MECHANISM DESIGN FOR STOPPING PROBLEMS WITH TWO ACTIONS

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Abstract. I analyze a class of dynamic mechanism design problems in which a single agent privately observes a time-varying state, chooses a stopping time, and upon stopping, chooses between one of two actions. The principal designs transfers that depend only on the time the agent stops and on the alternative the agent chooses. The analysis provides sufficient and necessary conditions for implementability in this environment. In particular, I show that any stopping rule in which the agent stops the first time the state falls outside of an interval in the state space can be implemented if and only if a pair of monotonicity conditions is satisfied. This result extends Kruse and Strack (2015) to problems with two alternatives.

1. Introduction

In many economic situations, an agent wants to choose between two alternatives but is not required to make the choice right away. Often, the agent can postpone the choice until later in order to either gather more information or wait for more favorable conditions. Additionally, another party, called a principal, has an interest in the timing and choices made by the agent and can use transfers to provide incentives.

I model such a situation as a stopping problem with two actions. The agent observes a Markov state, decides when to stop, and upon stopping, chooses between one of the two actions. The agent's payoffs are a sum of an intrinsic payoff, which depend on the time, action, state, and a quasi-linear transfer received from the principal, which depends on the time and the action choice. The goal of this analysis is to describe the range of choice rules that can be implemented and to determine whether and how the principal can incentivize the agent with quasi-linear transfers.

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Economic applications include:

- (Job Search) A worker observes a changing demand for his services and chooses between taking up a job, continuing to search, or leaving the market (possibly to retire, or to go back to school). The government decides on the amount of employment insurance it offers to workers in order to achieve its own policy goals. This example expands on the main example from Kruse and Strack (2015) by adding the possibility that the worker can choose to leave the market.
- (Hypothesis Testing) The principal hires an econometrician to conduct a hypothesis test. The econometrician performs a Bayesian sequential test of H_0 vs H_1. After obtaining each sample, she can choose to reject one of the two hypotheses or obtain an additional sample. Payments are based on sample size to incentivize the econometrician to perform the designed test.
- (Project Funding) The principal, for instance, a public authority, a government agency sponsoring research, or a city council, is deciding which one of two projects it will invest extra resources into and wants to hire an investigator for advice. The investment decision affects the utility of the investigator, so the principal would like to find a payment scheme to incentivize the investigator to report truthfully.

In the above examples, the strategies for the agent consist of:

- (A stopping rule) The stopping rule can be complicated, but it often involves a sequence of pairs of thresholds that may vary over time. If the state falls inside the thresholds, the agent will continue; otherwise, she stops.
- (A choice of alternative) After the agent stops, she makes a choice between the two alternatives.

The above class of strategies is defined formally as the two-sided threshold rules. I analyze the implementability of this class of strategies.

The main result describes the necessary and sufficient conditions for the intrinsic utility and Markov process that ensure the implementability of all two-sided threshold rules. The conditions are similar to the single crossing condition in Kruse and Strack (2015) and the monotonicity condition from Pavan, Segal, and Toikka (2014). In addition, closed form formulas for the transfers that implement these stopping rules can be found.

The agent's problem has been studied since Wald (1947) and Arrow, Blackwell, and Girshick (1949) as the optimal stopping problem. The principal's problem, which is the focus of this analysis, is most closely related to Kruse and Strack (2015), where the principal chooses transfers to influence the stopping rule used by the agent. Kruse and Strack (2015) define a cut-off stopping rule as a strategy where the agent stops the first time the state she privately observes is above a certain threshold. They make a single crossing assumption (KS-SCC) that is a sufficient and necessary condition for the implementability of cut-off rules. It requires the expected difference between utilities in consecutive periods, which they call marginal incentives, to be weakly decreasing in the state. They also give closed form solutions for the transfers.

The model is also related to Pavan, Segal, and Toikka (2014). In their paper, there are multiple agents, and each agent can choose an allocation after each period. They show that a sequence of allocations is implementable if and only if an integral monotonicity condition holds. They also have a stronger condition called the single crossing condition (PST-SCC), which is a sufficient condition for implementability. PST-SCC requires the expected sums of discounted marginal utilities to be monotonic in state. The discount factor they use is the impulse response function. They give a characterization of the transfers but not closed form formulas for the transfers.

Due to the assumption that the agent can choose one of two actions only once, not only am I able to simplify PST-SCC, but the conditions in this model will also hold for utility functions that are non-monotonic in a particular way that does not satisfy KS-SCC. If the utilities are increasing in state for one alternative and decreasing in state for the other, then these conditions imply implementability as well.

