ECO2020 Tutorial 3

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1 Question 1

(Comprehensive Exam June 2005 Q3)

1.1 Part a

Competitive Equilibrium requires:

1. Consumers maximize utility:

$$q_1(\alpha) \geq \theta q_2(\alpha) \text{ for } \theta \in \Theta_1$$

$$q_1(\alpha) \leq \theta q_2(\alpha) \text{ for } \theta \in \Theta_2$$

2. Markets clear:

$$\int_{\Theta_1} f(\theta)\, d\theta = \alpha$$

Consider $\hat{\theta} = \frac{q_1(\alpha)}{q_2}(\alpha)$,
If $\theta < \hat{\theta}$,

$$q_1(\alpha) = q_2(\alpha)\hat{\theta} > q_2(\alpha)\theta$$

$$\Rightarrow \theta \in \Theta_1$$

If $\theta > \hat{\theta}$,

$$q_1(\alpha) = q_2(\alpha)\hat{\theta} < q_2(\alpha)\theta$$

$$\Rightarrow \theta \in \Theta_2$$

Therefore,

$$\Theta_1 = \left[0, \hat{\theta}\right]$$
$$\Theta_2 = \left[\hat{\theta}, 1\right]$$

1.2 Part b

$$\begin{aligned} \hat{\theta} &= \frac{q_1(\alpha)}{q_2}(\alpha) \\ &= \frac{2a+\alpha}{2a+\alpha+1} \end{aligned}$$

Then market clearing condition becomes:

$$\begin{aligned} &\int_{\Theta_1} f(\theta)\, d\theta = \alpha \\ &\Rightarrow \int_0^{\hat{\theta}} f(\theta)\, d\theta = \alpha \\ &\Rightarrow F\left(\hat{\theta}\right) = \alpha \\ &\Rightarrow F\left(\frac{2a+\alpha}{2a+\alpha+1}\right) = \alpha \end{aligned}$$

Let $G(\alpha) = F\left(\frac{2a+\alpha}{2a+\alpha+1}\right)$:
G is continuous since is composite of continuous functions on $\alpha \in [0,1]$.

The support is $[0,1]$, compact convex,

Apply Brower fixed point theorem:

$$\exists \alpha^\star \text{ such that } G(\alpha^\star) = \alpha^\star$$

2 Question 2

(Comprehensive Exam June 2007 Q3)

2.1 Part a

Let allocations be functions $x : [0,1] \to \mathbb{R}_+$ and $y : [0,1] \to \mathbb{R}_+$

Pareto Efficient allocation requires:

1. Individual feasibility:

$$x(\theta) \geq 0 \text{ and } y(\theta) \geq 0 \text{ for } \theta \in [0,1]$$

2. Aggregate feasibility:

$$\int_0^1 x(\theta) = 1$$
$$\int_0^1 y(\theta) = z$$

3. Optimality:

   There does not exist $(\tilde{x}, \tilde{y})$ such that:

$$(\tilde{x}(\theta), \tilde{y}(\theta)) \succsim_\theta (x(\theta), y(\theta) \text{ for } \theta \in [0,1]$$

   with strict inequality on some non-empty open interval $(a,b) \in [0,1]$.

2.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x(\theta), y(\theta)} x(\theta)^\theta y(\theta)^{1-\theta} \text{ such that } p_x x(\theta) + p_y y(\theta) = p_x + p_y z \text{ for } \theta \in [0,1]$$

2. Markets clear:

$$\int_0^1 x(\theta) = 1$$
$$\int_0^1 y(\theta) = z$$

2.3 Part c

Normalize $p_y = 1$, then

$$x(\theta) = \theta \frac{p_x + z}{p_x}$$
$$y(\theta) = (1-\theta)(p_x + z)$$

Check X market clearing:

$$\int_0^1 \theta \frac{p_x + z}{p_x} = 1$$
$$\Rightarrow \frac{1}{2} \frac{p_x + z}{p_x} = 1$$
$$\Rightarrow p_x = z$$

Or check Y market clearing:

$$\int_0^1 (1-\theta)(p_x + z) = z$$
$$\Rightarrow \frac{1}{2}(p_x + z) = z$$
$$\Rightarrow p_x = z$$

Therefore, WE has allocations:

$$x(\theta) = \theta \frac{z + z}{z} = 2\theta$$
$$y(\theta) = (1-\theta)(z + z) = (1-\theta) 2z$$

3 Question 3

(Comprehensive Exam June 2007 Q4)

