Second Year Paper
Mechanism Design with Stopping Problem

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General Problem

- Agent observes Markov stochastic process X_t
- Agent chooses (1) when to stop τ ; (2) action $q_t(x)$
- Principal pays $p_t(q_t)$
- What behavior is implementable?

Investment Example

- Agent collects information about the value of a project X_t until time τ
- Agent then decides how much to invest q_τ(money or effort)
- Principal has possibly different preference on the amount of information to collect, and the amount of investment.
- Principal wants to implement particular stopping rules by paying the agent $p_t(q_t)$

Advice Example

- Unknown state θ in {-1, 1}; payoff linear in state.
- Agent has private information: X_t is the belief of the state
- Agent recommends action cancel project ($q_t = -1$) or start production ($q_t = 1$), or continue research (wait until next period to update X_t).
- Principal wants to set a price for each advice at each time $p_t(-1)$ and $p_t(1)$ to incentivize the agent to provide advice that aligns with the preference of the principal (i.e. maybe biased).

Agent's Problem

- Observe Markov Process $\{X_t\}_{t=0}^{T}, X_t \in X = [\underline{x}, \bar{x}]$
- Choose stopping rule + terminal action $(\tau, \{q_t\}_{t=0}^{T})$
- where $\tau : X^T \to \{0, 1, 2, ..., T\}$ is a (predictable) stopping rule
- and $q : X \to Q$, Q is the set of terminal actions (could be continuous or discrete)
- Maximize $U_0(\tau, x) = \mathbb{E}\left[u_\tau(q_\tau, X_\tau) - p_\tau(q_\tau) | X_0 = x\right]$

Principal's Problem

- Observe time of decision τ and decision q_τ
- Choose a mechanism, a set of prices, $\{p_t(q_t)\}_{t=0}^T$ to incentivize the agent to use a particular $(\tau, \{q_t\}_{t=0}^T)$
- Here, (τ, q_t) is implementable means given the prices, the agent's optimal stopping rule + terminal decision is (τ, q_t)

Utility Functions

- Stopping value (utility if stopped at t):

$$U_t(x) = \max_{q_t \in Q} u_t(q_t, x) - p_t(q_t)$$

- Continuation value:

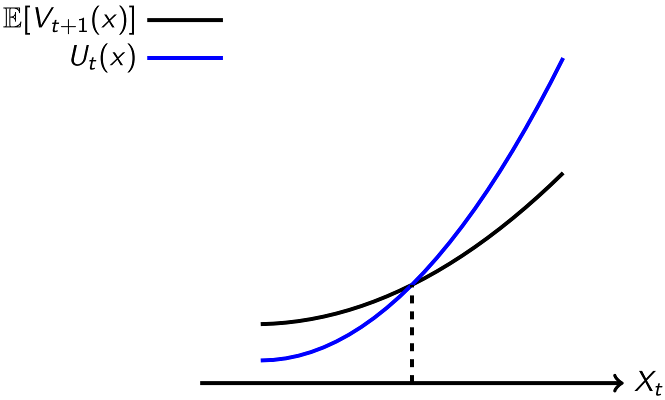
$$\mathbb{E}\left[V_{t+1}(X_{t+1}) | X_t = x\right] = \sup_{\tau : t+1 \leq \tau \leq T} \mathbb{E}\left[U_\tau(X_\tau) | X_t = x\right]$$

- Value function: $V_t(x) = \max\{U_t(x), \mathbb{E}\left[V_{t+1}(X_{t+1}) | X_t = x\right]\}$

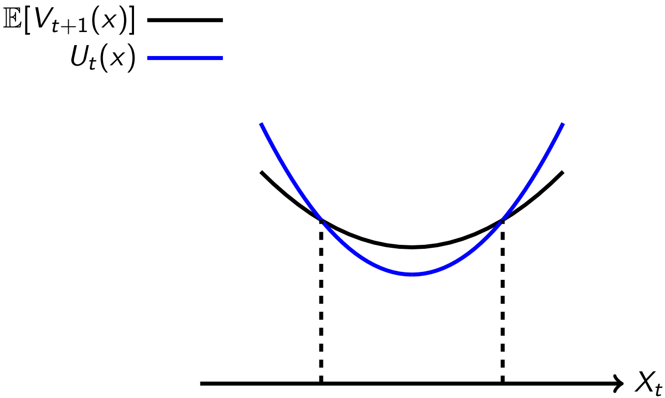
Interesting Stopping Rules

- Single Cutoff Stopping Rule:
  $\tau = \min_{t \leq T}\{x_t \in [b_t, \bar{x}]\}$; fix $b_T = \underline{x}$
- Binary Cutoff Stopping Rule:
  $\tau = \min_{t \leq T}\{x_t \notin [a_t, b_t]\}$; fix $a_T = b_T$

