ECO2020 Tutorial 2

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1 Question 1

1.1 Part a

Let $f(i) = (f_1(i), f_2(i), ..., f_L(i))$

Pareto Efficient allocation requires:

1. Individual feasibility:

$$f_j(i) \geq 0 \; \forall \; i \in [0,1], j \in \{1,2,...,L\}$$

2. Aggregate feasibility:

$$\int_0^1 f(i)\, di = \omega$$

   where $\omega = \displaystyle\int_0^1 \omega(i)\, di$

3. Optimality:

   There does not exist $\left(\tilde{f}\right)$ such that:

$$\tilde{f}(i) \succsim_i f(i) \text{ for } i \in [0,1]$$

   with strict inequality on some non-empty open interval $(a,b) \in [0,1]$.

   It is not enough to have strict inequality for one single consumer (or on any measure-zero subset of $[0,1]$).

1.2 Part b

Consider the WE (p, f),

Suppose, for a contradiction, there exists $\tilde{f}$ such that

$$\tilde{f}(i) \succsim_i f(i) \text{ for } i \in [0,1]$$
$$\tilde{f}(i) \succ_i f(i) \text{ for } i \in (a,b) \text{ for some } a, b$$

with strict inequality on some non-empty open interval $(a,b) \in [0,1]$.

Then it can be shown using LNS that,

$$p \cdot \tilde{f}(i) \geq p \cdot f(i) \ \forall \ i \in [0,1]$$
$$p \cdot \tilde{f}(i) > p \cdot f(i) \ \forall \ i \in (a,b)$$

Proof:

Suppose not, meaning $p \cdot \tilde{f} < p \cdot f$,

By continuity, there is a neighborhood $B_\varepsilon\left(\tilde{f}\right)$ such that $\tilde{\tilde{f}} \in B_\varepsilon\left(\tilde{f}\right) \Rightarrow p \cdot \tilde{\tilde{f}} < p \cdot f$

By LNS, there is an $\tilde{\tilde{f}} \in B_\varepsilon\left(\tilde{f}\right)$ with the property that $\tilde{\tilde{f}} \succ \tilde{f}$

Combining them implies: $\tilde{\tilde{f}} \succ f$ with $p \cdot \tilde{f} < p \cdot f$, contradicting the fact that f is a part of WE.

The second line with strict inequality follows directly from definition of WE.

End Proof.

Which implies,

$$\int_{[a,b]} p \cdot \tilde{f}(i)\, di > \int_{[a,b]} p \cdot f(i)\, di$$
$$\Rightarrow \int_{[0,1]} p \cdot \tilde{f}(i)\, di > \int_{[0,1]} p \cdot f(i)\, di$$
$$\Rightarrow p \cdot \omega > p \cdot \omega$$

Contradiction.

2 Question 2

2.1 Part a

Consider the Pareto optimal allocation $(x, \omega - x) \in X^2$ where $X = \mathbb{R}^L +$

Take two sets:

$$U_x^A = \{\tilde{x} \in X^o : \tilde{x} \succ_A x\}$$
$$U_x^B = \{\omega - \tilde{x} \in X^o : \omega - \tilde{x} \succ_B x\}$$

By continuity, these two sets are open (since their complements are closed, and interior of X is open).

Also, by Pareto optimality,

$$U_x^A \cap U_x^B = \emptyset$$

(since otherwise anything in the intersection would Pareto dominate x).

By Separating Hyperplane Theorem,

$$\exists\,(p, w) \text{ such that } p \cdot \tilde{x} \geq w \;\forall\, \tilde{x} \in U_x^A \text{ and } p \cdot \tilde{x} \leq w \;\forall\, \tilde{x} \in U_x^B$$

It can be shown that $p \cdot x = w$.