I show that the condition of implementability can be simplified for the previously discussed examples:

- (Job Search) All stopping rules with two-sided thresholds are implementable if the expected change in utility from choosing the same action in any two consecutive periods is weakly decreasing in the current state.
- (Hypothesis Testing) As long as the payoff from accepting the more likely hypothesis is higher, the Bayesian sequential test with any size is implementable.
- (Project Funding) Suppose the investigator observes a state process that is additive, the payoff is linear in the states, and the slopes have different signs for different projects. Then, every strategy where the investigator stops when the state is either too high or too low is implementable.

Section 2 introduces the model, Section 3 gives sufficient and necessary conditions when threshold rules are implementable, and Section 4 provides simplifications of these conditions for examples with special utilities or stochastic processes of the states.

2. Model

2.1. **Agent's Problem.** Consider a stopping problem with two possible terminal actions. In each period $t = 0, 1, ..., T$, an agent observes the state of a one-dimensional Markov process, $x_t \in [\underline{x}, \bar{x}]$. Next, she chooses between three options: stop and choose action -1, stop and choose action +1, or continue. If the agent stops at time $\tau = t$ and chooses $q_t \in \{-1, +1\}$, she receives utility, $u_t(q_t, x_t)$.

I make the following monotonicity assumption on the utility function.

Assumption 1. The period t utility functions are partially differentiable with respect to x for all $t \in \{0, 1, ..., T\}$, and,

$$\frac{\partial u_t(-1, x)}{\partial x} \leqslant 0 \text{ and } \frac{\partial u_t(+1, x)}{\partial x} \geqslant 0, \ \forall\, x \in [\underline{x}, \bar{x}].$$

A strategy (or decision rule) is a sequence of mappings $q_t : [\underline{x}, \bar{x}] \to Q$ where $Q = \{-1, 0, 1\}$ is the set of decisions. The decision q_t is independent of previous states $x_0, x_1, ..., x_{t-1}$ because the utility only depends on the current x_t the agent observes. Each strategy induces a stopping time, which is a mapping $\tau : [\underline{x}, \bar{x}]^T \to \{0, 1, ..., T\}$ defined so that,

$$\tau(x_0, ..., x_T) = \min\{t : q_t(x_t) \neq 0\}.$$

2.2. **Implementation Problem.** The principal chooses transfers, $p_t(-1)$ and $p_t(+1)$, in order to provide the agents with incentives to pick a particular strategy (or decision rule), q_t. The agent observing state x_t in period t gets payoff $u_t(q_t, x_t) - p_t(q_t)$ if she stops and makes a decision, $q_t \in \{-1, +1\}$.

The expected utility at time 0 for the agent is,

$$U_0(q, x) = \mathbb{E}[u_\tau(q_\tau, X_\tau) - p_\tau(q_\tau) | X_0 = x]$$

where τ is the stopping time implied by the strategy q.

Definition 1. A strategy, $q = \{q_t\}_0^T$, is implementable if there are transfers $\{p_t\}_{t=0}^T$ such that for an agent observing state $x \in [\underline{x}, \bar{x}]$ at time $t = 0$,

$$\{q_t\}_{t=0}^T = \arg\max_{\{q'_t\}_{t=0}^T} \mathbb{E}[u_{\tau'}(q'_{\tau'}(X_{\tau'}), X_{\tau'}) - p_{\tau'}(q'_{\tau'}(X_{\tau'})) | X_0 = x].$$

A strategy (with its implied stopping time) is implementable if it is possible to provide the agent with incentives to choose that strategy.

2.3. **Threshold Rules.** I am interested in threshold strategies where the agent stops whenever the state falls outside of a sequence of intervals, $\{[a_t, b_t]\}_{t=0}^T$.

Definition 2. A strategy, q, is a (two-sided) threshold strategy if there are sequences $\{a_t, b_t\}_{t=0}^T$ such that $a_t \leqslant b_t, a_T = b_T$ and,

$$q_t = \begin{cases} -1 & \text{if } x_t \in [\underline{x}, a_t) \\ 0 & \text{if } x_t \in [a_t, b_t] \\ +1 & \text{if } x_t \in (b_t, \bar{x}]. \end{cases}$$

I denote a threshold strategy with thresholds $\{a_t, b_t\}_{t=0}^{T}$ as $q^{(a,b)}$. Kruse and Strack (2015) uses cut-off stopping rules with only one sequence of thresholds, and such rules are special cases where $a_t = \underline{x}$.

2.4. **Stochastic Process of the States.** I follow assumptions similar to Kruse and Strack (2015) on the stochastic process, $\{X_t\}_{t=0}^{T}$. These assumptions are needed for the model to be tractable.