Single Cutoff Stopping Rule - Diagram



Binary Cutoff Stopping Rule - Diagram



Assumptions on Stochastic Process

- Markov
- Monotonic transition (equivalent to X_{t+1} FOSD X_t)
- Continuous transition
- Full support
- Examples: additive and multiplicative random walks (proof in Kruse and Strack (2014))
- In some examples: martingale

Kruse and Strack (2014) - Results

- Assumption: $q_t \in Q = \{1\}$
- Main result:

$$z_t(x) = \mathbb{E}\left[u_{t+1}(q_{t+1} = 1, X_{t+1}) | X_t = x\right] - u_t(q_t = 1, x)$$

- If $z_t(x)$ is non-increasing (Single Crossing Condition):
- Then a stopping rule is implementable if and only if it is a (single) cutoff rule
- Found closed form expressions for prices

Single Cutoff Rule - Assumptions on Utility Function

- Assumption: $q_t \in Q$ is an interval $\left[0, \bar{Q}\right]$ in $\mathbb{R}^+$
- Assumption: $\dfrac{\partial^2 u_t(q,x)}{\partial q \partial x} \geq 0$

Single Cutoff Rule - Conditions

- Sufficient and Necessary conditions for implementability of all single cutoff rules:
- Spence-Mirrlees Condition (SM) : $\frac{dq_t(x)}{dx} \geq 0$
- Pavan-Segal-Toikka's Single Crossing Condition (SC $_{PST}$):

$$\frac{\partial u_t(q_t, x)}{\partial x} \geq \mathbb{E}\left[\frac{\partial u_{t+1}(q_{t+1}, X_{t+1})}{\partial x}\mathcal{I}(X_{t+1}, X_t)|X_t = x\right]$$

- Impulse response function: $\mathcal{I}(x_{t+1}, x_t) = -\frac{\partial F_{t+1}(x_{t+1}|x_t)}{\partial x_t}\frac{1}{f_{t+1}(x_{t+1}|x_t)}$
- From PST: $\mathcal{I}(x_{t+1}, x+t)$ captures "marginal effects of the current type on future ones"

Single Cutoff Rule - Main Result

Theorem

Under assumptions on the stochastic process and the utility function stated in the previous slides:
$\{q_t\}_{t=0}^{T}$ *satisfies conditions SM and SC* $_{PST}$ *if and only if* $(\tau, \{q_t\}_{t=0}^{T})$ *is implementable for all single cutoff rules* τ.

Investment Example (from Slide 2)

- Agent collects information about the value of a project X_t until time τ
- Agent then decides how much to invest q_τ(money or effort)
- Assumption: $u_t(q, x) = \beta^t qx - \sum_{s=0}^{t} c_s$
- Assumption: $X_{t+1} = X_t + \varepsilon_t, \varepsilon_t \sim G_t$ independent

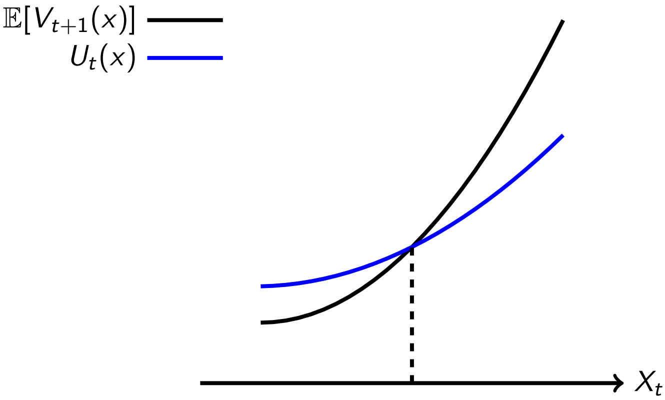
Investment Example

- Assumption: $u_t(q, x) = \beta^t qx - \sum_{s=0}^{t} c_s$
- Assumption: $X_{t+1} = X_t + \varepsilon_t, \varepsilon_t \sim G_t$ independent
- Then SM: $q_t'(x) \geq 0$
- And SC: $q_t(x) \geq \beta \mathbb{E}\left[q_{t+1}(X_{t+1}) | X_t = x\right]$
- A modified version of the closed form formula for prices in KS still applies

Investment Example - Violate SC

- For simplicity, let $T = 2, \beta = 1$ and G_1 has mean 0 and variance σ_G^2
- Consider implementing $q_0(x) = x$ and $q_1(x) = 2x$
- This does not satisfy SC
- Then $U_0(x) = \frac{x^2}{2} - p_0(b_0)$
- and $\mathbb{E}\left[V_1(X_1)|X_0 = x\right] = x^2 + \sigma_G^2$

