DESIGN OF SEARCH BY COMMITTEES

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Abstract. I apply a mechanism design approach to committee search problems, such as a department's hiring process or a couple's search for a house. A special class of simple dynamic decision rules has agents submit, in each period, one of three votes: veto, approve, or recommend; the current option is adopted whenever no agent vetoes and at least one agent recommends. I show that every implementable payoff can be attained by randomizing among these simple rules. This result dramatically simplifies the design problem.

1. Introduction

A hiring committee with two members receives job applications and conducts interviews until a position is filled. A decision is made right after each interview and is irreversible. Every committee member obtains private value from hiring a candidate. A common decision rule is the unanimity rule, according to which a candidate is hired whenever he is preferred by both members. Another possibility is a rule according to which each member submits a numerical score for a candidate and hiring occurs whenever the sum or average score is above a preset threshold.

A married couple looks for a house until they decide on purchasing one. The Canadian housing market is competitive, and a house is gone before a new one becomes available. The husband and the wife have different views regarding the ideal house, and their preferences with respect to its style and size may differ. In many households, decisions are not made collectively, and one member can establish dictatorship. In other cases, each member of the couple rates a house as "ideal", "acceptable" or "uninhabitable", and they purchase the house if it is not uninhabitable for either member and ideal for at least one.

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Two coauthors periodically gain access to new data sets. The authors have differing opinions about whether a certain data set could lead to interesting results. Data sets are costly to obtain and process, and they are likely to be used by other authors before a new one appears. A reasonable decision rule could be adopting a data set whenever at least one of the coauthors thinks the potential results are interesting. The threshold for a data set to be deemed interesting can vary between coauthors and and over time.

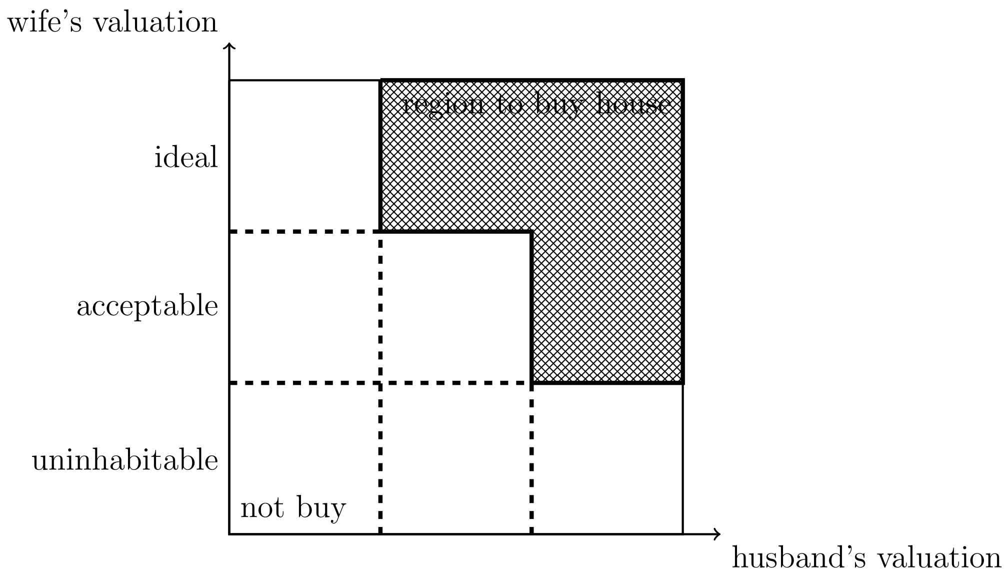

Figure 1. A possible decision rule in the couple's housing example

In all of these scenarios, two agents have to make a collective and irreversible decision regarding an option to adopt when facing options that are presented to them sequentially. Monetary transfers are not feasible. All the decision rules in these examples are plausible ones that are used widely in various collective search problems. In this paper, I focus on a class of simple voting rules, called ternary rules, that give each agent the power to veto, approve, or recommend an option, and the current option is adopted whenever no one vetoes and at least one of the agents recommends. It is similar to the "ideal-acceptable-uninhabitable" rule described in the house-searching example and in Figure 1. The shaded region in Figure 1 depicts the set of value pairs that lead to the purchase of the current house. The contribution of this paper is showing that every implementable decision rules is payoff-equivalent to some randomization among ternary rules, and as a result, a designer can restrict attention to the use of only these simple voting rules.

In particular, I consider a model in which two agents observe private values of, for example, hiring a candidate, in every period. The values are independent over time, but the values from different agents in the same period can be correlated. In each period, either the candidate is hired, in which case, every agent gets her private value, or the decision is delayed to the next period. The mechanism incentivizes the agents to report their values truthfully without using transfers, and it uses differing future decision rules to link incentives over time.

I assume a finite time horizon, and each agent obtains a fixed value from an outside option if no candidate is ever hired. In the last period, the only implementable decisions are,

- unanimity, where the candidate is hired when both agents have values higher than their outside options,
- dictatorship, where the candidate is hired as long as the dictator agent has value higher her outside option,
- reverse unanimity, where the candidate is not hired when both agents have values lower than their outside options, and
- constant, where the candidate is either always hired or never hired.

In the second to last period, a ternary rule specifies a lower threshold of value below which an agent vetoes and an upper threshold above which an agent recommends. It also needs to specify which rule to use in the last period after each pair of reported values. The following is an example of an implementable ternary rule.

- If one agent recommends and the other does not veto, the candidate is hired, and the search ends.
- If one agent recommends and the other vetoes, the candidate is not hired, and the agent who recommends becomes the dictator in the last period.
- If both agents choose to either approve or veto, the candidate is not hired, and the agents use the unanimity in the last period.

The reason for using the dictator rule is to give the agent who vetoes another agent's preferred candidate a lower expected future payoff as punishment to ensure that veto power will be exercised with discretion.

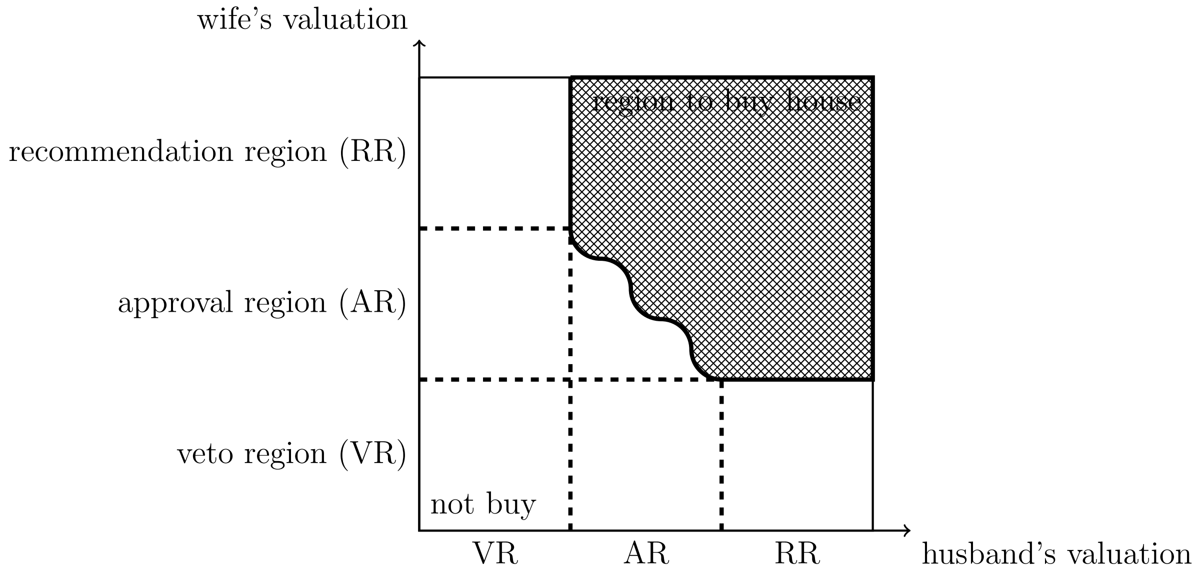

Figure 2. Another possible decision rule in the couple's housing example

Many other decision rules are implementable in the periods before the last such as the one depicted in Figure 2. Given an implementable rule, the domain of each agent's value can be partitioned into three regions:

- the region where the candidate is not hired regardless of the other agent's value, called the veto region,
- the region where the candidate is hired as long as the other agent does not veto, called the recommendation region, and,
- the remaining region, called the approval region.

A candidate may or not may be hired in the region where both agents approve. Among the rules with the same partitions, the one that always rejects the candidate in the region where both agents approve is a ternary rule, and it yields the highest payoffs for both agents. This is the case because, intuitively, when considering a candidate whom both agents do not recommend, no agent will have a significant loss from continuing the search, and on average, they can both be better off if they continue.

For the same reason, the rule that always hires the candidate in the region where both agents approve is another ternary rule that yields lower payoffs for both agents. The dictator rule, according to which the candidate is hired whenever the dictator's value is above some threshold, is another special case of a ternary rule. With a carefully chosen threshold, dictatorship gives one agent higher payoff and the other agent lower payoff compared to the original rule. Therefore, any implementable decision rule is payoff-equivalent to the randomization between these ternary rules.

Technically, the model involves the implementation of an incentive compatible mechanism. Dynamic ex-post incentive compatibility is used as the definition of implementability so that the solution is robust to private communication between agents and robust to correlation between values and beliefs of the agents. I also restrict attention to mechanisms that are deterministic in every period, where either the candidate is hired for certain, or the decision is delayed. These assumptions make the model tractable. I make other assumptions for the purpose of presentation. For example, there are only two agents and a finite number of periods, and the private values are drawn from distributions with full support on some compact set and are independent over time. Those assumptions can be relaxed without significantly changing the main results.

This paper is closely related to the literature on dynamic mechanism design without transfers. Guo and Hörner (2015) investigate the problem without transfers, and in which goods can be allocated in multiple periods. They characterize the optimal mechanism and provide a simple implementation in which the agents have virtual budgets. Lipnowski and Ramos (2016) develop a similar model with similar results, in which the principal has less commitment power. In this model, the principal can decrease the veto threshold in the next period if she exercises her veto power. This is similar to the budget mechanism from Guo and Hörner (2015) in the sense that the agent has a virtual budget of veto power, and if she vetoes excessively, she will gradually lose her power to veto.

