac{1}{1-\alpha} p_x} \right)^{\left(\frac{1}{1-\alpha} + \frac{1}{\alpha}\right)^{-1}}$$

5 (1) $B_1 = \{(f, c) : f + c \le w + 10s\}$

$B_2 = \{(f, c) : f + c \le w + fs\}$

we want to show that

if $f \geq 10$, $B(w, s) = B_1$

if $f < 10$, $B(w, s) = B_2$

Notice that when $f \geq 10$, $B_1 \subseteq B_2$

when $f < 10$, $B_2 \subseteq B_1$

$$B(w, s) = B_1 \cap B_2 = \begin{cases} B_1 & \text{if } f \geq 10 \\ B_2 & \text{if } f < 10 \end{cases}$$

(2) $u(f, c) = \alpha \log f + \log c$, $\alpha > 0$, $w > 0$

$$\text{s.t.} \quad \begin{cases} f + c \le w + 10s \\ f + c \le w + fs \end{cases}$$

$$\mathcal{L} = \alpha \log f + \log c + \lambda_1 (w + 10s - f - c) + \lambda_2 (w + fs - f - c)$$

FOC: $\dfrac{\partial \mathcal{L}}{\partial f} = \alpha \cdot \dfrac{1}{f} - \lambda_1 + \lambda_2 (s - 1) = 0$

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda_1 - \lambda_2 = 0$$

$$\lambda_1 (w + 10s - f - c) = 0 \qquad \text{with} \quad \lambda_1 \geq 0$$

$$\lambda_2 (w + f(s-1) - c) = 0 \qquad \lambda_2 \geq 0$$

(i) $\lambda_1 > 0, \lambda_2 > 0$

(a) $s \neq 0$, $\begin{cases} f = 10 \\ c = w + 10(s-1) \end{cases}$

(b) $s = 0$ $\begin{cases} f = \frac{\alpha w}{1+\alpha} \\ c = \frac{w}{1+\alpha} \end{cases}$

(ii) $\lambda_1 > 0, \lambda_2 = 0$ $\begin{cases} f = \frac{\alpha(w + 10s)}{1+\alpha} \\ c = \frac{w + 10s}{1+\alpha} \end{cases}$

Need conditions on parameters α, w, s such that $(f, c) \in B$

(iii) $\lambda_1 = 0, \lambda_2 > 0$

(a) $s \neq 1$ $\begin{cases} f = \frac{\alpha}{1+\alpha} \cdot \frac{w}{1-s} \\ c = \frac{w}{1+\alpha} \end{cases}$

(b) $s = 1$ $w = 0$, which is a contradiction.

(iv) $\lambda_1 = 0, \lambda_2 = 0$ impossible.