Proof:

Use the closure sets:

$$\bar{U}_x^A = \{\tilde{x} \in X : \tilde{x} \succsim_A x\}$$
$$\bar{U}_x^B = \{\omega - \tilde{x} \in X : \omega - \tilde{x} \succsim_B x\}$$

Apply Separating Hyperplane Theorem on $\bar{U}_x^A$ with U_x^B to get:

$$p \cdot x \geq w$$

and on U_x^A with $\bar{U}_x^B$ to get:

$$p \cdot x \leq w$$

End Proof.

It can be shown that the inequalities are strict (use Lemma 2)

Proof:

Suppose $p \cdot \tilde{x} = w$ for a contradiction, By continuity, there is a neighborhood $B_\varepsilon(\tilde{x})$ such that $\tilde{\tilde{x}} \in B_\varepsilon(\tilde{x}) \Rightarrow \tilde{\tilde{x}} \succ x$.

Also, $p \cdot (\alpha\tilde{x}) < p \cdot x \; \forall \; \alpha < 1$ (linear combination with the 0 vector and $\tilde{x} \neq 0$)

For α close to 1, take $\alpha\tilde{x} \in B_\varepsilon(\tilde{x}), \alpha\tilde{x} \succ x$

Contradiction (since $\alpha\tilde{x} \in U_x^A$ but $p \cdot (\alpha\tilde{x}) < w$).

End Proof.

There with initial endowments $\omega_A = x$ and $\omega_B = \omega - x$, the prices p and the allocations $(x, \omega - x)$ forms a WE.

2.2 Part b

Lemma 2 will not apply here because $\alpha\tilde{x} = 0 \; \forall \; 0 < \alpha < 1$.

If $x_A = 0$, take $\bar{U}_x^A$ and U_x^B and apply the Separating Hyperplane Theorem.

Alternatively, use Supporting Hyperplane Theorem on U_x^B.

If $x_B = 0$, take U_x^A and $\bar{U}_x^B$ and apply the Separating Hyperplane Theorem.

Alternatively, use Supporting Hyperplane Theorem on U_x^A.

The only thing left is to show $p >> 0$ using strict monotonicity:

Proof:

Suppose $p_i = 0$, consider $x + \varepsilon e_i$ where e_i the vector with 1 on the i-th coordinate and 0 everywhere else.

For $\varepsilon > 0, x + \varepsilon e_i \succ x$ by strict monotonicity, but $p \cdot x = p \cdot (x + \varepsilon e_i)$ since $p_i = 0$. Contradiction.

End Proof.

3 Question 3

(Comprehensive Exam August 2014 Q3)

3.1 Part a

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_{A1}, y_{A1}} \sqrt{x_{A1}, y_{A1}} \text{ such that } p_A^x x_{A1} + p_A^y y_{A1} = 3p_A^x + p_A^y$$

$$\max_{x_{A2}, y_{A2}} \sqrt{x_{A2}, y_{A2}} \text{ such that } p_A^x x_{A2} + p_A^y y_{A2} = 3p_A^x + p_A^y$$

$$\max_{x_{B1}, y_{B1}} \sqrt{x_{B1}, y_{B1}} \text{ such that } p_B^x x_{B1} + p_B^y y_{B1} = p_B^x + 2p_B^y$$

$$\max_{x_{B2}, y_{B2}} \sqrt{x_{B2}, y_{B2}} \text{ such that } p_B^x x_{B2} + p_B^y y_{B2} = p_B^x + 2p_B^y$$

2. Firm maximizes (zero) profit:

$$\alpha\left(-p_B^x + p_B^y\right) = 0$$

3. Markets clear:

$$x_{A1} + x_{A2} = 3 + 3$$

$$y_{A1} + y_{A2} = 1 + 1$$

$$x_{B1} + x_{B2} = 1 + 1 - \alpha$$

$$y_{B1} + y_{B2} = 2 + 2 + \alpha$$

Normalize $p_A^y = 1$, then

$$x_{A1} = x_{A2} = \frac{3p_A^x + 1}{2p_A^x}$$
$$y_{A1} = y_{A2} = \frac{3p_A^x + 1}{2}$$