Kovác, Krähmer, and Tatur (2013) examine the stopping problem in which only a single good is allocated. In their model, the optimal mechanism is one in which the principal chooses different probabilities for assigning the good at different times to incentivize the agent's use of the stopping rule the principal prefers. I focus on allocation rules that are deterministic in each period, so instead of change probabilities, the principal changes the amount of control an agent has over the allocation in every period to incentivize an agent. The size of the region where an agent has more control over the allocation, for example, to veto or recommend it, is different in every period, and this is used in place of the probabilities in the deadline mechanism proposed by Kovác, Krähmer, and Tatur (2013). Both Guo and Hörner (2015) and Kovác, Krähmer, and Tatur (2013) focus on single-agent problems, whereas this paper focuses on problems with more than one agent.

Johnson (2014) considers the problem with multiple agents, but the good is allocated to only one of the agents. He uses the promised utility approach where the principal promises future allocation rules that correspond to different expected payoffs to incentivize truth-telling. I use the same approach, but I consider the problem with a public good. In his model, the agents trade favors in a virtual market for decision rights, and as a result, they take turns obtaining their favorite private allocation. In this model, the agents also trade decision rights, but the goal is to wait for the public allocation that is preferable to every agent.

Moldovanu and Shi (2013) study the committee search problem similar to mine. They solve the stationary voting rules that have a single threshold for each agent. Compte and Jehiel (2010) and Albrecht, Anderson, and Vroman (2010) develop a similar model and focus on majority voting rules. I consider all direct revelation mechanisms, and one result states that if decisions cannot be linked over time, then all implementable decision rules are the voting rules with a single threshold, as in Moldovanu and Shi (2013). Additional decision rules that are not voting rules can be implementable in this environment through the linking of decisions over time, but I show that every implementable rule is payoff-equivalent to some randomization among ternary voting rules, and this justifies the restriction to using only simple voting rules when solving committee search problems.

This paper is organized as follows. Section 1.2 introduces the model. Section 1.3 characterizes implementability. Section 1.4 states and explains the main result: every implementable payoff can be attained by randomizing among ternary rules. Section 1.5 concludes.

2. Model

In this section, I describe the agent's stopping problem and the principal's design problem. In particular, I define implementability as dynamic ex-post incentive compatibility, and explain why it is appropriate for this model.

2.1. **Valuations.** In this subsection, I describe the payoffs and the class of mechanisms.

I describe the model with two agents. A principal hires a new employee through a committee. In each period, a new candidate appears and each agent in the committee observes the value of hiring the candidate. The number of periods T is finite and there is no discounting. I assume that the values in period t, $v_t = (v_{t,1}, v_{t,2}) \in \mathcal{V} = [\underline{v}, \bar{v}]^2$ are independently distributed over time with continuous density f_t and full support $\mathcal{V}$, and that the means are normalized to 0 in each coordinate.

I use v^t to denote a history of values, or reports, from period 1 to period t,

$$\begin{aligned} v^t &= (v_1, v_2, ..., v_t) \\ &= ((v_{1,1}, v_{1,2}), (v_{2,1}, v_{2,2}), ..., (v_{t,1}, v_{t,2})) . \end{aligned}$$

I assume independence over time but not between agents. Independence is not necessary for the results in this paper to hold, I make the assumption only for simpler presentation. In particular, the assumption simplifies the notations and the calculations in the examples. The value of hiring a candidate to each agent may be correlated.

Due to the revelation principle for deterministic mechanisms with ex-post constraints in Jarman, Meisner et al. (2017), I restrict attention to direct mechanisms, in which the agents report only their own valuation in every period, which indicates whether to hire the candidate in each period. I define the mapping $q : \mathcal{V} \to [0, 1]$ as a stage mechanism after some history of reports, $v^{t-1} = (v_1, v_2, ..., v_{t-1})$. Here, $q(v_t|v^{t-1})$ is the conditional probability that the candidate is hired in period t if the agents report v_t after the history v^{t-1}, condition on the candidate not hired in periods before t. When the context is clear, I omit the history after which the stage mechanism is used and use the expression $q(v_t)$. The grand mechanism, Q, is the collection of stage mechanisms following every possible history,

$$Q : \bigcup_{t \leqslant T} \mathcal{V}^t \to [0,1] .$$

The mechanism can depend on historical reports. As a result, the principal can choose different allocations in the future as rewards or punishments to incentivize truthful reports.

If no candidate is hired by the end of the last period, the agents get values from the outside option in period $T+1, v_{T+1} = v^\star = (v_1^\star, v_2^\star)$, in the interior of $\mathcal{V}$. For each terminal history in the form $v^{T+1} = (v_1, v_2, ..., v_T, v^\star) \in \mathcal{V}^{T+1}$, the total probability of hiring the candidate should add up to 1,

$$\sum_{t=1}^{T} q(v_t) \prod_{s=1}^{t-1} (1 - q(v_s)) = 1.$$

Given a grand mechanism Q, let $w_{i,t}(Q; v^{t-1})$ denote the ex-ante continuation value of agent i in period t after some history $v^{t-1} \in \mathcal{V}^{t-1}$, before $v_t, v_{t+1}, ..., v_T$ are realized, assuming both agents report truthfully.

$$w_{i,t}\left(Q; v^{t-1}\right) = \mathbb{E}\left[\sum_{s=t}^{T+1} q(v_s) \prod_{s'=1}^{s-1} (1 - q(v_{s'}))\, v_{i,s}\right].$$

For a mechanism, Q, the *continuation value after history* v^{t-1} is a pair,

$$w_t\left(Q; v^{t-1}\right) = \left(w_{1,t}\left(Q; v^{t-1}\right), w_{2,t}\left(Q; v^{t-1}\right)\right).$$

In particular, the total payoff an agent, i, gets from a mechanism, Q, is $w_{i,1}(Q)$.

I also define two grand mechanisms, Q_1, Q_2, to be *payoff-equivalent* after history v^{t-1} if,

$$w_t\left(Q_1; v^{t-1}\right) = w_t\left(Q_2; v^{t-1}\right).$$

In particular, the payoff-equivalence between two dynamic mechanism Q_1 and Q_2 after history v^{t-1} is defined in terms of the ex-ante continuation payoff in period t before the vector of period-t valuations $(v_{1,t}, v_{2,t})$ is realized.

2.2. **Implementability.** In this subsection, I define dynamic ex-post implementability and outline the reasons it is used for this problem. Moreover, I define and restrict attention to a subset of random mechanisms I call quasi-deterministic mechanisms, where the mechanism is deterministic in each stage but between-period randomization is allowed.

Definition 1. A mechanism, Q, is (dynamic ex-post) incentive compatible, if in each period $t \in \{1, 2, ..., T\}$, after each history $v^{t-1} \in \mathcal{V}^{t-1}$, every $i \in \{1, 2\}$, every $v_{i,t} \in [\underline{v}, \bar{v}]$, and every $v_{-i,t} \in [\underline{v}, \bar{v}]$,

$$v_{i,t} \in \arg\max_{\hat{v}_{i,t} \in [\underline{v}, \bar{v}]} v_{i,t} q\left(\hat{v}_{i,t}, v_{-i,t}\right) + \left(1 - q\left(\hat{v}_{i,t}, v_{-i,t}\right)\right) w_{i,t+1}\left(Q; v^{t-1}, \hat{v}_{i,t}, v_{-i,t}\right).$$

A mechanism is dynamic ex-post incentive compatible (or the corresponding decision rule is dynamic ex-post implementable) if, in every period, it is optimal to report the true valuation given the other agent's report.

This definition is the same as the one from Noda (2016), but without transfers and discounting. Noda (2016) terms this type of implementability *within-period ex-post incentive compatibility,* or wp-EPIC. Bergemann and Välimäki (2010), Parkes, Cavallo, Constantin, and Singh (2010),and Athey and Segal (2013) use a more complicated version of ex-post implementability since valuations and types in their models can be correlated over time.

Ex-post implementability is used because it is robust to private communication between agents, within-period correlation between valuations, and variation in agents' beliefs about each other's types. This definition of incentive compatibility also makes the model tractable. Bergemann and Morris (2005) provide details regarding the use of ex-post incentive compatibility.

I omit individual rationality constraints, and agents are forced to participate.

I focus on mechanisms that are deterministic in every stage in which either the candidate is hired for certain or the decision is delayed to the next period.

Definition 2. A stage mechanism, q, is *deterministic* if $q(\cdot) \in \{0, 1\}$. A grand mechanism, Q, is *quasi-deterministic* if after every history $v^{t-1} \in \mathcal{V}^{t-1}$, only randomization among multiple deterministic stage mechanisms is used.

For example, after $(v_{1,t}, v_{2,t})$ are reported, the principal must either hire the candidate with probability 1 or delay hiring to the next period, but in the event that the principal chooses to delay, he can randomize among multiple stage mechanisms in period $t + 1$.

3. Implementability

In this section, I characterize the conditions for implementability. It is convenient to divide the analysis into a few steps. In the first step, I characterize implementable stage mechanisms in the last period. Next, I explain that the implementability in the dynamic setting can be reduced to a version of the static problem with appropriately chosen continuation values. Finally, I explain why the dynamic problem is substantially richer than a sequence of static ones.

3.1. **Static Implementation.** In this subsection, I consider the problem when $T = 1$. In this case, the continuation value is fixed at the outside option $(v_1^\star, v_2^\star)$. I describe a class of mechanisms called binary stage mechanisms that characterizes implementability. The same characterization applies to the last period when $T > 1$.