3.1 Part a

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_A,y_A} \min\{2x_A, y_A\} \text{ such that } p_x x_A + p_y y_A = p_x + p_y$$

$$\max_{x_B,y_B} \min\{x_B, 2y_B\} \text{ such that } p_x x_B + p_y y_B = p_x + p_y$$

$$\max_{x_C,y_C} \min\{2x_C, y_C\} \text{ such that } p_x x_C + p_y y_C = p_x + p_y$$

2. Markets clear:

$$x_A + x_B + x_C = 3$$

$$y_A + y_B + y_C = 3$$

Maybe not normalize $p_y = 1$ this time:

$$\text{set } 2x_A = y_A \text{ and } x_B = 2y_B \text{ and } 2x_C = y_C$$

Substitute them into the budget constraints:

$$\begin{aligned}
x_A &= \frac{p_x + p_y}{p_x + 2p_y} \\
y_A &= \frac{2(p_x + p_y)}{p_x + 2p_y} \\
x_B &= \frac{2(p_x + p_y)}{2p_x + p_y} \\
y_B &= \frac{p_x + p_y}{2p_x + p_y} \\
x_C &= \frac{p_x + p_y}{p_x + 2p_y} \\
y_C &= \frac{2(p_x + p_y)}{p_x + 2p_y}
\end{aligned}$$

Check X market clearing:

$$\begin{aligned}
&\frac{p_x + p_y}{p_x + 2p_y} + \frac{2(p_x + p_y)}{2p_x + p_y} + \frac{p_x + p_y}{p_x + 2p_y} = 3 \\
&\Rightarrow \frac{2p_y}{p_x + 2p_y} = \frac{p_y}{2p_x + p_y} \\
&\Rightarrow p_y = 0 \text{ or } 4p_x + 2p_y = p_x + 2p_y \\
&\Rightarrow p_y = 0 \text{ or } p_x = 0
\end{aligned}$$

AND check Y market clearing:

$$\begin{aligned}
&\frac{2\left(p_x+p_y\right)}{p_x+2p_y}+\frac{p_x+p_y}{2p_x+p_y}+\frac{2\left(p_x+p_y\right)}{p_x+2p_y}=3\\
&\Rightarrow \frac{2p_x}{p_x+2p_y}=\frac{p_x}{p_x+2p_y}\\
&\Rightarrow p_x=0 \text{ or } 2p_x+4p_y=p_x+2p_y\\
&\Rightarrow p_x=0
\end{aligned}$$

Note that $p_y = 0$ is NOT a solution here.

Therefore, WE has allocations:

$$x_A=\frac{1}{2}, x_B=2, x_C=\frac{1}{2}$$

$$y_A=1, y_B=1, y_C=1$$

3.2 Part b

Core allocation requires:

$$\begin{aligned}
\min\left\{2\cdot\frac{1+\varepsilon}{2}, 1+\varepsilon\right\} &\geq \min\left\{2x_A, y_A\right\}\\
\min\left\{2-\varepsilon, 2\cdot(1-2\varepsilon)\right) &\geq \min\left\{x_B, 2y_B\right\}\\
\min\left\{2\cdot\frac{1+\varepsilon}{2}, 1+\varepsilon\right\} &\geq \min\left\{2x_C, y_C\right\}
\end{aligned}$$

1. Not blocked by $\{A\},\{B\},\{C\}$:

$$x_A=y_A=1$$

$$x_B=y_B=1$$

$$x_C=y_C=1$$

2. Not blocked by $\{A,B\},\{A,C\},\{B,C\}$:

$$x_A+x_B=y_A+y_B=2$$

$$x_A+x_C=y_A+y_C=2$$

$$x_B+x_C=y_B+y_C=2$$

3. Not blocked by $\{A, B, C\}$:

$$x_A + x_B + x_C = y_A + y_B + y_C = 3$$

First simplify the conditions:

$$\begin{aligned} 1+\varepsilon &\geq \min\{2x_A, y_A\} \\ 2-4\varepsilon &\geq \min\{x_B, 2y_B\} \text{ this assumes } \varepsilon \geq 0 \\ 1+\varepsilon &\geq \min\{2x_C, y_C\} \end{aligned}$$

One person coalition:

$$\begin{aligned} &1+\varepsilon \geq 1 \text{ and } 2-4\varepsilon \geq 1 \\ &\Rightarrow 0 \leq \varepsilon \leq \frac{1}{4} \end{aligned}$$

Two people coalition:

$$\begin{aligned} &2-4\varepsilon \geq \min\left\{2-\frac{1+\varepsilon}{2}, 2\cdot(2-(1+\varepsilon))\right\} \\ &\Rightarrow 2-4\varepsilon \geq \min\left\{\frac{3}{2}-\frac{\varepsilon}{2}, 2-2\varepsilon\right\} \\ &\Rightarrow 2-4\varepsilon \geq \frac{3}{2}-\frac{\varepsilon}{2} \left(\text{since } \varepsilon \leq \frac{1}{4}\right) \\ &\Rightarrow \frac{1}{2} \geq \frac{7}{2}\varepsilon \\ &\Rightarrow \varepsilon \leq \frac{1}{7} \end{aligned}$$

Three people coalition:

It is Pareto optimal.