Assumption 2. (Regular Transitions) The process $\{X_t\}_{t=0}^{T}$ satisfies,

(1) For any continuous $\phi : [\underline{x}, \bar{x}] \to \mathbb{R}, \mathbb{E}\left[\phi\left(X_{t+1}\right)|X_t = x\right]$ is continuous in x.
(2) For any weakly decreasing $\phi : [\underline{x}, \bar{x}] \to \mathbb{R}, \mathbb{E}\left[\phi\left(X_{t+1}\right)|X_t = x\right]$ is non-increasing in x.
(3) For any interval $[a, b) \subseteq [\underline{x}, \bar{x}], \mathbb{E}\left[1_{X_{t+1} \in [a,b)}|X_t = x\right] > 0$ for each $x \in [\underline{x}, \bar{x}]$.

The continuity-preserving and monotonicity-preserving properties are used to ensure that the shape of the value function in the future periods stays the same after taken expectations given the state of the current period. The full support property is included to ensure the uniqueness of the representation of a threshold rule. Without full support, multiple thresholds can be used to represent the same stopping time.

3. Implementability

Theorem 1 finds conditions under which all threshold rules are implementable.

Definition 3. Define the impulse response function as,

$$\mathcal{I}\left(x_{t+1}, x_t\right) = -\frac{\partial F_{t+1}\left(x_{t+1}|x_t\right)}{\partial x_t} \frac{1}{f_{t+1}\left(x_{t+1}|x_t\right)}.$$

The impulse response function is used in Pavan, Segal, and Toikka (2014) to state their monotonicity conditions, too. It is used as a discount factor for marginal utilities so that they can be added over time.

Theorem 1. *The following are equivalent:*

(1) *(Implementability) Every two-sided threshold rule, $q^{(a,b)}$, is implementable.*

(2) *(Monotonic Marginal Incentive Condition) The following marginal incentive function is weakly decreasing for each* $t \in \{0, 1, ..., T\}$ *and* $x \in [\underline{x}, \bar{x}]$.

$$\mathbb{E}\left[u_{t+1}\left(s, X_{t+1}\right) | X_t = x\right] - u_t\left(s, x\right), \ \textit{for } s \in \{-1, +1\}$$

(3) *(Single Crossing Condition) The following single crossing condition is satisfied for each* $t \in \{0, 1, ..., T\}$ *and* $x \in [\underline{x}, \bar{x}]$.

$$s \cdot \mathbb{E}\left[\frac{\partial u_{t+1}\left(s, X_{t+1}\right)}{\partial x} \mathcal{I}\left(X_{t+1}, X_t\right) | X_t = x\right] \leqslant s \cdot \frac{\partial u_t\left(s, x\right)}{\partial x}, \ \textit{for } s \in \{-1, +1\}$$

Both the marginal incentive condition and the single crossing condition are sufficient to implement any threshold rule but are not necessary to implement a particular threshold rule. They are only necessary to implement all possible threshold rules at the same time.

The two conditions for implementability are equivalent due to a simple integration by parts. The sufficiency of the single crossing condition (3) $\Rightarrow$ (1) can be proven using results in Pavan, Segal, and Toikka (2014). In particular, the assumptions about the utility function and the single crossing conditions imply strong monotonicity, which is a sufficient condition for implementation from their paper. I present an alternative proof for this specific model, which, in addition to proving implementability, also gives a closed form formula for the transfers and contains parts that are useful for the proof of the necessity of the single crossing conditions in implementing threshold rules. The sufficiency of the marginal incentives (2) $\Rightarrow$ (1) for one-sided thresholds with $a_t = \underline{x}$ is shown in Kruse and Strack (2015), and I modify and extend their proof for two-sided thresholds, where the principal needs to prevent the agent from choosing both actions -1 and +1 in future periods if he wants the agent to stop in the current period.

The proof for the necessity of the two conditions for implementing all threshold rules (1) $\Rightarrow$ (2) or (3) is new and does not rely on the techniques or results from Kruse and Strack (2015) or Pavan, Segal, and Toikka (2014).

In order to describe these transfers, additional notations are needed.

Definition 4. Define the modified marginal incentive functions as,

$$z_t(x) = \mathbb{E}\left[\max_{s'}\left\{u_{t+1}\left(s', X_{t+1}\right)\right\} | X_t = x\right] - \max_{s'}\left\{u_t\left(s', x\right)\right\},$$
$$\tilde{z}_t(s, x) = \mathbb{E}\left[\max_{s'}\left\{u_{t+1}\left(s', X_{t+1}\right)\right\} | X_t = x\right] - u_t(s, x), \text{ for } s \in \{+1, -1\}.$$

The function $z_t(x)$ is the expected gain in utility if the agent continues for another period relative to stopping in period t. The function $\tilde{z}_t(s, x)$ is the expected gain in utility if the agent continues for another period relative to reporting the state she is supposed to.