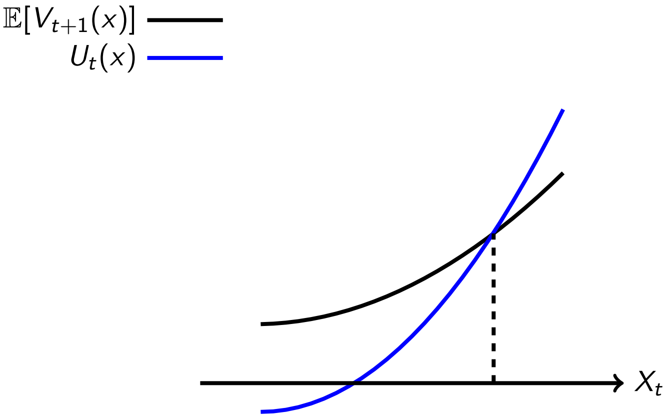
Investment Example - Violate SC Diagram



Investment Example - Not Violate SC

- Consider implementing $q_0(x) = x$ and $q_1(x) = \frac{x}{2}$
- This does satisfy SC
- Then $U_0(x) = \frac{x^2}{2} - p_0(b_0)$
- and $\mathbb{E}\left[V_1(X_1)|X_0 = x\right] = \frac{x^2 + \sigma_G^2}{4}$

Investment Example - Not Violate SC Diagram



Main Result - Proof Sufficiency

SM

SC

Standard static mechanism design

q_t implementable

τ implementable

Z_t increasing

$U_t - V_t$ increasing

Main Result - Proof Sufficiency

SM

SC

Some algebra

Z_t increasing

Standard static mechanism design

q_t implementable

τ implementable

$U_t - V_t$ increasing

Proof - SC $_{PST}$ to Monotonic Z_t

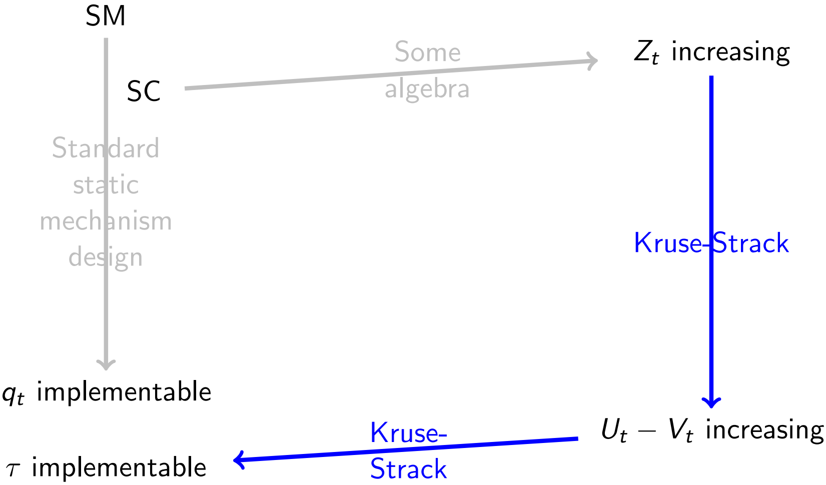
- Marginal Incentives: $Z_t(x) = \mathbb{E}\left[U_{t+1}(X_{t+1})|X_t = x\right] - U_t(x)$
- After some integration by parts:

$$\frac{dZ_t(x)}{dx} = \mathbb{E}\left[\frac{\partial u_{t+1}(q_{t+1}(X_{t+1}), X_{t+1})}{\partial x}\mathcal{I}(x)\right] - \frac{\partial u_t(q_t(x), x)}{\partial x}$$

- Which is the SC $_{PST}$ condition:

$$\frac{\partial u_t(q_t, x)}{\partial x} \geq \mathbb{E}\left[\frac{\partial u_{t+1}(q_{t+1}, X_{t+1})}{\partial x}\mathcal{I}(X_{t+1}, X_t)|X_t = x\right]$$