Therefore, any $\varepsilon \in \left[0, \frac{1}{7}\right]$ implies the allocation is in the Core.

4 Question 4

(Comprehensive Exam August 2008 Q4)

4.1 Part a

Let c_a^b represent the a-th period consumption of consumer b.

Core allocation requires:

1. Not blocked by $\{1\},\{2\},...,\{N\}$:

$$\log\left(c_n^n\right)+\log\left(c_{n+1}^n\right)\geq\log\left(\theta\right)+\log\left(\theta'\right)\text{ for }n>0$$

$$\log\left(c_1^0\right)\geq\log\left(\theta'\right)$$

$$\begin{aligned}
& c_{N+1}^N\leq\theta'\Rightarrow c_N^N<\theta\text{ are blocked by }\{N\}\\
& \Rightarrow c_N^N\geq\theta\Rightarrow c_N^{N-1}\leq\theta'\Rightarrow c_{N-1}^{N-1}<\theta\text{ are blocked by }\{N-1\}\\
& \Rightarrow c_{N-1}^{N-1}\geq\theta\Rightarrow c_{N-1}^{N-2}\leq\theta'\Rightarrow c_{N-2}^{N-2}<\theta\text{ are blocked by }\{N-2\}\\
& \Rightarrow\ ...\\
& \Rightarrow c_2^2\geq\theta\Rightarrow c_2^1\leq\theta'\Rightarrow c_1^1<\theta\text{ are blocked by }\{1\}
\end{aligned}$$

Now,

$$\begin{aligned}
& c_1^0<\theta'\text{ are blocked by }\{0\}\\
& \Rightarrow c_1^1\leq\theta\Rightarrow c_1^1=\theta\Rightarrow c_2^1<\theta'\text{ are blocked by }\{1\}\Rightarrow c_2^1=\theta'\\
& \Rightarrow c_2^2\leq\theta\Rightarrow c_2^2=\theta\Rightarrow c_3^2<\theta'\text{ are blocked by }\{2\}\Rightarrow c_3^2=\theta'\\
& \Rightarrow\ ...\\
& \Rightarrow c_N^N=\theta
\end{aligned}$$

Therefore, only consuming endowment is in the Core.

4.2 Part b

Walrasian equilibrium requires:

1. Consumers maximize utility:

$$\max_{c_n^n, c_{n+1}^n} \log(c_n^n) + \log\left(c_{n+1}^n\right) \text{ such that } .p_n c_n^n + p_{n+1} c_{n+1}^n = p_n \theta + p_{n+1} \theta'$$

$$\max_{c_1^0} \log\left(c_1^0\right) \text{ such that } p_0 c_1^0 = p_0 \theta'$$

2. Markets clear:

$$c_n^{n-1} + c_n^n = \theta + \theta'$$

$$c_{N+1}^N = \theta'$$

Consider any sequence of prices $\{p_n\}_{n=0}^N$:

Start from consumer 0, her budget constraint implies:

$$c_1^0 = \theta'$$

Then market clearing condition of c_1 implies:

$$c_1^1 = \theta + \theta' - c_1^0 = \theta + \theta' - \theta' = \theta$$

Then consumer 1's budget constraint implies:

$$c_2^1 = \theta'$$

Then market clearing condition of c_2 implies:

$$c_2^2 = \theta + \theta' - c_2^1 = \theta + \theta' - \theta' = \theta$$

$$\ldots$$

$$c_N^N = \theta$$

$$c_{N+1}^N = \theta'$$

4.3 Part c

The argument in Part b is forward induction. Therefore it applies to $N \to \infty$.

4.4 Part d

The argument in Part a is backward induction. Therefore it does not apply.