Corollary 1. *The following sequence of transfers implement* $q^{(a,b)}$.

$$p_t(-1) = \tilde{z}_t(-1, a_t) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s\left(\min\left\{\max\left\{a_t, X_s\right\}, b_t\right\}\right) | X_t = a_t\right]$$
$$p_t(+1) = \tilde{z}_t(+1, b_t) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s\left(\min\left\{\max\left\{a_t, X_s\right\}, b_t\right\}\right) | X_t = b_t\right]$$

The expressions are similar to the ones given in Kruse and Strack (2015), but the formulas for both z_t and $\tilde{z}_t$ are different from the one defined in their paper and are also different from the expression in the marginal incentive condition in Theorem 1. The monotonicity of the price functions is the key part in proving (2) and (3) $\Rightarrow$ (1) in Theorem 1.

4. Examples

Example 1. A worker observes job offers with wage $x_t \in [0, \bar{x}]$ in period t and decides whether to accept the offer, reject the offer and permanently leave the market, or keep searching. The government pays lump sum employment insurance , p_τ, in period τ to incentivize the worker to accept offers above some upper threshold and to exit the market when offers are below some lower threshold. Alternatively, the government can use other tax or transfer schemes to influence the worker into accepting offers earlier or to stop searching and get additional education. This example extends the one in Kruse and Strack (2015) to include the possibility of the worker exiting the market.

For example, if the utility the worker gets is equal to the wage when she accepts the offer, and 0 when she stops searching permanently, the conditions in Theorem 1 become,

$$z_t(x) = \mathbb{E}[X_{t+1}|X_t = x] - x \text{ is weakly decreasing in } x.$$

If the above condition holds, then the following transfers can be used to incentivize the worker to choose a job search strategy described by any thresholds $\{(a_t, b_t)\}_{t=0}^{T}.$

$$p_t(\text{ exit market }) = \mathbb{E}[X_{t+1}|X_t = a_t] + \sum_{s=t+1}^{T-1} \mathbb{E}[z_s(\min\{\max\{a_t, X_s\}, b_t\})|X_t = a_t]$$

$$p_t(\text{ accept offer }) = z_t(b_t) + \sum_{s=t+1}^{T-1} \mathbb{E}[z_s(\min\{\max\{a_t, X_s\}, b_t\})|X_t = b_t]$$

The transfers are different from the expressions in Kruse and Strack (2015) due to the addition of lower thresholds $\{a_t\}_{t=0}^{T}$. If $a_t = 0$ for $t = 0, 1, 2, ..., T$, then the condition and transfer functions coincide with the ones in their paper.

Example 2. A Bayesian statistician must distinguish between the two possible states of the world, $s = -1$ or $s = +1$. The statistician starts with a belief, x_0, that the state is +1. The statistician observes data and uses it to update the belief to posterior x_t. In each period t, she can either decide to collect more data at cost c_t, or choose one of two possible states. If in period t, she decides that the state of the world is $q_t = s$ and the true state of the world is also s, her payoff is equal to $\alpha_t(s)$. Otherwise, if the true state is -s, she gets $\beta_t(s)$, for each t and $s \in \{-1, +1\}$. Define the following constants,

$$m_t(-1) = \alpha_t(-1) - \beta_t(-1) \text{ and } m_t(+1) = \beta_t(+1) - \alpha_t(+1),$$

$$k_t(-1) = \beta_t(-1) - \sum_{i=0}^{t} c_i \text{ and } k_t(+1) = \alpha_t(+1) - \sum_{i=0}^{t} c_i.$$

Then, the utility function is linear in the posterior belief,

$$u_t(q_t, x_t) = m_t(q_t) x_t + k_t(q_t),$$

and Assumption 1 is equivalent to $\beta_t(s) < \alpha_t(s)$ for each t.