Either check X market clearing:

$$\frac{3p_A^x + 1}{2p_A^x} + \frac{3p_A^x + 1}{2p_A^x} = 3 + 3$$
$$p_A^x = \frac{1}{3}$$

Or (easier) check Y market clearing:

$$\frac{3p_A^x + 1}{2} + \frac{3p_A^x + 1}{2} = 1 + 1$$
$$p_A^x = \frac{1}{3}$$

Then,

$$x_{A1} = x_{A2} = 3$$

$$y_{A1} = y_{A2} = 1$$

Normalize $p_B^y = 1$, then

$$x_{B1} = x_{B2} = \frac{p_B^x + 2}{2p_B^x}$$
$$y_{B1} = y_{B2} = \frac{p_B^x + 2}{2}$$

Case $1 : \alpha = 0$

Either check X market clearing:

$$\frac{p_B^x + 2}{2p_B^x} + \frac{p_B^x + 2}{2p_B^x} = 1 + 1$$
$$p_B^x = 2$$

Or (easier) check Y market clearing:

$$\frac{p_B^x + 2}{2} + \frac{p_B^x + 2}{2} = 2 + 2$$
$$p_B^x = 2$$

Check zero profit condition:

$$-p_B^x + p_B^y = -2 + 1 = -1 < 0$$

Therefore there is a WE with the above prices and allocations:

$$x_{B1} = x_{B2} = 1$$

$$y_{B1} = y_{B2} = 2$$

Case 2 : $\alpha > 0$

$$p_B^x = p_B^y = 1$$

Check X market clearing:

$$\frac{p_B^x + 2}{2p_B^x} + \frac{p_B^x + 2}{2p_B^x} = 1 + 1 - \alpha$$

$$\Rightarrow 3 = 2 - \alpha$$

$$\Rightarrow \alpha = -1$$

Not feasible.

Or (not much easier) check Y market clearing:

$$\frac{p_B^x + 2}{2} + \frac{p_B^x + 2}{2} = 2 + 2 = 2 + 2 + \alpha$$

$$\Rightarrow 3 = 4 + \alpha$$

$$\Rightarrow \alpha = -1$$

Not feasible.

No WE.

3.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_{A1},y_{A1}} \sqrt{x_{A1}, y_{A1}} \text{ such that } p^x x_{A1} + p^y y_{A1} = 3p^x + p^y$$
$$\max_{x_{A2},y_{A2}} \sqrt{x_{A2}, y_{A2}} \text{ such that } p^x x_{A2} + p^y y_{A2} = 3p^x + p^y$$
$$\max_{x_{B1},y_{B1}} \sqrt{x_{B1}, y_{B1}} \text{ such that } p^x x_{B1} + p^y y_{B1} = p^x + 2p^y$$
$$\max_{x_{B2},y_{B2}} \sqrt{x_{B2}, y_{B2}} \text{ such that } p^x x_{B2} + p^y y_{B2} = p^x + 2p^y$$

2. Firm maximizes (zero) profit:

$$\alpha\left(-p^x + p^y\right) = 0$$

3. Markets clear:

$$x_{A1} + x_{A2} + x_{B1} + x_{B2} = 3 + 3 + 1 + 1 - \alpha$$
$$y_{A1} + y_{A2} + y_{B1} + y_{B2} = 1 + 1 + 2 + 2 + \alpha$$

Normalize $p^y = 1$, then

$$x_{A1} = x_{A2} = \frac{3p^x + 1}{2p^x}$$
$$y_{A1} = y_{A2} = \frac{3p^x + 1}{2}$$
$$x_{B1} = x_{B2} = \frac{p^x + 2}{2p^x}$$
$$y_{B1} = y_{B2} = \frac{p^x + 2}{2}$$

Case 1 : $\alpha = 0$

Check X market clearing:

$$\frac{3p^x + 1}{p^x} + \frac{p^x + 2}{p^x} = 8$$
$$\Rightarrow p^x = \frac{3}{4}$$

Or (easier) check Y market clearing:

$$(3p^x + 1) + (p^x + 2) = 6$$
$$\Rightarrow p^x = \frac{3}{4}$$

Check zero profit:

$$-p^x + p^y = \frac{-3}{4} + 1 > 0 \text{ NO!}$$

No WE.