Definition 3. A stage mechanism, q, is binary, with outside option (a_1, a_2), if for each $i \in \{1, 2\}$, for every $v_{-i,t} \in [\underline{v}, \bar{v}]$, either $q(v_{i,t}, v_{-i,t})$ is constant (0 or 1) for every $v_{i,t} \in [\underline{v}, \bar{v}]$, or,

$$q(v_{i,t}, v_{-i,t}) = \begin{cases} 0 & \text{if } v_{i,t} < a_i \\ 1 & \text{if } v_{i,t} > a_i \end{cases}.$$

As shown in Figure 3, there are only six mechanisms that are binary for a fixed outside option. The shaded regions are the acceptance regions, $\{v_t : q(v_t) = 1\}$, where the candidate

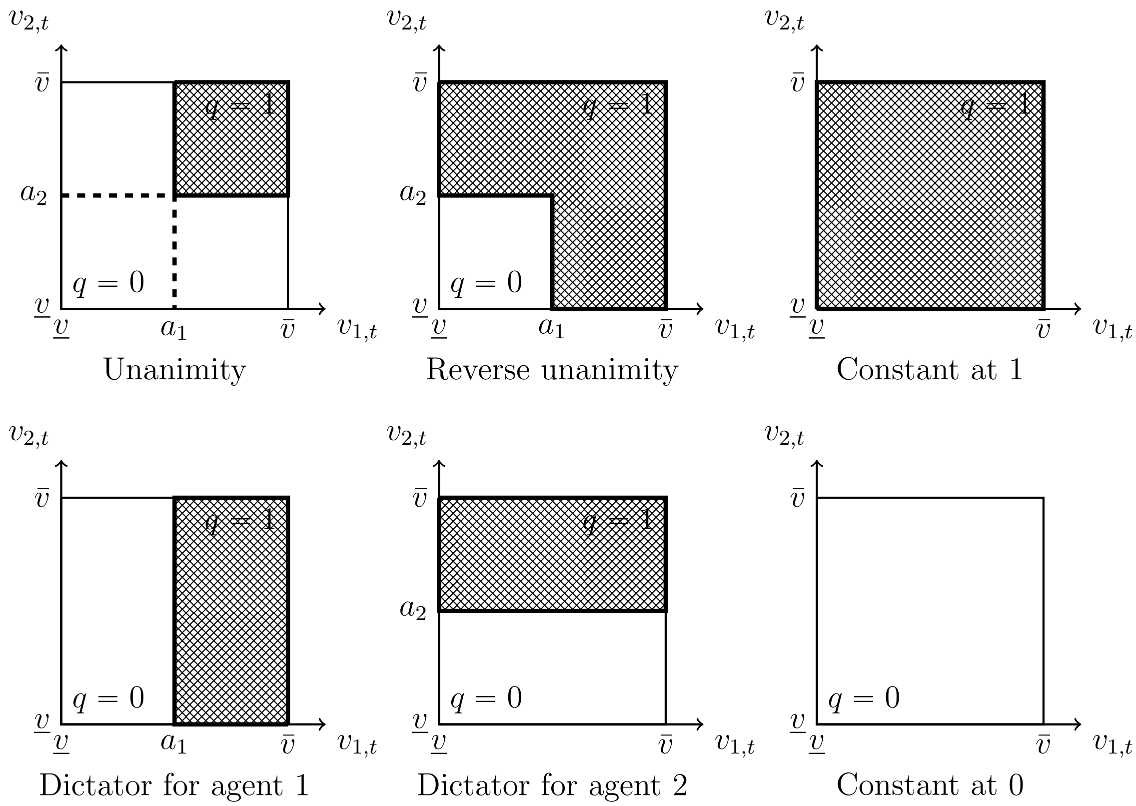

FIGURE 3. All six binary stage mechanisms

is hired. The remaining region is where the candidate is not hired and the outside option (a_1, a_2) is given to the agents.

Lemma 1. *For $T = 1$, if a quasi-deterministic mechanism, Q, is incentive compatible, then the (only) stage mechanism $q(\cdot|\varnothing)$ must be binary with outside option $(v_1^\star, v_2^\star)$.*

The intuition for Lemma 1 is as follows. Note that in each of the six binary stage mechanisms, whenever there is a threshold above which the candidate is hired and below which the candidate is not, the threshold value must be $v_1^\star$ for agent 1 and $v_2^\star$ for agent 2. This is because, in the last period, if the candidate is still not hired, the agents get $(v_1^\star, v_2^\star)$ in period $T+1$. For an agent and a fixed report from the other agent, if the mechanism always hires the candidate or never hires the candidate, this agent will be indifferent between reporting any value, and as a result, she will not misreport. Otherwise, if the mechanism hires the candidate when she reports a value lower than the outside option, she will misreport a higher value and obtain the outside option instead; and if the mechanism does not hire the candidate when she reports a value higher than the outside option, she will misreport a lower value to get the candidate hired. Therefore, the only incentive-compatible mechanism hires the candidate if and only if an agent reports a value higher than the outside option.

3.2. **Dynamic Implementation.** In this subsection, I show how the dynamic problem can be reduced to a static one but with more than one possible outside options. I then state a monotonicity condition as a characterization of all quasi-deterministic incentive-compatible mechanisms in this environment.

An incentive-compatible grand mechanism consists of a collection of incentive-compatible stage mechanisms, one after each history v^{t-1}, with continuation value pairs for all reports in the rejection region, $\{v_t : q(v_t) = 0\}$, where each continuation value pair corresponds to some sequence of incentive-compatible stage mechanisms in periods $t+1, t+2, ..., T$.

Given a stage mechanism $q(\cdot|v^{t-1})$ and the report of every agent other than i, define the following threshold function,

$$R_i(v_{-i,t}) = \inf\{v_{i,t} : q(v_t) = 1\}, \tag{1}$$

with the convention that $\inf \varnothing = \infty$.

For deterministic stage mechanisms, there must thresholds (R_1, R_2) such that the candidate is hired if and only if an agent reports a value above the threshold. By incentive compatibility, the continuation value function must be constant whenever the value observed is below the threshold, since if it is not, an agent can misreport and obtain a different, and possibly higher, continuation value. This observation is stated as the following lemma.

Lemma 2. *A quasi-deterministic mechanism, Q, is incentive compatible if and only if after every history v^{t-1}, for each $i \in \{1,2\}$, and every $v_{-i,t} \in [\underline{v}, \bar{v}]$,*

$$q(v_t) = \begin{cases} 0 & \text{if } v_{i,t} < R_i(v_{-i,t}) \\ 1 & \text{if } v_{i,t} > R_i(v_{-i,t}) \end{cases},$$

and for every $v_t = (v_{i,t}, v_{-i,t})$ *such that* $q(v_t) = 0$,

$$w_{i,t+1}\left(Q; v^{t-1}, v_t\right) \text{ is independent of } v_{i,t}, \text{ and,}$$

$$w_{i,t+1}\left(Q; v^{t-1}, v_t\right) = R_i(v_{-i,t}) \text{ if } R_i(v_{-i,t}) \neq \infty.$$

The property is stated recursively: the payoff $w_{i,t+1}$ is the payoff from some sequence of stage mechanisms in periods $t+1, t+2, ...T$, and those stage mechanisms also satisfy the above conditions.

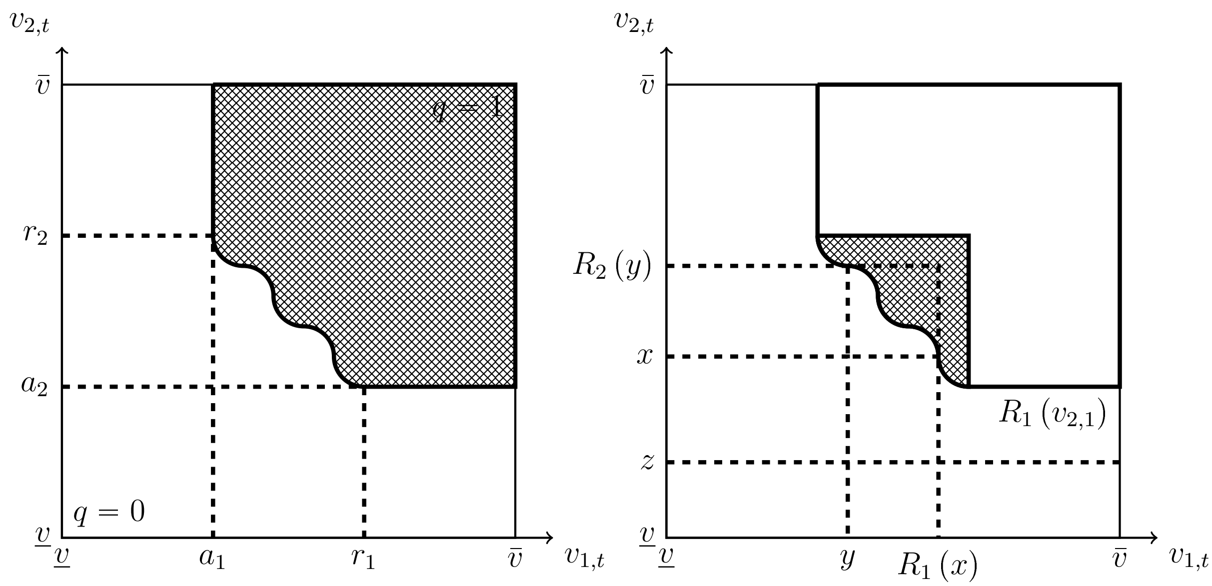

FIGURE 4. Example of a stage mechanism that is possibly incentive compatible

An example of a stage mechanism that satisfies the condition of Lemma 2 is depicted in the diagram on the left-hand side of Figure 4. According to the first part of Lemma 2, the acceptance region has the monotonicity property: if the candidate is accepted at some value pair v_t, he is also accepted at the value pairs that dominate v_t.

The continuation values for agent 1 along the line $v_{2,t} = z$ in the diagram on the right-hand side of Figure 4 are constant; for example,

$$w_{1,t+1}\left(v^{t-1}, (v_{1,t}, z)\right) = a_1 \ \forall \ v_{1,t} \in [\underline{v}, \bar{v}].$$

If continuation values for agent 1 are not the same along this line, then the type of agent 1 who has a value that leads to a lower continuation value will misreport and get a different and higher continuation value.

The continuation values of agent 1 along $v_{2,t} = x$ in the diagram on the right-hand side of Figure 4 are equal to the lowest valuation for which the candidate is accepted $R_1(x)$, or,

$$w_{1,t+1}\left(v^{t-1}, (v_{1,t}, x)\right) = R_1(x) \;\; \forall \; v_{1,t} \text{ such that } q(v_{1,t}, x) = 0.$$

Not only must the continuation value be constant for every $v_{1,t}$, it must be equal to $R_1(x)$, because if the continuation value is higher, say $R_1(x) + \delta$, then the type of agent 1 with value $R_1(x) + \dfrac{\delta}{2}$ will have an incentive to misreport a lower value and delay hiring to get a higher continuation value. Similarly, if the continuation value is lower, say $R_1(x) - \delta$, then an agent 1 observing $R_1(x) - \dfrac{\delta}{2}$ will have an incentive to misreport a higher value to get the candidate hired in the current period.

Similar requirements apply to agent 2 along the vertical line segments in the diagram. For example, the continuation value at the point (x, y) is fixed for both players at,

$$w_{t+1}\left(v^{t-1}, (x, y)\right) = \left(R_1(y), R_2(x)\right).$$

3.3. **Linking Decisions.** The principal can link payoffs over time by using mechanisms that are history dependent. In this subsection, I demonstrate, with two examples, that such mechanisms can lead to Pareto improvements over history-independent ones.

As demonstrated in Figure 4, there are many incentive-compatible stage mechanisms in periods $1, 2, ..., T-1$ that look different from the ones in the last period, T. A key reason that more allocation rules are implementable is the linking of decisions: the principal can use different randomizations among stage mechanisms after different reports to punish or reward the agents. I call these mechanisms history-dependent mechanisms.