The single crossing condition in Theorem 1 is simplified to,

$$\mathbb{E}\left[\mathcal{I}\left(X_{t+1}, X_t\right)|X_t = x\right] \leqslant \min\left\{\frac{m_t\left(+1\right)}{m_{t+1}\left(+1\right)}, \frac{m_t\left(-1\right)}{m_{t+1}\left(-1\right)}\right\}.$$

If the belief process forms a martingale and $\alpha_t\left(s\right) = \alpha\left(s\right), \beta_t\left(s\right) = \beta\left(s\right)$ for each t, then the single crossing condition in Theorem 1 is always satisfied due to the following derivation,

$$\mathbb{E}\left[\mathcal{I}\left(X_{t+1}, X_t\right)|X_t = x\right] = \frac{d}{dx}\left(\mathbb{E}\left[X_{t+1}|X_t = x\right]\right) = 1,$$

because of the martingale property, and since α and β are constant over time.

$$\frac{m_{t+1}\left(-1\right)}{m_t\left(-1\right)} = \frac{m_{t+1}\left(+1\right)}{m_t\left(+1\right)} = 1$$

Therefore, the single crossing condition is always satisfied. Bayesian hypothesis test of any size can be implemented by adding the $\{p_t\}_{t=0}^{T}$ from Theorem 1 to the loss function or, equivalently, to the cost of the samples.

Example 3. A government agency sponsoring research is choosing which one of two universities to invest in and hires an investigator for advice. The investigator observes an additive state process, $X_{t+1} = X_t + \varepsilon_t$, where $\varepsilon_t \sim G_t$, for some independently distributed G_t and the utility is linear in the state with a constant discount factor $\delta < 1$.

$$u_t\left(q_t, x_t\right) = \delta^t\left(m_t\left(q_t\right)x_t + k_t\left(q_t\right)\right)$$

Then, the single crossing condition in Theorem 1 is always satisfied due to the following derivation,

$$\mathcal{I}\left(x_{t+1}, x\right) = -\left(\frac{\partial G_t\left(x_{t+1} - x\right)}{\partial x}\right)\frac{1}{g_t\left(x\right)} = 1,$$

because of the additivity of the process, and,

$$\frac{m_t\left(-1\right)}{m_{t+1}\left(-1\right)} = \frac{m_t\left(+1\right)}{m_{t+1}\left(+1\right)} = \frac{1}{\delta}.$$

Therefore, linear utility with either martingale processes or additive processes leads to implementability.

5. Proof of Theorem 1

Define the expression in Theorem 1 as $\tilde{\tilde{z}}$.

$$\tilde{\tilde{z}}_t(s,x) = \mathbb{E}\left[u_{t+1}(s, X_{t+1}) \,|X_t = x\right] - u_t(s,x)\,, \text{ for } s \in \{-1,+1\}$$

This is the expected gain in utility for waiting for one more period while being forced to report the same state, s.

Equivalence (2) ⇔ (3): I start by converting the single crossing condition in Theorem 1 to the monotonicity condition of $\tilde{\tilde{z}}_{t.}$

Lemma 1. *The single crossing condition in Theorem 1 holds if and only if $s \cdot \tilde{\tilde{z}}_t(s,x)$ is weakly decreasing for $s \in \{-1,+1\}$.*

Implication (2) ⇒ (1): I prove the monotonicity of $s \cdot \tilde{\tilde{z}}_t(s,x)$ implies implementability, by starting with the following observation on $\tilde{z}_{t.}$

Lemma 2. *If $s \cdot \tilde{\tilde{z}}_t(s,x)$ is weakly decreasing, then $s \cdot \tilde{z}_t(s,x)$ is always weakly decreasing.*

Define the following transfer functions. They are not the actual transfers because they also depend on x.

$$p_t(s,x) = \tilde{z}_t(s,x) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s\left(\min\left\{\max\left\{a_t, X_s\right\}, b_t\right\}\right) |X_t = x\right]$$

The following lemma shows the monotonicity of $p_t(s,x)$.

Lemma 3. *If $s \cdot \tilde{\tilde{z}}_t(s,x)$ is weakly decreasing, then $s \cdot p_t(s,x)$ is weakly decreasing.*

The value function has the following form by induction.

Lemma 4. *The value function is,*

$$V_t(x) = u_t(x) + \sum_{s=t}^{T-1} \mathbb{E}\left[z_s\left(\min\left\{\max\left\{a_s, X_s\right\}, b_s\right\}\right) |X_t = \min\left\{\max\left\{a_t, x\right\}, b_t\right\}\right].$$

At the end, I show that, given the above value function, any threshold strategy is implementable.

Lemma 5. *Given the value function from Lemma 4, $q^{(a,b)}$ is implementable.*

Implication (1) $\Rightarrow$ (2): I prove implementability implies the monotonicity of $s \cdot \tilde{\tilde{z}}_t(s,x)$.

Lemma 6. *Suppose $q^{(a,b)}$ is implementable for any thresholds (a,b), then $s \cdot \tilde{\tilde{z}}_t(s,x)$ is weakly decreasing.*

The following are the proofs of the lemmas used.