Case 2 : $\alpha > 0$

$$p^x = p^y = 1$$

Check X market clearing:

$$\frac{3p^x + 1}{p^x} + \frac{p^x + 2}{p^x} = 8 - \alpha$$
$$\Rightarrow \alpha = 1$$

Or (not much easier) check Y market clearing:

$$(3p^x + 1) + (p^x + 2) = 6 + \alpha$$
$$\Rightarrow \alpha = 1$$

Therefore there is a WE with the above prices and allocations:

$$x_{A1} = x_{A2} = 2$$
$$y_{A1} = y_{A2} = 2$$
$$x_{B1} = x_{B2} = \frac{3}{2}$$
$$y_{B1} = y_{B2} = \frac{3}{2}$$

3.3 Part c

No trade:

$$u_{A1} = u_{A2} = \sqrt{3 \cdot 1} = \sqrt{3}$$
$$u_{B1} = u_{B2} = \sqrt{1 \cdot 2} = \sqrt{2}$$

Trade:

$$u_{A1} = u_{A2} = \sqrt{2 \cdot 2} = 2 > \sqrt{3}$$
$$u_{B1} = u_{B2} = \sqrt{\frac{3}{2} \cdot \frac{3}{2}} = \frac{3}{2} > \sqrt{2}$$

All consumers are better off.

Method 1: (Solve the whole thing again)

No trade:

$$x_{A1} = x_{A2} = 2$$
$$y_{A1} = y_{A2} = 2$$
$$x_{B1} = 1$$
$$y_{B1} = 2$$
$$x_{B2} = 1$$
$$y_{B2} = 2$$

Trade:

$$x_{A1} = x_{A2} = 2$$
$$y_{A1} = y_{A2} = 2$$
$$x_{B1} = 1$$
$$y_{B1} = 1$$
$$x_{B2} = 2$$
$$y_{B2} = 2$$

Consumer B1 worse off.

Consumer B2 better off.

Method 2: (Argue with prices)

Prices for x decreases with free trade:

Consumers owning more X are worse off.

Consumers owning more Y are better off.

4 Question 4

(Comprehensive Exam August 2012 Q3)

4.1 Part a

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_A, y_A} x_A - \frac{1}{y_A} \text{ such that } p_x x_A + p_y y_A = 2p_x$$
$$\max_{x_B, y_B} -\frac{1}{x_B} + y_B \text{ such that } p_x x_B + p_y y_B = 2p_y$$

2. Markets clear:

$$x_A + x_B = 2$$
$$y_A + y_B = 2$$

Normalize $p_y = 1$, then consumer's problem:

$$\max_{x_A, y_A} x_A - \frac{1}{y_A} \text{ such that } p_x x_A + y_A = 2p_x$$
$$\Rightarrow \max_{y_A} 2 - \frac{y_A}{p_x} - \frac{1}{y_A}$$
$$\Rightarrow \text{ FOC } : -\frac{1}{p_x} + \frac{1}{y_A^2} = 0$$
$$\Rightarrow y_A = \sqrt{p_x}, x_A = 2 - \frac{1}{\sqrt{p_x}}$$

Need feasibility:

$$x_A \geq 0$$
$$\Rightarrow p_x \geq \frac{1}{4}$$

If $p_x < \frac{1}{4}$

$$y_A = 2p_x, x_A = 0$$

and

$$\max_{x_B, y_B} -\frac{1}{x_B} + y_B \text{ such that } p_x x_B + y_B = 2$$
$$\Rightarrow \max_{y_B} -\frac{1}{x_B} - 2 - p_x x_B$$
$$\Rightarrow \text{ FOC } : \frac{1}{x_B^2} - p_x = 0$$
$$\Rightarrow x_B = \frac{1}{\sqrt{p_x}}, y_B = 2 - \sqrt{p_x}$$