Definition 4. A grand mechanism is *history independent* if the distributions over the stage mechanisms after any two histories with the same length are the same, that is, if

$$q\left(\cdot | v^t\right) = q\left(\cdot | \tilde{v}^t\right) \text{ for every } v^t, \tilde{v}^t \in \mathcal{V}^t, \text{ and every } t \in \{1, 2, ..., T\}.$$

In general, if the principal is restricted to the use of history-independent mechanisms, then every stage mechanism in every period must be binary. This observation is stated in the following corollary to Lemma 1.

Corollary 1. *If a quasi-deterministic incentive-compatible mechanism is history independent, then the stage mechanisms chosen with strictly positive probabilities after every history must be binary.*

Proof. After every history v^t, only one continuation value is allowed in a history-independent mechanism, because there is only one sequence of stage mechanisms in periods $t{+}1, t{+}2, ..., T$. Then, every period is similar to the last period, with the exception that the outside option may differ from $(v_1^\star, v_2^\star)$. The set of stage mechanisms that is incentive compatible with a single outside option is the set of binary stage mechanisms for that outside option. $\square$

The example below with $T = 2$ illustrates the difference between a history-dependent and a history-independent mechanism.

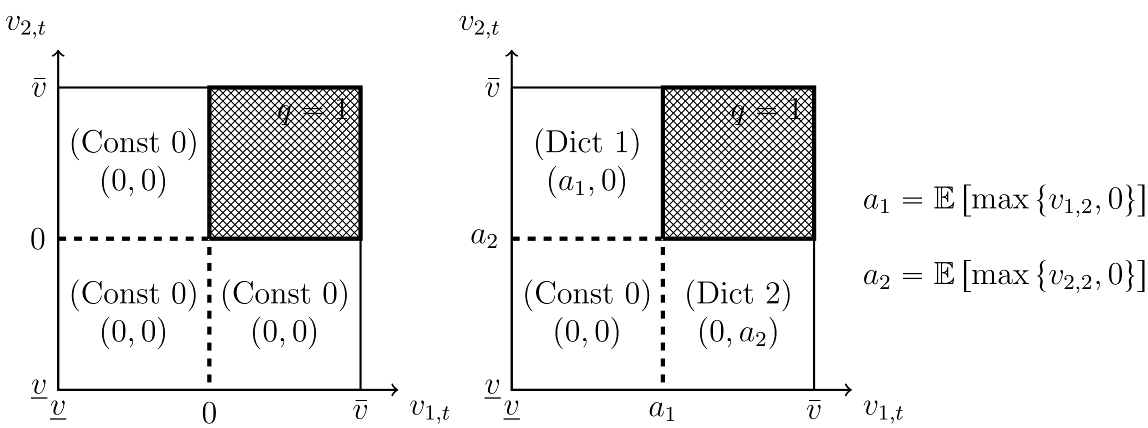

FIGURE 5. History independent (left) vs dependent (right) unanimity in period 1

Example 1. Suppose the value distributions are symmetric, the outside option is $v^\star = (0, 0)$, and the stage mechanism in period 1 involves symmetric unanimity rules. The two choices for the stage mechanisms in period 2 are

(1) constant 0 (or unanimity or reverse unanimity) after all histories in which the candidate is not hired in period 1 or,

(2) dictatorship by agent 1 after histories in which the candidate is not hired because agent 1 reports a value lower than a_1 but agent 2 reports a value higher than a_2; dictatorship by agent 2 after histories in which the candidate is not hired because agent 2 reports a value lower than a_2 but agent 1 reports a value higher than a_1; and constant 0 in period 2 after histories in which the candidate is not hired because both agents report values lower than their respective $a_{i.}$

These two mechanisms are depicted in the diagrams in Figure 5. Mechanism (1) is history independent and mechanism (2) is history dependent. Both diagrams depict the unanimity stage mechanism in period 1, and the name of the stage mechanism and its resulting continuation value pair from period 2 is written in each region in which that continuation mechanism is used.

Note that $a_1 = \mathbb{E}\left[\max\left\{v_{1,2}, 0\right\}\right]$ and $a_2 = \mathbb{E}\left[\max\left\{v_{2,2}, 0\right\}\right]$ because the property in Lemma 2 must be satisfied and the $\left(\mathbb{E}\left[\max\left\{v_{1,2}, 0\right\}\right], 0\right)$ and $\left(0, \mathbb{E}\left[\max\left\{v_{2,2}, 0\right\}\right]\right)$ are the continuation value generated by the two dictator stage mechanisms in the second period.

In the next example, I show that there are rules that are not binary and give both agents higher payoffs than any binary rules.

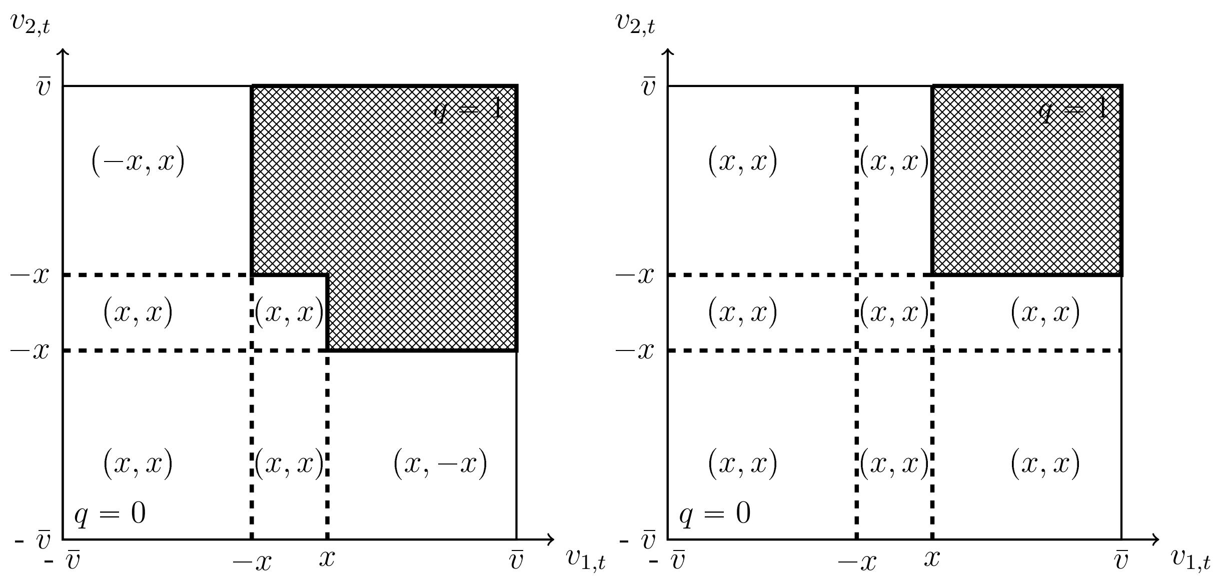

FIGURE 6. A mechanism, not binary, (left) that yields higher payoffs for both agents than the binary mechanism (right)

Example 2. Suppose the values in some period $t < T$ are independently piecewise uniformly distributed such that the probability in each of the nine rectangular regions in the diagram on the left-hand side of Figure 6 is $\frac{1}{9}$. I assume that in period $t+1$, the highest possible symmetric payoff pair that corresponds to some incentive-compatible mechanism is (x, x). For simplicity, let $\underline{v} = -\bar{v}$ and suppose $x < \frac{1}{7}\bar{v}$.

The difference in period t agent i payoff from the two stage mechanisms in Figure 6, after the same history v^{t-1}, is,

$$
\begin{aligned}
& w_{i,t}\left(Q^{left}; v^{t-1}\right) - w_{i,t}\left(Q^{right}; v^{t-1}\right) \\
&= \frac{1}{9}\left(-x + \frac{x+\bar{v}}{2}\right) - \frac{1}{9}\left(2x + x\right) \\
&> \frac{1}{9}\left(-x + \frac{x+7x}{2}\right) - \frac{1}{9}\left(2x + x\right) \\
&= 0.
\end{aligned}
$$

It can be shown that this stage mechanism yields higher payoff for both agents than do the other symmetric binary stage mechanisms, including the reverse unanimity and constant rules. In the next section, I show that this observation is not a coincidence and that all Pareto optimal stage mechanisms have the same shape as the one depicted in the diagram on the left-hand side of Figure 6.

4. Ternary Mechanisms

This section presents the main result of the paper. I start by defining ternary mechanisms, an important class of mechanisms with a very simple interpretation. The main result shows that any quasi-deterministic mechanism can be constructed by randomization among ternary stage mechanisms. The proof is divided into two steps. In the first subsection, I show that every Pareto optimal stage mechanism must be ternary. In the second subsection, I show that the rest of the boundary of the set of payoffs that can be generated by quasi-deterministic incentive-compatible mechanisms, that is not Pareto optimal, consists of only randomizations among binary mechanisms. The main result is stated and discussed in the last subsection.

Definition 5. A stage mechanism, q, is ternary if there are a_1, r_1, a_2, r_2, where $\underline{v} \leqslant a_i \leqslant r_i \leqslant \bar{v}$ for each $i \in \{1, 2\}$ such that,

$$q(v_t) = \begin{cases} 0 & \text{if } v_{i,t} < a_i \text{ for some } i \text{ or } a_i \leqslant v_{i,t} < r_i \text{ for all } i \\ 1 & \text{if } v_{i,t} > r_i \text{ for some } i \text{ and } v_{i,t} > a_i \text{ for all } i \end{cases}.$$

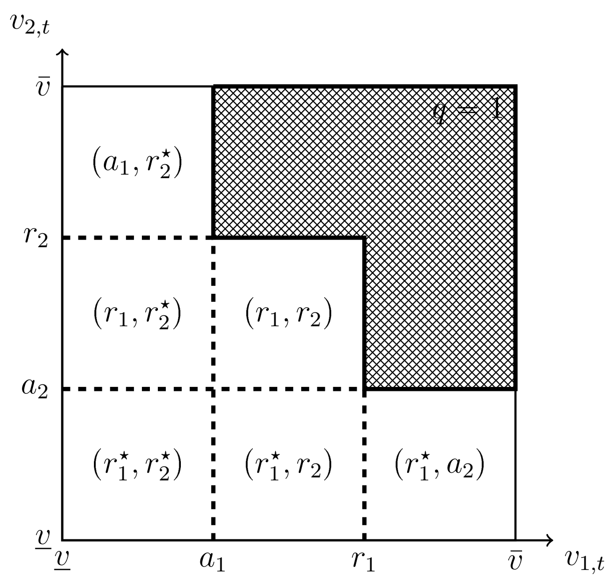

FIGURE 7. An example of a ternary stage mechanism

Figure 7 illustrates a typical ternary stage mechanism. The continuation values are placed in each rectangular region instead of the names of the continuation stage mechanisms from the next period. For general $T > 2$, the complete sequence of stage mechanisms after the current period is irrelevant and difficult to state explicitly; therefore, I write only the pair of continuation values, such as (r_1, r_2), in each rectangular region. This means that if the agents report v_t in this region, the candidate will not be hired and a sequence of stage mechanisms that results in payoff $w_t(Q; v^{t-1}) = (r_1, r_2)$ will be used in periods $t+1, t+2, ..., T$. As shown in Figure 7, the continuation values without a star are fixed due to Lemma 2, and the values with a star simply examples of possible continuation values.