Proof of Lemma 1:

$$\begin{aligned} s \cdot \frac{d\tilde{\tilde{z}}_t(s,x)}{dx} &= s \cdot \frac{d}{dx}\mathbb{E}\left[u_{t+1}(s, X_{t+1}) | X_t\right] - s \cdot \frac{\partial u_t(s,x)}{\partial x} \\ &= -s \cdot \int_{\underline{x}}^{\bar{x}} \frac{\partial u_{t+1}(s, x_{t+1})}{\partial x} \frac{\partial F_{t+1}(x_{t+1}|x)}{\partial x} dx_{t+1} - s \cdot \frac{\partial u_t(s,x)}{\partial x} \\ &= s \cdot \int_{\underline{x}}^{\bar{x}} \frac{\partial u_{t+1}(s, x_{t+1})}{\partial x} \mathcal{I}(X_{t+1}, x) f_{t+1}(x_{t+1}|x) dx_{t+1} - s \cdot \frac{\partial u_t(s,x)}{\partial x} \\ &= s \cdot \mathbb{E}\left[\frac{\partial u_{t+1}(s, X_{t+1})}{\partial x} \mathcal{I}(X_{t+1}, X_t) | X_t = x\right] - s \cdot \frac{\partial u_t(s,x)}{\partial x} \end{aligned}$$

Therefore, $s \cdot \tilde{\tilde{z}}_t(s,x)$ is weakly decreasing if and only if,

$$s \cdot \mathbb{E}\left[\frac{\partial u_{t+1}(s, X_{t+1})}{\partial x} \mathcal{I}(X_{t+1}, X_t) | X_t = x\right] \leqslant s \cdot \frac{\partial u_t(s,x)}{\partial x},$$

which is the condition in Theorem 1. □

Proof of Lemma 2: To shorten the notations, define,

$$u_t(x) = \max_{s'} \left\{u_{t+1}(s', X_{t+1})\right\}.$$

Due to Assumption 1, following inequality holds,

$$\frac{du_t(-1,x)}{dx} \leqslant \frac{du_t(x)}{dx} \leqslant \frac{du_t(+1,x)}{dx}.$$

Therefore, $s \cdot \tilde{z}_t(s,x)$ is also weakly decreasing if the condition in Theorem 1 holds.

$$s \cdot \frac{d\tilde{z}_t(s,x)}{dx} \leqslant s \cdot \frac{d\tilde{\tilde{z}}_t(s,x)}{dx} \leqslant 0$$

□

Proof of Lemma 3: When $t = T-1$,

$$s \cdot p_t(s,x) = s \cdot \tilde{z}_{T-1}(s,x),$$

which is weakly decreasing from Lemma 2.

The rest of the proof goes by backward induction on t. Assuming $s \cdot p_{t+1}(s,x)$ is weakly decreasing,

$$\begin{aligned}
s \cdot p_t(s,x) &= s \cdot \left(\tilde{z}_t(s,x) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s, X_s\}, b_s\}) | X_t = x\right]\right) \\
&= s \cdot \left(\tilde{z}_t(s,x) + \sum_{s=t+2}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s, X_s\}, b_s\}) | X_t = x\right]\right) \\
&\quad + s \cdot \mathbb{E}\left[z_{t+1}(\min\{\max\{a_{t+1}, X_{t+1}\}, b_{t+1}\}) | X_t = x\right] \\
&= s \cdot \mathbb{E}\left[p_{t+1}(s, X_{t+1}) | X_t = x\right] \\
&\quad - s \cdot \mathbb{E}\left[u_{t+1}(s, \min\{\max\{a_t, X_{t+1}\}, b_t\}) | X_t = x\right] - s \cdot u_t(s,x) \\
&= \mathbb{E}\left[s \cdot p_{t+1}(s, X_{t+1}) | X_t = x\right] + s \cdot \tilde{\tilde{z}}_t(s,x).
\end{aligned}$$

Here, $s \cdot p_{t+1}(s, X_{t+1})$ is weakly decreasing by induction hypothesis, $s \cdot \tilde{\tilde{z}}_t(s,x)$ is weakly decreasing by assumption, and taking conditional expectations on X_t preserves monotonicity by Assumption 2. Therefore, $s \cdot p_t(s,x)$ is weakly decreasing in x for $s \in \{-1, +1\}$. □

Proof of Lemma 4: The base case when $t = T$ is,

$$V_T(x) = u_T(x) \text{ with } p_t(-1, a_t) = p_t(+1, b_t) = 0.$$

If $p_T \neq 0$ at $a_T = b_T$, redefine $u_t(s,x)$ by subtracting the $p_t(-1, a_t)$ from it for all t.