Main Result - Proof Sufficiency



Main Result - Proof Necessity

SM

SC

Standard static mechanism design

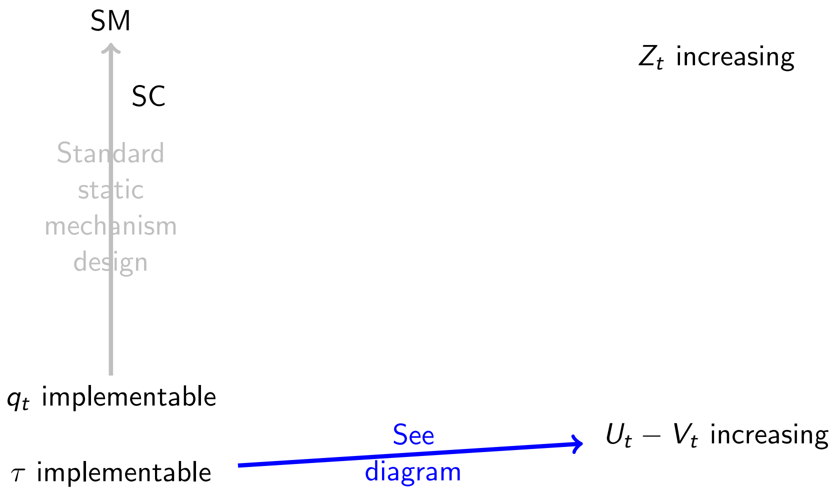
Z_t increasing

q_t implementable

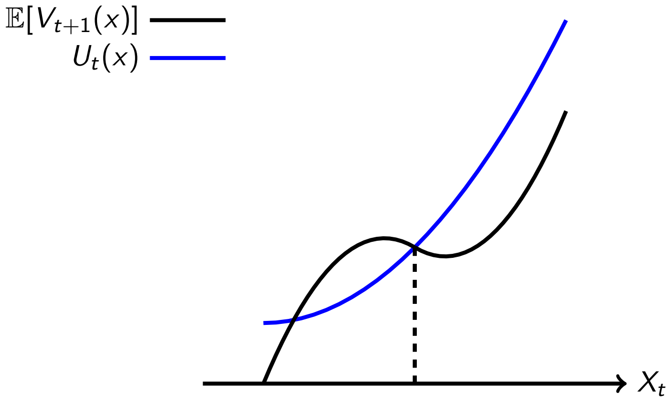
$U_t - V_t$ increasing

τ implementable

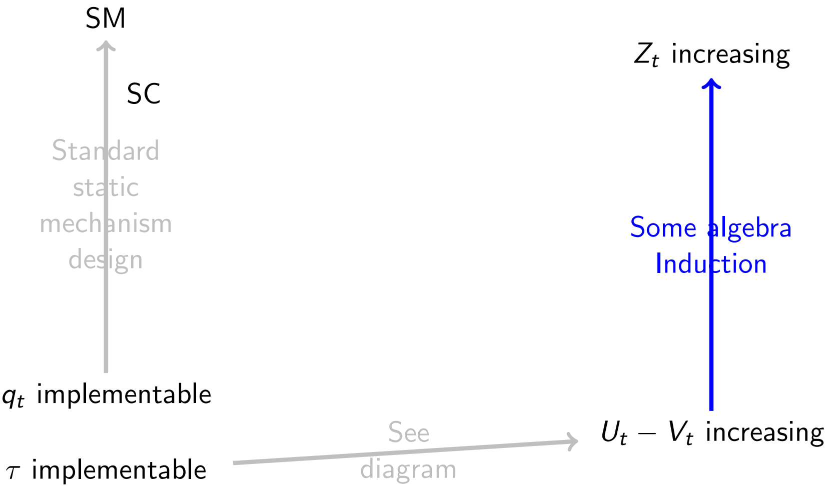
Main Result - Proof Necessity



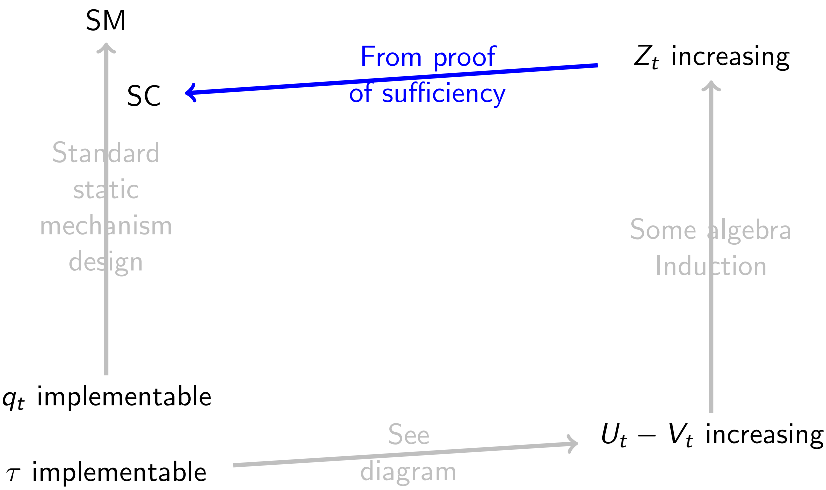
Proof - Implementable to Monotonic $V_t - U_t$