Ternary stage mechanisms are voting mechanisms in which each agent can cast one of three votes: veto, approve, or recommend. Every agent can veto the candidate, and in order to hire the candidate, the principal needs at least one person to recommend the candidate.

In particular, this voting rule does not hire the candidate if both agents approve and neither recommends. Each agent has three intervals separated by a_i and r_i, the smallest (lower than a_i) where the agent can veto the candidate, the largest (higher than r_i) where the agent recommends and the candidate is hired as long as the other agent does not veto, and the one in the middle where the candidate is hired only when the other agent recommends.

4.1. **Pareto Optimal Mechanisms.** In this subsection, I define Pareto optimal mechanisms as the ones that are restricted Pareto optimal within the set of quasi-deterministic incentive-compatible mechanisms. I briefly explain why if a stage mechanism is not ternary after some history, then it is Pareto dominated by one that is ternary after the same history.

Definition 6. For incentive-compatible mechanisms Q and $\tilde{Q}$, the mechanism Q *Pareto dominates* $\tilde{Q}$ after history v^{t-1} if

$$w_{i,t}\left(Q; v^{t-1}\right) \geqslant w_{i,t}\left(\tilde{Q}; v^{t-1}\right) \text{ for each } i \in \{1,2\},$$

with strict inequality for at least one agent.

An incentive-compatible mechanism, Q, is *Pareto optimal after history* v^{t-1}, if it is not Pareto dominated by any other incentive compatible mechanism, $\tilde{Q}$, after the same history v^{t-1}.

Lemma 3. *If a stage mechanism is Pareto optimal after some history, then it is payoff-equivalent to a randomization among ternary stage mechanisms after the same history.*

For an arbitrary incentive-compatible stage mechanism like the one in the diagram on the left-hand side of Figure 8, I define a_i and r_i as follows:

$$a_i = \sup_{v_{i,t}} \left\{ v_{i,t} : q\left(v_{i,t}, v_{-i,t}\right) = 0 \text{ for all } v_{-i,t} \in \mathcal{V}_{-i} \right\}, \tag{2}$$

If agent i observes a value lower than a_i, the candidate will not be hired, regardless of what value the other agent reports.

$$r_i = \lim_{\varepsilon \to 0^+} R_i\left(a_{-i} + \varepsilon\right), \tag{3}$$

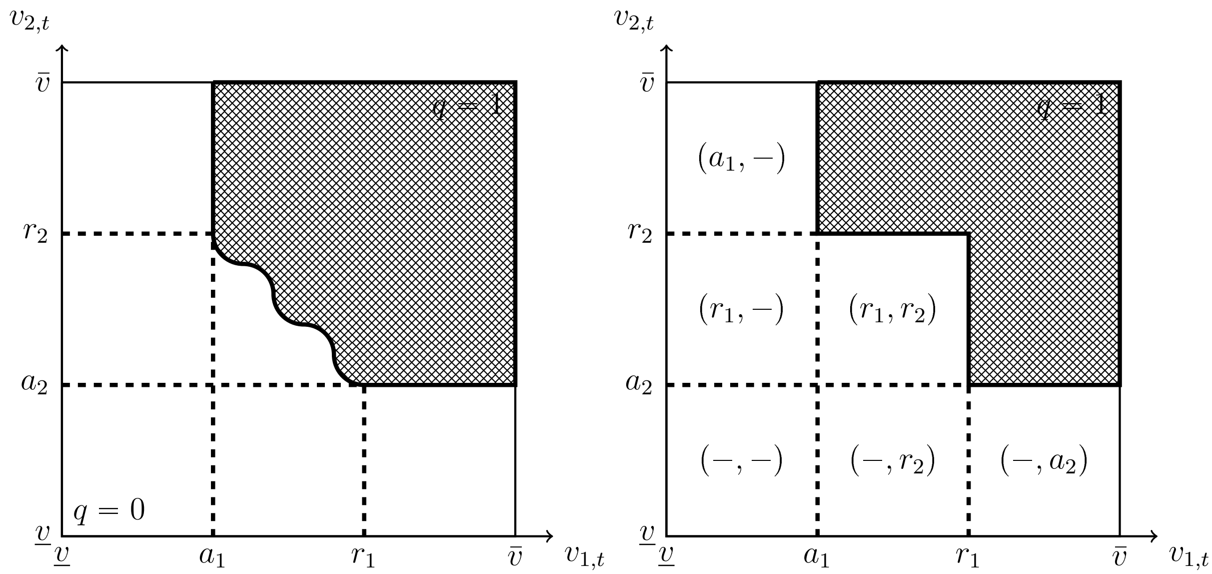

FIGURE 8. An arbitrary mechanism (left) and an ternary mechanism that Pareto dominates it (right)

If agent i observes a value higher than r_i, the candidate will be hired as long as the other agent reports a value higher than a_{-i}. Recall that R_i is the threshold function defined in Equation 1.

In the case in which the acceptance region is closed,

$$r_i = R_i\left(a_{-i}\right) < \infty. \tag{4}$$

I call a_i the approval threshold and r_i the recommendation threshold for an agent i. When agent i observes a value lower than a_i, the candidate will never be hired for any value the other agent reports, meaning agent i vetoes hiring. When agent i observes a value higher than r_i, the candidate will be hired as long as the other agent does not veto, meaning agent i recommends hiring. When both agents approve but none recommends, the stage mechanism can choose an arbitrary hiring rule such as the one in the left-hand-side diagram in Figure 4.

I compare the original stage mechanism in the diagram on the left-hand side of Figure 8 with one in which the candidate is never hired in the region $[a_1, r_1] \times [a_2, r_2]$ on the right-hand side of Figure 8. The latter stage mechanism is ternary. In the ternary mechanism, when $v_{2,t}$ is between a_2 and r_2, agent 1 gets constant continuation value a_1 if the candidate is not hired, whereas in the original mechanism, agent 1 gets continuation values that are lower than or equal to a_1 whenever she has a value lower than a_1. When $v_{2,t}$ is lower than a_2 or higher than r_2, the ternary mechanism yields the same expected payoff for agent 1 by construction.

4.2. **Non-Pareto Optimal Mechanisms.** The boundary of the set of continuation values that correspond to some incentive-compatible mechanism consists of Pareto optimal mechanisms and the ones that are the worst for one agent fixing the payoff of the other agent. In this subsection, I explain why the part of the boundary that is not Pareto optimal is made up of ternary mechanisms and conclude that every incentive-compatible mechanism is payoff-equivalent to some randomization among ternary mechanisms.

For an incentive-compatible mechanism, Q, it is *on the non-Pareto optimal boundary* after history v^{t-1}, if for some $i \in \{1, 2\}$, there does not exist another incentive-compatible mechanism, $\tilde{Q}$, with the property that

$$\begin{aligned} w_{i,t}\left(\tilde{Q}; v^{t-1}\right) &= w_{i,t}\left(Q; v^{t-1}\right) \text{ and,} \\ w_{-i,t}\left(\tilde{Q}; v^{t-1}\right) &< w_{-i,t}\left(Q; v^{t-1}\right). \end{aligned}$$

The shaded region in Figure 9 represents a set of continuation values that are achievable by some incentive-compatible mechanisms. The Pareto optimal boundary is highlighted, and the remaining boundary is the non-Pareto optimal boundary.

Lemma 4. *If the grand mechanism is on the non-Pareto optimal boundary after some history, then the stage mechanism after that history is payoff-equivalent to a randomization among binary stage mechanisms after the same history.*

I show that for an arbitrary stage mechanism like the one shown in Figure 8, a binary mechanism such as the one depicted in the left-hand-side diagram in Figure 10, with carefully constructed continuation value pairs in each rectangular region, results in a larger payoff for one agent but a smaller one for the other agent. Intuitively, the stage mechanism in the

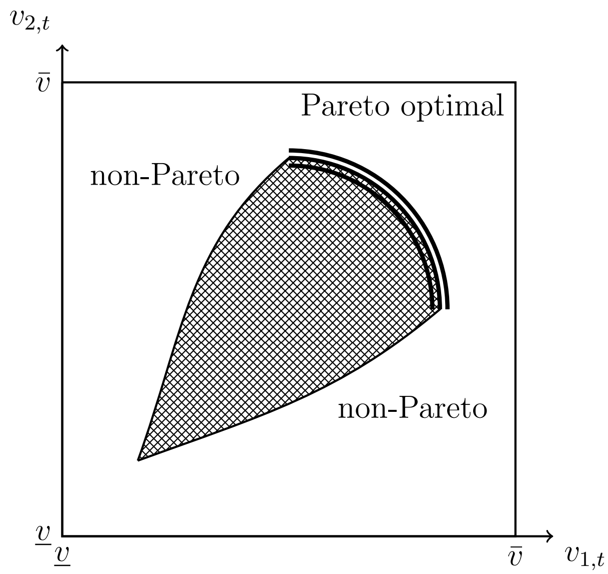

FIGURE 9. Boundary of a set of continuation values that corresponds to some incentive-compatible mechanism

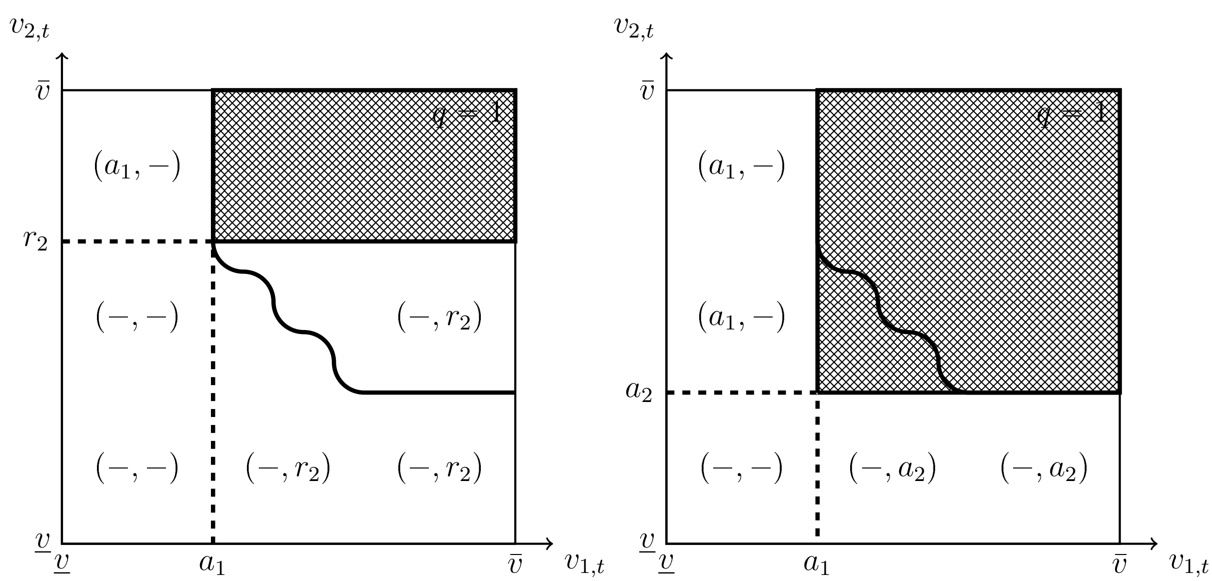

FIGURE 10. A binary mechanism better for 1 and worse for 2 (left) and a binary mechanism worse for both (right)

diagram is close to the dictator mechanism for agent 1, so it leads to a payoff that is higher for agent 1 and lower for agent 2. A ternary mechanism depicted in the left-hand side diagram in Figure 10 results in a smaller payoff for both agents. As explained at the end of the previous subsection, when both agents approve, never hiring the candidate is better for both agents than sometimes hire and sometimes not. For the same reason, always hiring is worse for both agents.