Need feasibility:

$$y_B \geq 0$$
$$\Rightarrow p_x \leq 4$$

If $p_x > 4$

$$x_B = \frac{2}{p_x}, y_B = 0$$

Market clearing condition:

Case 1 : $p_x < \frac{1}{4}$

Check X market clearing:

$$0 + \frac{1}{\sqrt{p_x}} = 2$$
$$\Rightarrow p_x = \frac{1}{4}$$

Not feasible.

No WE.

Case 2 : $p_x > 4$

Check Y market clearing:

$$\sqrt{p_x} + 0 = 2$$
$$\Rightarrow p_x = 4$$

Not feasible.

No WE.

Case 3 : $\frac{1}{4} \leq p_x \leq 4$

Check X market clearing:

$$2 - \frac{1}{\sqrt{p_x}} + \frac{1}{\sqrt{p_x}} = 2$$

Always satisfied.

Or check X market clearing:

$$\sqrt{p_x} + 2 - \sqrt{p_x} = 2$$

Always satisfied.

Therefore there are WE for all prices $p_x \in \left[\frac{1}{4}, 4\right]$

4.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_A, y_A} x_A - \frac{1}{y_A} \text{ such that } p_x x_A + p_y y_A = 2p_x$$
$$\max_{x_B, y_B} -\frac{1}{x_B} + y_B \text{ such that } p_x x_B + p_y y_B = 2p_y$$

2. Firms maximize (zero) profit:

$$\alpha_1 (p_x - 2p_y) = 0$$
$$\alpha_2 (-5p_x + p_y) = 0$$

3. Markets clear:

$$x_A + x_B = 2 + \alpha_1 - 5\alpha_2$$
$$y_A + y_B = 2 - 2\alpha_1 + \alpha_2$$

Case 1 : $\alpha_1 = \alpha_2 = 0$

Zero profit condition:

$$p_x - 2p_y \leq 0 \text{ and } -5p_x + p_y \leq 0$$
$$\Rightarrow p_x \leq 2 \text{ and } p_x \geq \frac{1}{5}$$

Therefore there are WE (same allocations as in Part a) for all prices $p_x \in \left[\frac{1}{4}, 2\right]$

4.3 Part c

Case 2 : $\alpha_1 > 0, \alpha_2 = 0$

Zero profit condition:

$$p_x - 2p_y = 0 \text{ and } -5p_x + p_y \leq 0$$
$$\Rightarrow p_x = 2$$

Check X market clearing:

$$2 - \frac{1}{\sqrt{p_x}} + \frac{1}{\sqrt{p_x}} = 2 + \alpha_1$$
$$\Rightarrow 2 = 2 + \alpha_1$$
$$\Rightarrow \alpha_1 = 0$$

Not feasible.

No WE.

Case 3 : $\alpha_1 = 0, \alpha_2 > 0$

Zero profit condition:

$$
\begin{aligned}
& p_x - 2p_y \leq 0 \text{ and } -5p_x + p_y = 0 \\
& \Rightarrow p_x = \frac{1}{5}
\end{aligned}
$$

Check X market clearing:

$$
\begin{aligned}
& 0 + \frac{1}{\sqrt{p_x}} = 2 - 5\alpha_2 \\
& \Rightarrow \sqrt{5} = 2 - 5\alpha_2 \\
& \Rightarrow \alpha_2 = \frac{2 - \sqrt{5}}{5} < 0
\end{aligned}
$$

Or (harder) check Y market clearing:

$$
\begin{aligned}
& 2p_x + 2 - \sqrt{p_x} = 2 + \alpha_2 \\
& \Rightarrow \frac{2}{5} - \frac{\sqrt{5}}{5} = \alpha_2
\end{aligned}
$$

Same.

Not feasible.

No WE.