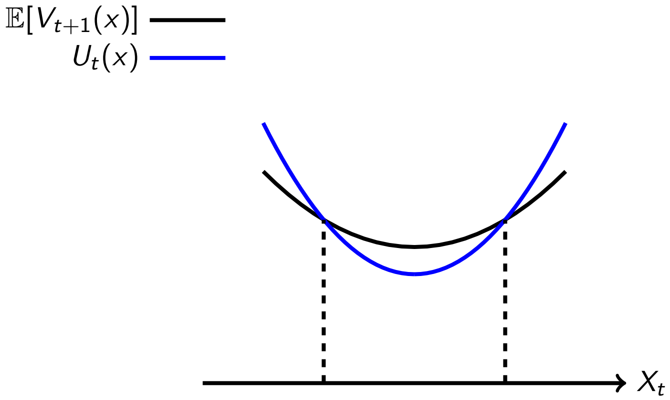
Main Result - Proof Necessity



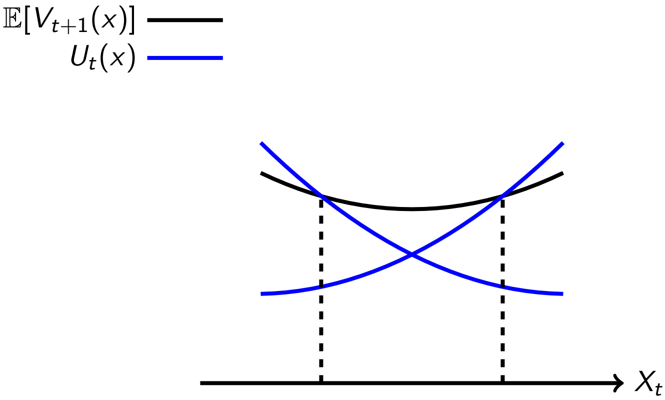
Main Result - Proof Necessity



Binary Cutoff Rule - Convexity Conditions (Not Feasible)



Binary Cutoff Rule from Two Types of Allocations



Binary Cutoff Rule - Assumptions and Notations

- Two types of allocations $q_t = -q_t^- \geq 0$ or $q_t = q_t^+ \geq 0$
- Assumption: $q_t \in Q$ is an interval $\left[\underline{Q}, \bar{Q}\right]$ in $\mathbb{R}$, $\underline{Q} < 0$ and $\bar{Q} > 0$

Binary Cutoff Rule - Conditions

- Sufficient and Necessary conditions for implementability of all binary cutoff rules:
- Spence-Mirrlees Condition (SM) : $\frac{dq_t^-(x)}{dx} \leq 0, \frac{dq_t^+(x)}{dx} \geq 0$
- Pavan-Segal-Toikka's Single Crossing Condition (SC $_{PST}$):

$$\frac{\partial u_t(-q_t^-, x)}{\partial x} \leq \mathbb{E}\left[\frac{\partial u_{t+1}(-q_{t+1}^-, X_{t+1})}{\partial x}\mathcal{I}(X_{t+1}, X_t)|X_t = x\right]$$

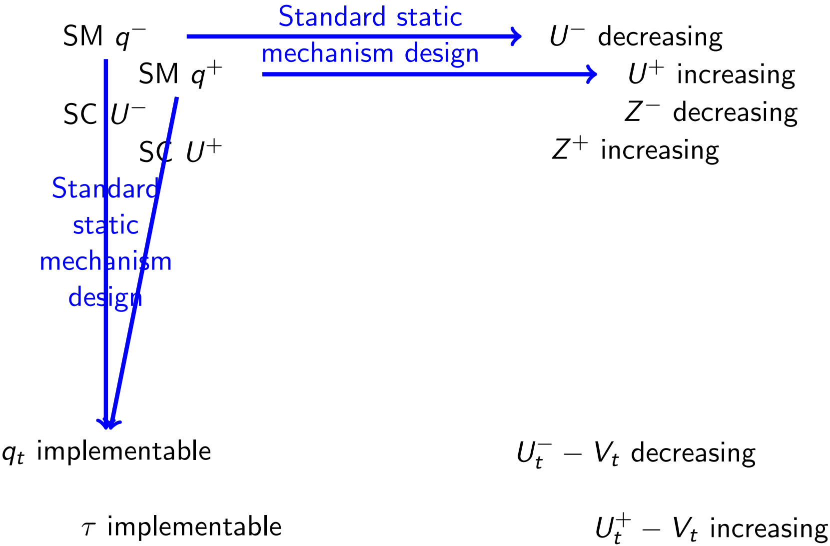
$$\frac{\partial u_t(q_t^+, x)}{\partial x} \geq \mathbb{E}\left[\frac{\partial u_{t+1}(q_{t+1}^+, X_{t+1})}{\partial x}\mathcal{I}(X_{t+1}, X_t)|X_t = x\right]$$