Theorem 1. *For every quasi-deterministic incentive-compatible mechanism, Q, there is a mechanism $\tilde{Q}$ that satisfies,*

(1) *Every stage mechanism of $\tilde{Q}$ is ternary, and*
(2) *Q is payoff-equivalent to $\tilde{Q}$.*

The result follows directly from Lemma 3 and Lemma 4, since the best and worst payoff for an agent are randomizations between ternary stage mechanisms, and this result applies to every agent and the stage mechanism after every history. Theorem 1 implies that a principal can use only ternary mechanisms to obtain every implementable payoff for the agents.

5. Discussion

In this section, I discuss possible extensions of the model.

The assumptions on the distributions over time are imposed for simpler presentation in the main text and the proofs, and they can easily be relaxed without significantly changing the proofs. In particular, the independence assumption is never used in the proofs due to the simplified definition of implementability used in the paper. However, a more complex definition of dynamic ex-post incentive compatibility should be used in the case with correlated values over time.

The definition of binary stage mechanisms can easily be extended to problems with more than two agents. These binary stage mechanisms remain the only incentive-compatible ones in the static case and in the last period for problems with $N > 2$ agents. The proofs of Lemma 1 and Corollary 1 applies directly to the general N agent problems. The definition of ternary stage mechanisms extend to problems with more than two agents. Lemma 3 and Lemma 4 apply also to the general agents problems, but since the proof requires additional notations that are complicated and do not provide additional insights, it is therefore not presented. However, the main result, Theorem 1, does not follow from Lemma 3 and Lemma 4, as in the two-agent case. This is because the mechanisms from Lemma 4 do not cover the entire non-Pareto boundary in N dimensions. Lemma 4 implies only that fixing the payoff of *one* agent makes it possible to decrease the payoffs of *all other* agents at the same time. I conjecture that Theorem 1 holds for N agent problems but that a different approach to the proof of Lemma 4 is required.

Another restrictive assumption is the requirement to use only quasi-deterministic mechanisms. A decomposability result similar to that provided by Pycia and Ünver (2015) is required to show that any random mechanism can first be decomposed into multiple quasi-deterministic mechanisms and then constructed through further randomization among ternary mechanisms. Decomposability is not obvious in this model for even the simplest random mechanisms; therefore, I restrict attention to quasi-deterministic mechanisms without full justification.

It is possible to characterize the Pareto frontier in very simple examples. For example, if the valuations are independently uniformly distributed on [-1, 1], the Pareto frontier in the first period can be described by the line segment connecting the payoffs from the dictator mechanisms where one agent is the always the dictator in all periods. For two period problems, it is also possible to find the Pareto frontier numerically, but it is very difficult in general to provide a characterization of the Pareto frontier. As a result, it is also difficult to find the optimal dynamic mechanism given an objective function. They are interesting questions that I am unable to answer at the moment and I will leave them to potential future research.

6. Proofs (for the Lemmas in Section 3)

In this section, I start by stating the monotonicity condition that is necessary but not sufficient for implementability. Then I provide the proof of Lemma 2, and I use the result to prove Lemma 1.

Lemma 5. *If a grand mechanism, Q, is incentive compatible, then after every history $v^{t-1}, q\left(v_t|v^{t-1}\right)$ is weakly increasing in $v_{i,t}$ for each agent i.*

Proof of Lemma 5: Let v^t and $\tilde{v}^t$ be two histories that are the same except for the value for agent i in period $t, v_{i,t}$ and $\tilde{v}_{i,t}$, respectively, such that $v_{i,t} > \tilde{v}_{i,t.}$

Ex post implementability implies, for a stage mechanism $q(\cdot) = q_{\cdot|v^{t-1}}$ of Q,

$$q(v_t)\, v_{i,t} + (1 - q(v_t))\, w_{i,t}\left(Q; v^t\right) \geqslant q(\tilde{v}_t)\, v_{i,t} + (1 - q(\tilde{v}_t))\, w_{i,t}\left(Q; \tilde{v}^t\right) \text{ and },$$
$$q(v_t)\, \tilde{v}_{i,t} + (1 - q(v_t))\, w_{i,t}\left(Q; v^t\right) \leqslant q(\tilde{v}_t)\, \tilde{v}_{i,t} + (1 - q(\tilde{v}_t))\, w_{i,t}\left(Q; \tilde{v}^t\right).$$

Taking the difference between the above inequalities gives the result.

$$(q(v_t) - q(\tilde{v}_t)) \cdot (v_{i,t} - \tilde{v}_{i,t}) \geqslant 0$$
$$q(v_t) - q(\tilde{v}_t) \geqslant 0$$

Since v^t and $\tilde{v}^t$ only differ in the component $v_{i,t} > \tilde{v}_{i,t}, q$ is monotonic in the component $v_{i,t}$ for every i and every t. □

Proof of Lemma 2: Fix a grand mechanism Q and its stage mechanism after the history $v^{t-1}, q(\cdot) = q(\cdot|v^{t-1})$, recall the definition of the threshold function in Equation 1, $R_i(v_{-i,t}) = \inf\{v_{i,t} : q(v_t) = 1\}$, with the convention that $\inf \varnothing = \infty$.

I first prove that the conditions (monotonicity on q and threshold condition on w_{t+1}) are necessary. Assume Q is a quasi-deterministic incentive compatible mechanism.

By Lemma 5, since q is monotonic in $v_{i,t}$, and,

$$q(v_{i,t}, v_{-i,t}) = \begin{cases} 0 & \text{if } v_{i,t} < R_i(v_{-i,t}) \\ 1 & \text{if } v_{i,t} > R_i(v_{-i,t}) \end{cases}.$$

The continuation value for agent i for fixed $v_{-i,t}, w_{i,t+1}(Q; v^{t-1}, (v_{i,t}, v_{-i,t}))$, must be constant for all values $v_{i,t}$ in the rejection region $\{v_t : q(v_t) = 0\}$. If not, the agent could always find it optimal to report the valuation with the largest possible continuation value. From now on, fix $v_{-i,t}$ and let $w_{i,t+1}(Q; v^{t-1}, (v_{i,t}, v_{-i,t})) = c$ be the constant continuation value.

For an agent observing $v_{i,t} < R(v_{-i,t})$, incentive compatibility implies reporting $v_{i,t}$ is preferred to reporting another $\hat{v}_{i,t} > R_i(v_{-i,t})$ to get the allocation $q = 1$, meaning,

$$c \geqslant v_{i,t}.$$

This is true for every $v_{i,t} < R(v_{-i,t})$,

$$c \geqslant R_i(v_{-i,t}).$$

For an agent observing $v_{i,t} > R_i(v_{-i,t})$, incentive compatibility implies reporting $v_{i,t}$ is preferred to reporting another $\hat{v}_{i,t} < R_i(v_{-i,t})$ to get the allocation $q = 0$, meaning,

$$c \leqslant v_{i,t}.$$

This is true for every $v_{i,t} > R(v_{-i,t})$,

$$c \leqslant R_i(v_{-i,t}).$$

Therefore,

$$c = R_i(v_{-i,t}),$$

and,

$$w_{i,t+1}\left(Q; v^{t-1}, (v_{i,t}, v_{-i,t})\right) = R_i(v_{-i,t}) \text{ for each } v_{i,t} \text{ such that } q(v_{i,t}, v_{-i,t}) = 0.$$

Now I prove that the monotonicity and threshold conditions are sufficient. Fix $v_{-i,t}$ and assume q is a stage mechanism that satisfy these conditions.

For an agent i with $v_{i,t} < R(v_{-i,t})$, reporting $\hat{v}_{i,t}$ results in payoff,

$$w_{i,t}\left(Q; v^{t-1}\right) = \begin{cases} R_i(v_{-i,t}) & \text{if } \hat{v}_{i,t} = v_{i,t} \\ R_i(v_{-i,t}) & \text{if } \hat{v}_{i,t} \neq v_{i,t}, \hat{v}_{i,t} \leqslant R(v_{-i,t}) \\ v_{i,t} & \text{if } \hat{v}_{i,t} > R(v_{-i,t}) \end{cases}.$$

Since $v_{i,t} < R_i(v_{-i,t})$, it is optimal to report truthfully.

For an agent i with $v_{i,t} > R(v_{-i,t})$, reporting $\hat{v}_{i,t}$ results in payoff,

$$w_{i,t}\left(Q; v^{t-1}\right) = \begin{cases} v_{i,t} & \text{if } \hat{v}_{i,t} = v_{i,t} \\ v_{i,t} & \text{if } \hat{v}_{i,t} \neq v_{i,t}, \hat{v}_{i,t} > R(v_{-i,t}) \\ R_i(v_{-i,t}) & \text{if } \hat{v}_{i,t} \leqslant R(v_{-i,t}) \end{cases}.$$

Since $v_{i,t} > R_i(v_{-i,t})$, it is optimal to report truthfully.