Consider a coalition of $\{0, 1, 2, ...\}$:

The allocation with:

$$c_n^n = c_n^{n-1} = \frac{\theta + \theta'}{2} \text{ for } n \geq 0$$

is preferred by all consumers since:

$$\log\left(\frac{\theta + \theta'}{2}\right) > \log(\theta') \text{ for consumer } 0$$
$$2\log\left(\frac{\theta + \theta'}{2}\right) > \log(\theta) + \log(\theta') \text{ for consumers } n > 0$$

by concavity of log.

5 Question 5

(Comprehensive Exam June 2004 Q4)

5.1 Part a

Z is blocked by coalition $\{A_1, A_2, B_1\}$, need to check:

$$u_{A1}(x_{A1}, y_{A1}) \geq u_{A1}(Z_A^x, Z_A^y)$$
$$u_{A2}(x_{A2}, y_{A2}) \geq u_{A2}(Z_A^x, Z_A^y)$$
$$u_{B1}(x_{B1}, y_{B1}) \geq u_{B1}(Z_B^x, Z_B^y)$$

with

$$x_{A1} + x_{A2} + x_{B1} = 2\omega_A^x + \omega_B^x$$
$$y_{A1} + y_{A2} + y_{B1} = 2\omega_A^y + \omega_B^y$$

Consider:

$$x_{A1} = x_{A2} = \frac{1}{2}\left(\omega_A^x + Z_A^x\right)$$
$$y_{A1} = y_{A2} = \frac{1}{2}\left(\omega_A^y + Z_A^y\right)$$
$$x_{B1} = Z_B^x$$
$$y_{B1} = Z_B^y$$

where

(x_{A1}, y_{A1}) and (x_{A2}, y_{A2}) are located at the midpoint between ω and Z (between Z and B), strictly preferred for A1 and A2.

(x_{B1}, y_{B1}) is located at Z, indifferent for B1.

and

$$x_{A1} + x_{A2} + x_{B1}$$
$$= \omega_A^x + Z_A^x + Z_B^x$$
$$= \omega_A^x + \omega_A^x + \omega_B^x$$
$$y_{A1} + y_{A2} + y_{B1}$$
$$= \omega_A^y + Z_A^y + Z_B^y$$

5.2 Part b

Suppose there are N players of each type, then:

Z is blocked by coalition $\{A_1, A_2, ..., A_N, B_1, B_2, ..., B_{N-1}\}$, need to check:

$$u_{Ai}\left(x_{Ai}, y_{Ai}\right) \geq u_{Ai}\left(Z_A^x, Z_A^y\right) \text{ for } i \in \{1, 2, ..., N\}$$
$$u_{Bj}\left(x_{Bj}, y_{Bj}\right) \geq u_{Bj}\left(Z_B^x, Z_B^y\right) \text{ for } j \in \{1, 2, ..., N-1\}$$

with

$$\sum_{i=1}^{N} x_{Ai} + \sum_{j=1}^{N-1} x_{Bj} = N\omega_A^x + (N-1)\,\omega_B^x$$
$$\sum_{i=1}^{N} y_{Ai} + \sum_{j=1}^{N-1} y_{Bj} = N\omega_A^y + (N-1)\,\omega_B^y$$

Consider:

$$x_{Ai} = \frac{1}{N}\omega_A^x + \frac{N-1}{N} Z_A^x \text{ and}$$
$$y_{Ai} = \frac{1}{N}\omega_A^y + \frac{N-1}{N} Z_A^y \text{ for } i \in \{1, 2, ..., N\}$$
$$x_{Bj} = Z_B^x \text{ and}$$
$$y_{Bj} = Z_B^y \text{ for } j \in \{1, 2, ..., N-1\}$$

where

(x_{Ai}, y_{Aj})s are located at some point between ω and Z for large N (between Z and B), strictly preferred for A_i.

(x_{Bj}, y_{Bj}) is located at Z, indifferent for Bj.

and

$$\sum_{i=1}^{N} x_{Ai} + \sum_{j=1}^{N-1} x_{Bj}$$
$$= \omega_A^x + (N-1)\left(Z_A^x + Z_B^x\right)$$
$$= \omega_A^x + (N-1)\left(\omega_A^x + \omega_B^x\right)$$
$$= N\omega_A^x + (N-1)\,\omega_B^x$$
$$\sum_{i=1}^{N} y_{Ai} + \sum_{j=1}^{N-1} y_{Bj}$$
$$= \omega_A^y + (N-1)\left(Z_A^y + Z_B^y\right)$$
$$= \omega_A^y + (N-1)\left(\omega_A^y + \omega_B^y\right)$$
$$= N\omega_A^y + (N-1)\,\omega_B^y$$