Binary Cutoff Rule - Main Result

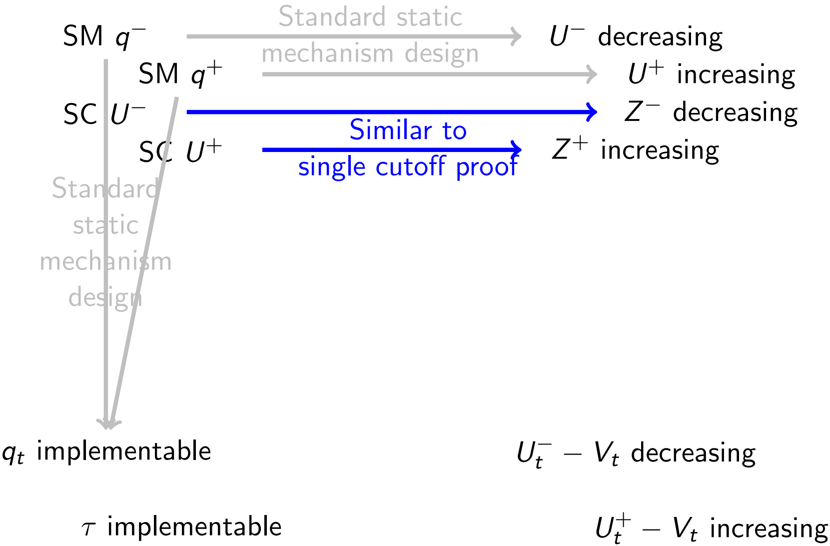
Theorem

Under assumptions on the stochastic process and the utility function stated in the previous slides:
$\{q_t\}_{t=0}^{T}$ *satisfies conditions SM and SC* $_{PST}$ *if and only if* $(\tau, \{q_t\}_{t=0}^{T})$ *is implementable for all binary cutoff rules* τ.

Main Result - Proof Sufficiency



Main Result - Proof Sufficiency



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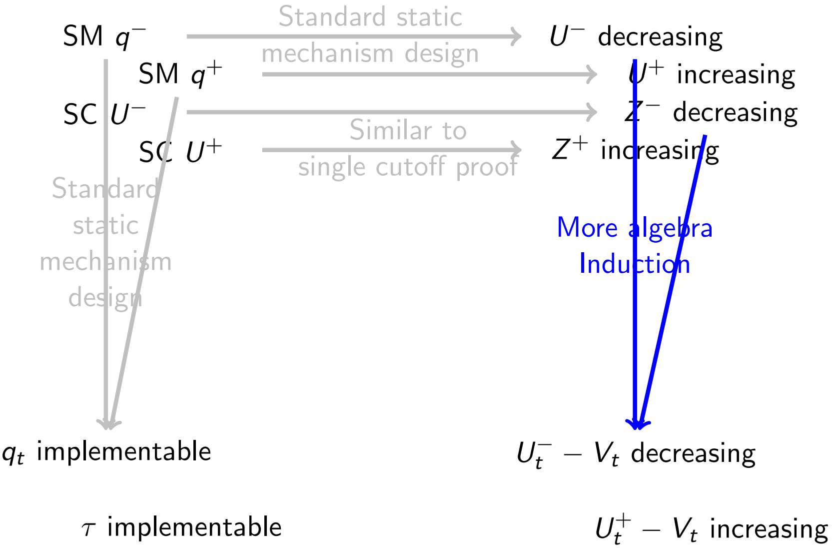
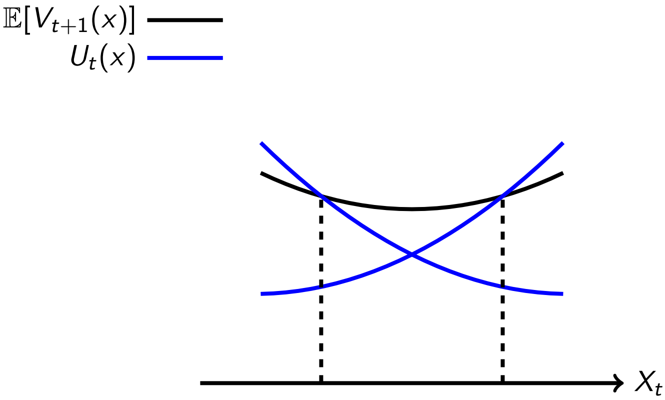


Proof - Plus Minus Notations

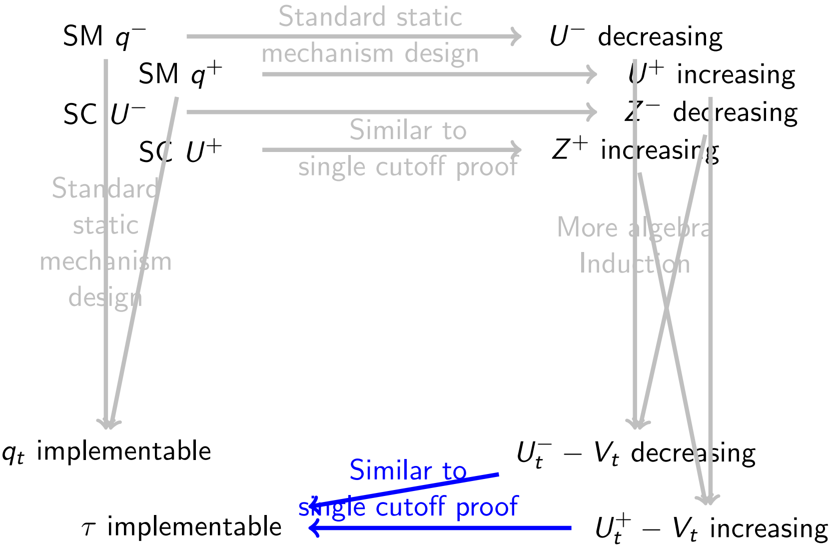
- For example, when there are only two actions {-1, 1}:
- Us: U^{-} means $u_t(q=-1,x)-p_t(-1)$
- and U^{+} means $u_t(q=1,x)-p_t(1)$
- Zs: Z^{-} means $\mathbb{E}\left[U_{t+1}(-1,X_{t+1})|X_t=x\right]-U_t(-1,x)$
- and Z^{+} means $\mathbb{E}\left[U_{t+1}(1,X_{t+1})|X_t=x\right]-U_t(1,x)$
- Monotonicity of these Zs implies CS $_{PST}$ and they are strong than the monotonicity of the following Marginal Incentives:

$$\mathbb{E}\left[\max\{U_{t+1}(-1,X_{t+1}),U_{t+1}(1,X_{t+1})\}|X_t=x\right]-U_t(-1,x)$$
$$\mathbb{E}\left[\max\{U_{t+1}(-1,X_{t+1}),U_{t+1}(1,X_{t+1})\}|X_t=x\right]-U_t(1,x)$$

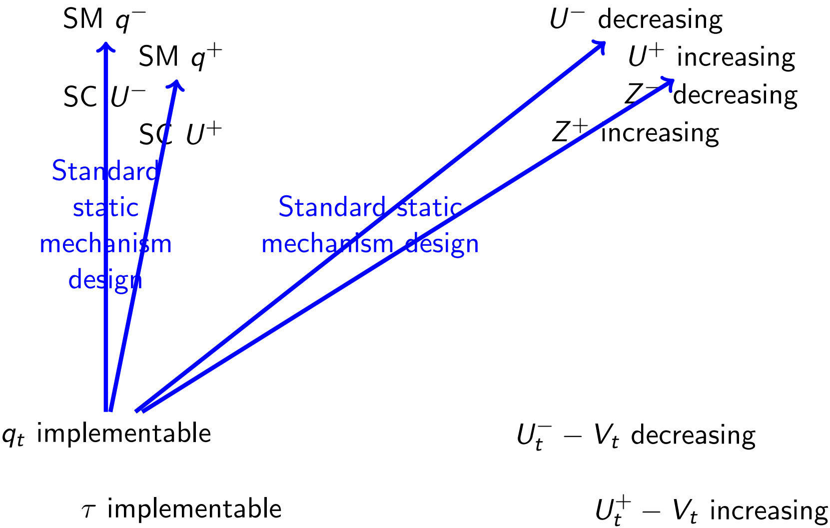
Proof - Plus Minus Zs Diagram



Main Result - Proof Sufficiency



Main Result - Proof Necessity



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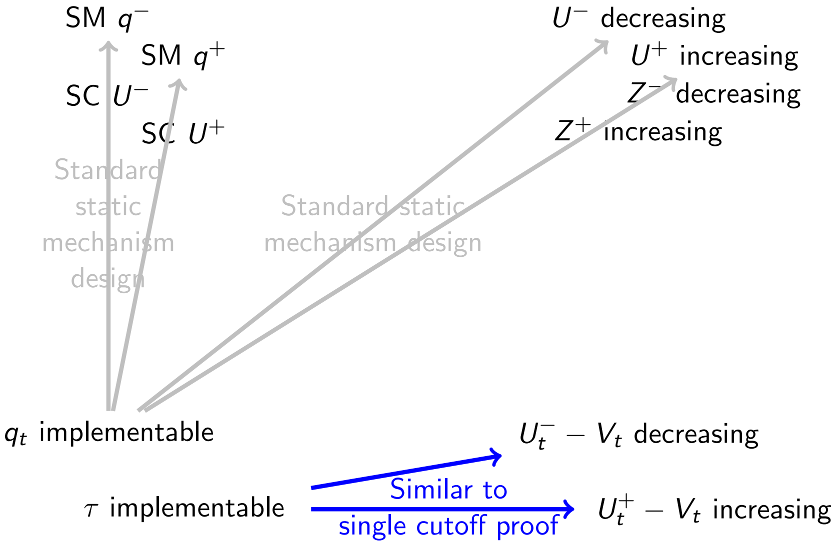