Therefore, truthful reports are optimal, q is incentive compatible. □

Proof of Lemma 1: Fix a grand mechanism, Q. Since for any history $v^T \in \mathcal{V}^T$,

$$w_{i,T+1}\left(Q; v^T\right) = v_i^\star,$$

by Lemma 2, for any $v_{-i,T}$, either $q\left(v_{i,T}, v_{-i,T} | v^{T-1}\right)$ is constant in $v_{i,T}$ or,

$$R_i(v_{-i,T}) = v_i^\star.$$

By Definition 3, these stage mechanisms are binary with outside option $v_i^\star$ for agent i. □

7. Proofs (for the Lemmas and Propositions in Section 4)

In this section, I start by defining some shorthand notations for the proofs in this section and some preliminary observations that simplify the shapes and continuation values of an incentive compatible mechanism. Then, I prove Lemma 3 and Lemma 4. Theorem 1 follows directly from Lemma 3 and Lemma 4.

From now on, I am going to fix the grand quasi-deterministic incentive compatible mechanism Q, period t and history v^{t-1}. To simplify the subsequent notation, I write,

$$q(v_t) = q\left(v_t | v^{t-1}\right).$$

I also write,

$$w_{t+1}(v_t) = w_{t+1}\left(Q; v^{t-1}, v_t\right),$$

and,

$$w_t = w_t\left(Q; v^{t-1}\right).$$

I define the following constants and sets, I assume the acceptance region $\{v_t : q(v_t) = 1\}$ is closed for these definitions.

(1) Recall from Equation 2 that the *approval threshold* , a_i, is the threshold below which the agent i has veto power: if $v_{i,t} < a_i$, the candidate is vetoed by i and will never be hired for any value of $v_{-i,t}$,

$$a_i = \sup_{v_{i,t}} \left\{v_{i,t} : q\left(v_{i,t}, v_{-i,t}\right) = 0 \text{ for all } v_{-i,t} \in \left[\underline{v}, \bar{v}\right]\right\}.$$

(2) The *non-threshold region* for agent i, is the set of $v_{-i,t}$ such that the candidate is never hired for any value of $v_{i,t}$,

$$\left\{v_{-i,t} : q\left(\tilde{v}_{i,t}, v_{-i,t}\right) = 0 \text{ for all } \tilde{v}_{i,t} \in \left[\underline{v}, \bar{v}\right]\right\} = \left[\underline{v}, a_{-i}\right).$$

The *average non-threshold continuation value* , c_i, is the expected continuation value within the non-threshold region for agent i,

$$c_i = \mathbb{E}\left[w_{i,t+1}\left(v_{i,t}, v_{-i,t}\right)|v_{-i,t} < a_i\right].$$

(3) The definition for the *recommendation threshold* , r_i, is simplified from Equation 4 due to the assumption that the acceptance region is closed,

$$r_i = R_i\left(a_{-i}\right).$$

One useful relation due to the definition of these constants is,

$$a_i \leqslant R_{v_{-i,t}} \leqslant r_i \text{ for all } v_{-i,t} \geqslant a_{-i}. \tag{5}$$

Then, I state an corollary to Lemma 2 that makes computation easier in all the following proofs.

Corollary 2. *Fix a history v^{t-1} and a stage mechanism of a quasi-deterministic incentive compatible Q after this history, $q(\cdot) = q(\cdot|v^{t-1})$,*

$$\mathbb{E}\left[\max\left\{v_{i,t}, R_i\left(v_{-i,t}\right)\right\}|v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} + c_i \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\}.$$

Proof of Corollary 2: It follows directly from Lemma 2. □

In the course of this section, I will construct alternative stage mechanisms with new continuation values. I say *a continuation value is incentive compatible* if they can be implemented in an incentive compatible quasi-deterministic grand mechanisms.

As an example, I show the following Compactness Lemma that ensures that the acceptance region is closed.

Lemma 6. *There exists an incentive compatible stage mechanism $\tilde{q}$, with incentive compatible continuation values, that is payoff equivalent to q, such that,*

$$\left\{v_t : \tilde{q}\left(v_t\right) = 1\right\} \text{ is closed}.$$

Recall that payoff equivalence in this section means $\tilde{w}_t = w_t$ where $\tilde{w}_t$ is the continuation value of mechanism that has $\tilde{q}$ after history v^{t-1} in place of q.

Proof. Define $\tilde{q}$ such that the set $\{v_t : \tilde{q}(v_t) = 1\}$ is the closure of the set $\{v_t : q(v_t) = 1\}$. Payoff equivalence is due to continuity and full support property of the value distributions. □

The only issue is how to define the continuation payoffs for player i when the other player says a_i. But use the continuity and just take the limit of continuation values when $v_{-i} > a_{i.}$

Also, I show the following Flattening Lemma that ensures that continuation values in the non-threshold region is constant.

Lemma 7. *There exists an incentive compatible stage mechanism $\tilde{q}$, with incentive compatible continuation values, that is payoff equivalent to q, such that,*

$$\tilde{w}_{t+1}\left(v_{i,t}, v_{-i,t}\right) = c_i \text{ for all } v_{-i,t} < a_{-i}.$$

Proof of Lemma 7: Let $\tilde{q}$ be the stage mechanism from replacing the continuation value of agent i in q in the non-threshold region $v_{-i,t} < a_{-i}$ by $c_{i.}$

I divide the proof into three main parts.

(1) $\tilde{q}$ is incentive compatible.

(2) $\tilde{q}$ payoff equivalent to q.

Part (1) There are two types of new continuation values that is different from the ones used in q.

(1) $(c_i, R_{-i}(v_{i,t}))$ from the region $v_{i,t} \geqslant a_i$ and $v_{-i,t} < a_{-i}$,

(2) (c_i, c_{-i}) from the region $v_{i,t} < a_i$ and $v_{-i,t} < a_{-i}$.

Fix $\tilde{v}_{i,t} \geqslant a_i, R_{-i}(\tilde{v}_{i,t})$ is finite due to the definition of $a_{i.}$

$$\begin{aligned}
(c_i, R_{-i}(\tilde{v}_{i,t})) &= \left(\mathbb{E}\left[w_{i,t+1}(v_{i,t}, v_{-i,t}) \left| v_{-i,t} < a_{-i}\right.\right], R_{-i}(\tilde{v}_{i,t})\right) \\
&= \left(\mathbb{E}\left[\mathbb{E}\left[w_{i,t+1}(v_{i,t}, v_{-i,t}) \left| v_{-i,t} < a_{-i}\right.\right], w_{-i,t+1}(\tilde{v}_{i,t}, v_{-i,t})\right)\right. \\
&= \mathbb{E}\left[w_{t+1}(\tilde{v}_{i,t}, v_{-i,t}) \left| v_{-i,t} < a_{-i}\right.\right].
\end{aligned}$$

The second equality is due to Lemma 2 which states that $w_{i,t+1}(v_{i,t}, v_{-i,t})$ is constant in $v_{i,t}$ in the non-threshold region $[\underline{v}, a_{-i})$.

Similarly,

$$\begin{aligned}
(c_i, c_{-i}) &= \left(\mathbb{E}\left[w_{i,t+1}(v_{i,t}, v_{-i,t}) \left| v_{-i,t} < a_{-i}\right.\right], \mathbb{E}\left[w_{-i,t+1}(v_{i,t}, v_{-i,t}) \left| v_{i,t} < a_{-i}\right.\right]\right) \\
&= \mathbb{E}\left[w_{t+1}(v_{i,t}, v_{-i,t}) \left| v_{i,t} < a_{-i}, v_{-i,t} < a_{-i}\right.\right].
\end{aligned}$$

The second equality is due to Lemma 2 which states that $w_{i,t+1}(v_{i,t}, v_{-i,t})$ is constant in $v_{i,t}$ in the non-threshold region $[\underline{v}, a_{-i})$.

Therefore, $(c_i, R_{-i}(v_{i,t}))$ and (c_i, c_{-i}) are the expected value other continuation value pairs so they are incentive compatible too. Since $\tilde{q}$ is the same as q, so incentive compatible of $\tilde{q}$ follows from the incentive compatibility of q.

Part (2) Note that,

$$\begin{aligned}\tilde{c}_i &= \mathbb{E}\left[\tilde{w}_{i,t+1}\left(v_{i,t}, v_{-i,t}\right)|v_{-i,t} < a_{-i}\right] \\ &= \mathbb{E}\left[c_i|v_{-i,t} < a_{-i}\right] \\ &= c_i.\end{aligned}$$

I check $\tilde{q}$ is payoff equivalent to q using Corollary 2,

$$\begin{aligned}\tilde{w}_{i,t} &= \mathbb{E}\left[\max\left\{v_{i,t}, \tilde{R}_i\left(v_{-i,t}\right)\right\}|v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} + \tilde{c}_i \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\} \\ &= \mathbb{E}\left[\max\left\{v_{i,t}, R_i\left(v_{-i,t}\right)\right\}|v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t}v_{-i,t} \geqslant a_{-i}\right\} + c_i \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\} \\ &= w_{i,t.}\end{aligned}$$

□

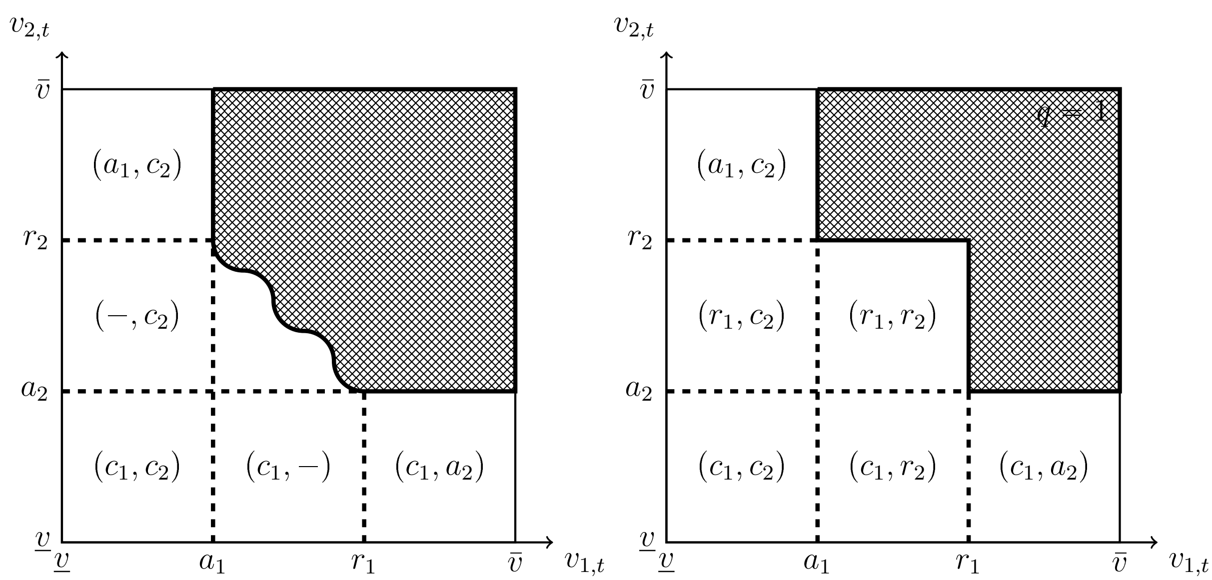

FIGURE 11. Mechanism $\tilde{q}$

Lemma 8. *Ternary stage mechanisms with continuation values described in diagram on the right of Figure 11 are incentive compatible as long as the continuation values are incentive compatible.*

Proof. It follows directly from Lemma 2. □

In the following proofs, by Lemma 6 and Lemma 7, without loss of generality, assume the acceptance region of q is closed and continuation value for each agent i in its non-threshold region $v_{-i,t} \in [\underline{v}, a_{-i})$ is constant at c_i. Let the continuation values and the shape of the acceptance region of q be ones as in .