For the induction, I assume that,

$$V_{t+1}(x) = u_{t+1}(x) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s, X_s\}, b_s\}) | X_{t+1} = \min\{\max\{a_t, x\}, b_t\}\right].$$

For $x \leqslant x_t^\star$, the value function can be simplified to,

$$\begin{aligned}
V_t(x) &= \max\left\{u_t(-1,x) - p_t(-1,a_t), u_t(1,x) - p_t(+1,b_t), \mathbb{E}\left[V_{t+1}(X_{t+1}) | X_t = x\right]\right\} \\
&= \max\left\{u_t(-1,x) - p_t(-1,a_t), \mathbb{E}\left[V_{t+1}(X_{t+1}) | X_t = x\right]\right\} \\
&= u_t(-1,x) + \max\Bigg\{\tilde{z}_t(-1,a_t) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_t, X_s\}, b_t\}) | X_t = a_t\right], \\
&\quad \tilde{z}_t(-1,x) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_t, X_s\}, b_t\}) | X_t = x\right]\Bigg\} \\
&= u_t(-1,x) + \tilde{z}_t(-1, \max\{a_t, x\}) \\
&\quad + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s, X_s\}, b_s\}) | X_t = \max\{a_t, x\}\right] \\
&= u_t(-1,x) + \mathbb{E}\left[u_{t+1}(X_{t+1}) | X_t = \max\{a_t, x\}\right] - u_t(-1, \max\{a_t, x\}) \\
&\quad + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s, X_s\}, b_s\}) | X_t = \max\{a_t, x\}\right].
\end{aligned}$$

The second to last line is obtained by substituting in the transfers and applying the induction hypothesis. □

Proof of Lemma 5: I define $U_t(x)$ as the utility the agent observing state x gets if she stops.

$$U_t(x) = \max\left\{u_t(-1,x) - p_t(-1,a_t), u_t(+1,x) - p_t(+1,b_t)\right\}$$

Let $x_t^\star$ be the x that satisfies $u_t(-1,x) - p_t(-1,a_t) = u_t(+1,x) - p_t(+1,b_t)$. Then due to the monotonicity of $u_t(s,x)$ from the assumption of the lemma, $U_t(x)$ can be rewritten as,

$$U_t(x) = \begin{cases} u_t(-1,x) - p_t(-1,a_t) & \text{if } x \leqslant x_t^\star \\ u_t(+1,x) - p_t(+1,b_t) & \text{if } x \geqslant x_t^\star. \end{cases}$$

If $x > x_t^\star$, using similar arguments, we obtain,

$$\begin{aligned} V_t(x) &= u_t(+1,x) + \mathbb{E}\left[u_{t+1}(X_{t+1})\,|X_t = \min\{x,b_t\}\right] - u_t(+1,\min\{x,b_t\}) \\ &\quad + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s,X_s\},b_s\})\,|X_t = \max\{a_t,x\}\right]. \end{aligned}$$

Combining the above two pieces results in the desired form,

$$\begin{aligned} V_t(x) &= u_t(x) + \mathbb{E}\left[u_{t+1}(X_{t+1})\,|X_t = \min\{\max\{a_t,x\},b_t\}\right] - u_t(\min\{\max\{a_t,x\},b_t\}) \\ &\quad + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s,X_s\},b_s\})\,|X_t = \min\{\max\{a_t,x\},b_t\}\right] \\ &= u_t(x) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s,X_s\},b_s\})\right] + z_t(\min\{\max\{a_t,x\},b_t\}) \\ &= u_t(x) + \sum_{s=t}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s,X_s\},b_s\})\right]. \end{aligned}$$

Therefore, if $x < a_t$,

$$\begin{aligned} V_t(x) - U_t(x) &= V_t(x) - (u_t(-1,x) - p_t(-1,a_t)) \\ &= \tilde{z}_t(-1,a_t) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s,X_s\},b_s\})\,|X_t = a_t\right] \\ &\quad - \tilde{z}_t(-1,a_t) - \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s(\min\{\max\{a_s,X_s\},b_s\})\,|X_t = a_t\right] \\ &= 0. \end{aligned}$$

The same expression can be found for $x > b_t$.