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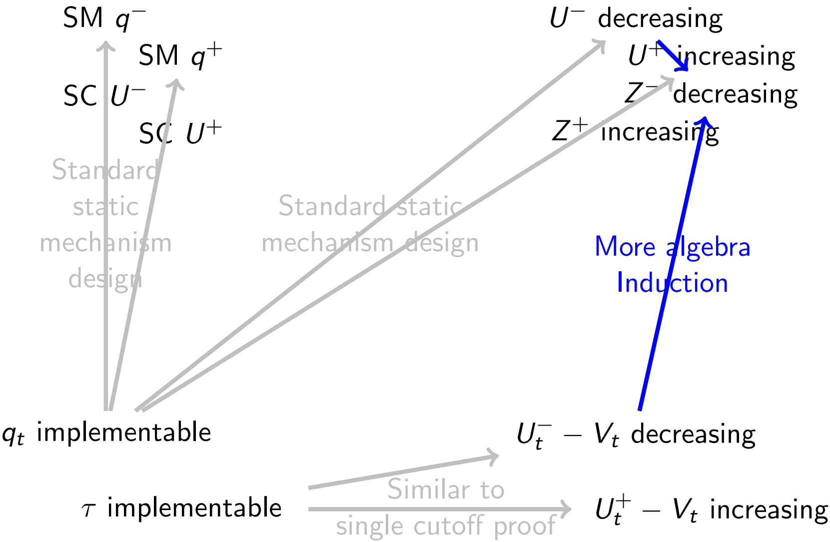
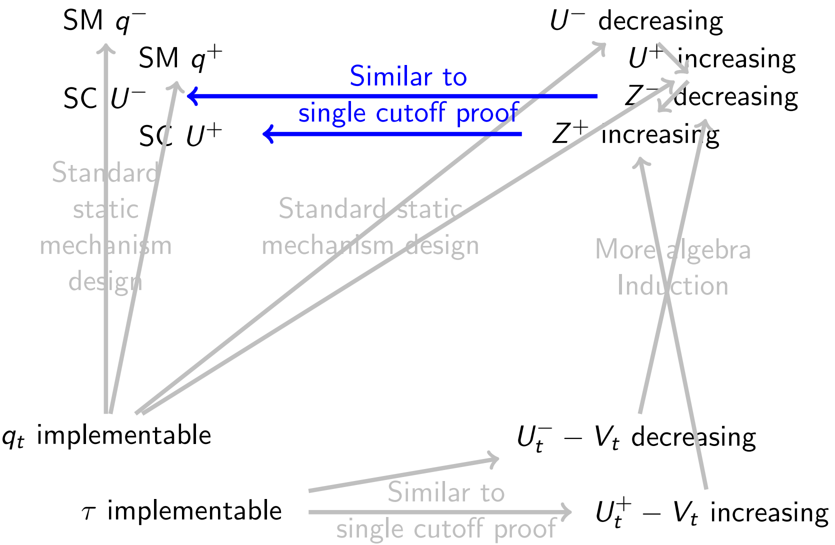


Main Result - Proof Necessity



Advice Example (from Slide 3)

- Unknown state θ in $\{$-1, 1$\}$; payoff linear in state.
- Agent has private information: X_t is the belief of the state
- Agent recommends action cancel project ($q_t = -1$) or start production ($q_t = 1$), or continue research (wait until next period to update X_t).
- Assumption: X_t form a martingale
- Assumption: $u_t(-1, x) = \alpha x - c_t$ and $u_t(1, x) = \beta x - c_t$ with $\alpha < 0$ and $\beta > 0$

Advice Example - Linear Utility + Martingale

- Assumption: $q_t \in Q = \{-1, (\text{ cancel project }); 1, (\text{ start production })\}$
- Assumption: X_t form a martingale
- Assumption: $u_t(-1, x) = \alpha x - c_t$ and $u_t(1, x) = \beta x - c_t$ with $\alpha < 0$ and $\beta > 0$
- Then $(\tau, \{q_t\}_{t=0}^{T})$ is implementable for all binary cutoff rule τ
- A modified version of the closed form formula for prices in KS still applies

Advice – Proof

- Value function V_t has derivatives bounded by α and β
- Martingale + Monotonic Transition preserves this property when taking conditional expectations

Comparison with Kruse and Strack (2014)

- Extension of KS to multiple terminal actions
- In KS, SC is an assumption (sufficient) for implementing single cutoff rules
- Here, SC is a sufficient and necessary condition for implementing all single cutoff rules

Comparison with Pavan Segal and Toikka (2014)

- Special case of PST where an agent could only choose q_t once
- Simpler expression for SC
- In PST, SC is sufficient (stronger than the necessary condition integral monotonicity) for implementing (q_t, τ)
- Here, SC is sufficient and necessary for implementing (q_t, τ) for all τ
- In PST, no closed form solution for prices
- Here, closed form solution modified from KS still applies

In addition to KS and PST

- Conditions for implementability of binary cutoff rules is not a direct extension from KS
- Model with two types of q that results in binary cutoff rules is not special case of PST
- Interpretable SC conditions for examples like investment and advice example.

Thanks

- Thank you for attending this presentation
- Thank Dr. Damiano for comments on the paper
- Thank Dr. Peski for all the help on the paper and the presentation