Proof of Lemma 3: Consider a ternary stage mechanism, $\tilde{q}$, with the thresholds a_i, r_i with continuation values specified in .

I divide the proof into three main parts.

(1) $\tilde{q}$ is incentive compatible.

(2) $\tilde{q}$ Pareto dominates q.

Part (1) I only need to show that the continuation values are incentive compatible, the rest follows from Lemma 8. There are three types of new continuation values that is different from the ones used in q.

(1) (r_i, r_{-i}) in the region $a_j \leqslant v_{j,t} < r_j$ for both $j \in \{1, 2\}$,

(2) (r_i, c_{-i}) in the region $v_{i,t} < a_i$ and $a_{-i} \leqslant v_{-i,t} < r_{-i}$.

$$
\begin{aligned}
(r_i, r_{-i}) &= (R_i(a_{-i}), R_{-i}(a_i)) \\
&= w_{t+1}(a_i, a_{-i}),
\end{aligned}
$$

and fix $a_i \leqslant v_{i,t} < r_i$,

$$
\begin{aligned}
(r_i, c_{-i}) &= (R_i(a_{-i}), c_{-i}) \\
&= w_{t+1}(v_{i,t}, a_{-i}),
\end{aligned}
$$

are both incentive compatible continuation values due to implementability of q.

Part (2) I use Corollary 2 to compare $\tilde{q}$ and q.

$$\begin{aligned}
&\tilde{w}_{i,t} - w_{i,t} \\
&= \mathbb{E}\left[\max\left\{v_{i,t}, \tilde{R}_{i,t}\left(v_{-i,t}\right)\right\} - \max\left\{v_{i,t}, R_{i,t}\left(v_{-i,t}\right)\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} \\
&\quad + \left(\tilde{c}_i - c_i\right) \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\} \\
&= \mathbb{E}\left[\max\left\{v_{i,t}, r_i\right\} - \max\left\{v_{i,t}, R_{i,t}\left(v_{-i,t}\right)\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} \\
&\geqslant \mathbb{E}\left[\max\left\{v_{i,t}, r_i\right\} - \max\left\{v_{i,t}, r_i\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} \\
&= 0.
\end{aligned}$$

The inequality is due to the observation Equation 5.

Therefore, q is Pareto dominated by $\tilde{q}$. □

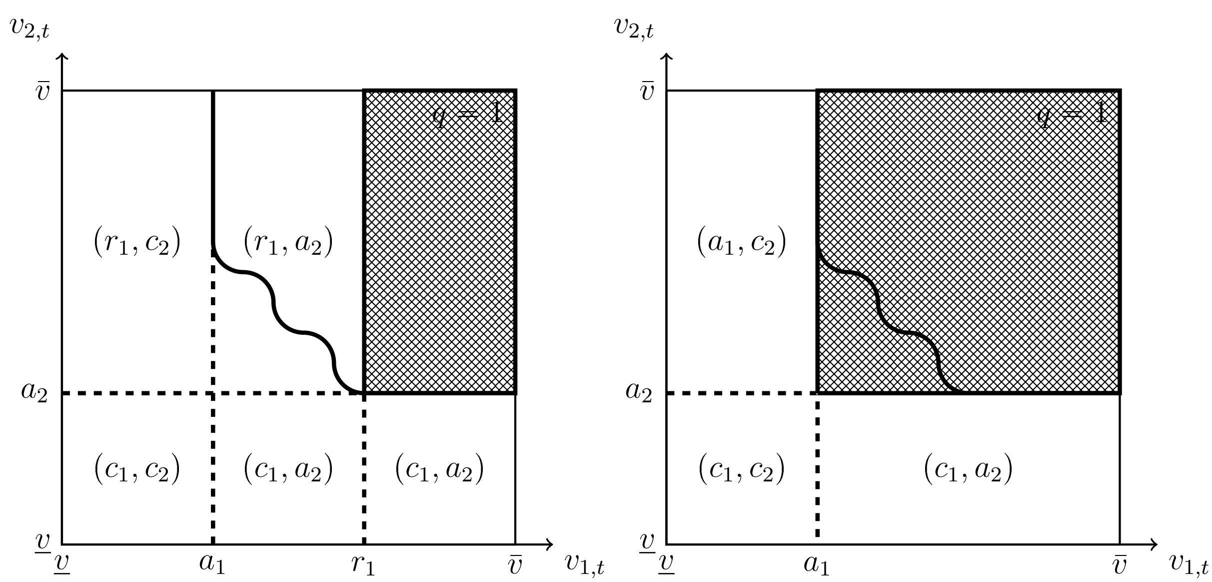

FIGURE 12. Mechanism q^{min}

Proof of Lemma 4: Consider two other stage mechanisms q^{min} and q^{max} in place of q after history v^{t-1}, where q^{max} is the binary mechanism with thresholds (a_i, r_{-i}) and q^{min} is the binary mechanism with thresholds (a_i, a_{-i}), and with continuation values specified in Figure

12. I add superscript min and max to denote the modified recommendation thresholds,

$$r_i^{min} = a_i,$$

$$r_{-i}^{min} = r_{-i},$$

and,

$$r_i^{max} = a_i,$$

$$r_{-i}^{max} = a_{-i}.$$

with continuation values specified in .

I divide the proof into two parts.

(1) q^{min} and q^{max} are incentive compatible.

(2) A randomization between Q^{min} and Q^{max} results in the same continuation value for i and is a smaller continuation value for $-i$.

Part (1) The continuation values are incentive compatible for similar reasons as the continuation values of $\tilde{q}$ are incentive compatible in the proof of Lemma 3. The rest follows from Lemma 8.

Part (2) There are three things to show.

(1) q^{min} is worse than q for every agent,

(2) q^{max} is better than q for i,

(3) q^{max} is worse then q for $-i$.

I use Corollary 2 for all three comparisons.

Comparison (1) : For every agent $j \in \{1, 2\}$,

$$
\begin{aligned}
& w_{j,t} - w_{j,t}^{min} \\
& = \mathbb{E}\left[\max\{v_{j,t}, R_{j,t}(v_{-j,t})\} - \max\left\{v_{j,t}, R_{j,t}^{min}(v_{-j,t})\right\} |v_{-j,t} \geqslant a_{-j}\right] \cdot \mathbb{P}\{v_{-j,t} \geqslant a_{-j}\} \\
& \quad + \left(c_j^{min} - c_j\right) \cdot \mathbb{P}\{v_{-j,t} < a_{-j}\} \\
& = \mathbb{E}\left[\max\{v_{j,t}, R_{j,t}(v_{-j,t})\} - \max\{v_{j,t}, a_j\} |v_{-j,t} \geqslant a_{-j}\right] \cdot \mathbb{P}\{v_{-j,t} \geqslant a_{-j}\} \\
& \geqslant \mathbb{E}\left[\max\{v_{j,t}, a_j\} - \max\{v_{j,t}, a_j\} v_{-j,t} \geqslant a_{-j}\right] \cdot \mathbb{P}\{v_{-j,t} \geqslant a_{-j}\} \\
& = 0.
\end{aligned}
$$

The inequality is due to the observation Equation 5.

Comparison (2) : For agent i,

$$
\begin{aligned}
& w_{i,t}^{max} - w_{i,t} \\
& = \mathbb{E}\left[\max\left\{v_{i,t}, R_{i,t}^{max}(v_{-i,t})\right\} - \max\{v_{i,t}, R_{i,t}(v_{-i,t})\} |v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\{v_{-i,t} \geqslant a_{-i}\} \\
& \quad + (c_i^{max} - c_i) \cdot \mathbb{P}\{v_{-i,t} < a_{-i}\} \\
& = \mathbb{E}\left[\max\{v_{i,t}, r_i\} - \max\{v_{i,t}, R_{i,t}(v_{-i,t})\} |v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\{v_{-i,t} \geqslant a_{-i}\} \\
& \geqslant \mathbb{E}\left[\max\{v_{i,t}, r_i\} - \max\{v_{i,t}, r_i\} |v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\{v_{-i,t} \geqslant a_{-i}\} \\
& = 0.
\end{aligned}
$$

The inequality is due to the observation Equation 5.

Comparison (3) : For agent $-i$,

$$
\begin{aligned}
& w_{-i,t} - w^{max_{-i,t}} \\
& = \mathbb{E}\left[\max\left\{v_{-i,t}, R_{-i,t}\left(v_{i,t}\right)\right\} - \max\left\{v_{-i,t}, R^{max_{-i,t}}\left(v_{i,t}\right)\right\} | v_{i,t} \geqslant a_i\right] \cdot \mathbb{P}\left\{v_{i,t} \geqslant a_i\right\} \\
& \quad + \left(c_j^{max} - c_j\right) \cdot \mathbb{P}\left\{v_{i,t} < a_i\right\} \\
& = \mathbb{E}\left[\max\left\{v_{j,t}, R_{j,t}\left(v_{-j,t}\right)\right\} - a_j | a_i \leqslant v_{i,t} < r_i\right] \cdot \mathbb{P}\left\{a_i \leqslant v_{i,t} < r_i\right\} \\
& \geqslant \mathbb{E}\left[\max\left\{v_{j,t}, a_j\right\} - a_j | a_i \leqslant v_{i,t} < r_i\right] \cdot \mathbb{P}\left\{a_i \leqslant v_{i,t} < r_i\right\} \\
& \geqslant 0.
\end{aligned}
$$

The first inequality is due to the observation Equation 5. Note that this abuses Corollary 2 to since the non-threshold regions are not the same.

Therefore, there is a randomization between q^{min} and q^{max} such that the continuation value for i is the same and lower for agent $-i$. □

Proof of Theorem 1: Since binary mechanisms in Lemma 4 are ternary and the set $-i$ is a singleton, the result follows from Lemma 4 and Lemma 3. □

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