If $a_t \leqslant x \leqslant x_t^\star$,

$$\begin{aligned} V_t(x) - U_t(x) &= V_t(x) - \left(u_t(-1, x) - p_t(-1, a_t)\right) \\ &= \tilde{z}_t(-1, x) + \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s\left(\min\left\{\max\left\{a_s, X_s\right\}, b_s\right\}\right) | X_t = x\right] \\ &\quad - \tilde{z}_t(-1, a_t) - \sum_{s=t+1}^{T-1} \mathbb{E}\left[z_s\left(\min\left\{\max\left\{a_s, X_s\right\}, b_s\right\}\right) | X_t = a_t\right] \\ &\geqslant 0. \end{aligned}$$

Similarly, if $x_t^\star \leqslant x \leqslant b_t$,

$$V_t(x) - U_t(x) \geqslant 0,$$

and if $x \geqslant b_t$,

$$V_t(x) - U_t(x) = 0.$$

Combining the four pieces for x in different parts of the domain, we get,

$$V_t(x) - U_t(x) \begin{cases} = 0 & \text{if } x \notin [a_t, b_t] \\ \geqslant 0 & \text{if } x \in [a_t, b_t]. \end{cases}$$

Therefore, the optimal strategy is $q^{(a,b)}$, and this is implementable by transfers $p_t(-1, a_t)$ and $p_t(+1, b_t)$, which are the transfers given in Lemma 3. □

Proof of Lemma 6: Consider the two sequences of thresholds $\{(a_t, \bar{x})\}_{t=0}^T$ and $\{(\underline{x}, b_t)\}_{t=0}^T$.

Given $\{(\underline{x}, b_t)\}_{t=0}^T$ is implementable, take $x < x'$ and assume $\tilde{\tilde{z}}_t(+1, x) < \tilde{\tilde{z}}_t(+1, x')$ for a contradiction.

For x, the value function in the recursive form is given by,

$$\begin{aligned} V_t(x) - U_t(x) &= \max\left\{U_t(x), \mathbb{E}\left[V_{t+1}(X_{t+1}) | X_t = x\right]\right\} - U_t(x) \\ &= \max\left\{0, \mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x\right] \right. \\ &\quad \left. + \mathbb{E}\left[u_{t+1}(X_{t+1}) | X_t = x\right] - u_t(+1, x) - p_t(-1, a_t)\right\} \\ &= \max\left\{0, \mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x\right] + \tilde{\tilde{z}}_t(+1, x) - p_t(-1, a_t)\right\}. \end{aligned}$$

All of the future transfers are fixed for different values of b_t. Therefore, $V_{t+1}(x)$ and $U_t(x)$ are fixed for different values of b_t. Consider two cases for b_t as follows:

For $x < x'$, if the following holds,

$$\mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x\right] \leqslant \mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x'\right],$$

I implement the stopping rule $b_t = x'$. This means that the agent observing state $x_t = x$ should continue at time t. However,

$$\begin{aligned} &\mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x\right] + \tilde{\tilde{z}}_t(+1, x) - p_t(-1, a_t) \\ &< \mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x'\right] + \tilde{\tilde{z}}_t(+1, x') - p_t(-1, a_t) \\ &= 0. \end{aligned}$$

The inequality shows that it is optimal for the agent observing state $x_t = x$ to stop, contradicting the definition of the stopping rule with $b_t = x' > x$.

Similarly, if the following holds,

$$\mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x\right] \geqslant \mathbb{E}\left[V_{t+1}(X_{t+1}) - u_{t+1}(X_{t+1}) | X_t = x'\right],$$

I implement the stopping rule $b_t = x$. It means that the agent observing state $x_t = x$ should stop at time t. However,

$$\begin{aligned}
&\mathbb{E}\left[V_{t+1}\left(X_{t+1}\right) - u_{t+1}\left(X_{t+1}\right)|X_t = x'\right] + \tilde{\tilde{z}}_t\left(+1, x'\right) - p_t\left(-1, a_t\right) \\
&> \mathbb{E}\left[V_{t+1}\left(X_{t+1}\right) - u_{t+1}\left(X_{t+1}\right)|X_t = x\right] + \tilde{\tilde{z}}_t\left(+1, x\right) - p_t\left(-1, a_t\right) \\
&= 0.
\end{aligned}$$

This shows that it is optimal for the agent observing state $x_t = x'$ to continue, contradicting the definition of the stopping rule with $b_t = x < x'$.

Therefore, $\tilde{\tilde{z}}_t\left(+1, x\right)$ is weakly decreasing in x.

Similarly, $\tilde{\tilde{z}}_t\left(-1, x\right)$ is weakly increasing in x since $\left\{\left(a_t, \bar{x}\right)\right\}_{t=0}^{T}$ is implementable.

Therefore, $s \cdot \tilde{\tilde{z}}_t\left(s, x\right)$ is weakly decreasing in x for $s \in \{+1, -1\}$. □

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