INTEGRATED-QUANTILE-BASED ESTIMATION FOR FIRST-PRICE AUCTION MODELS

YAO LUO† AND YUANYUAN WAN‡

UNIVERSITY OF TORONTO

ABSTRACT. This paper considers nonparametric estimation of first-price auction models under the monotonicity restriction on the bidding strategy. Based on an integrated-quantile representation of the first-order condition, we propose a tuning-parameter-free estimator for the valuation quantile function. We establish its cube-root-n consistency and asymptotic distribution under weaker smoothness assumptions than those typically assumed in the empirical literature. If the latter are true, we also provide a trimming-free smoothed estimator and show that it is asymptotically normal and achieves the optimal rate of Guerre, Perrigne, and Vuong (2000). We illustrate our method using Monte Carlo simulations and an empirical study of the California highway procurement auctions.

Key words: First Price Auctions, Monotone Bidding Strategy, Nonparametric Estimation, Tuning-Parameter-Free

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† Corresponding author. 150 St. George Street, Toronto ON M5S 3G7, Canada. yao.luo@utoronto.ca.

‡yuanyuan.wan@utoronto.ca.

1. INTRODUCTION

Since the seminal work of Guerre, Perrigne, and Vuong (2000, GPV hereafter), the nonparametric estimation of auction models has received enormous attention from both the perspectives of econometric analysis and empirical applications. In this paper, we revisit the first-price auction models and propose a novel estimation procedure for the valuation quantile function. Our approach is appealing both computationally and theoretically. We first construct a quantile estimator that is tuning-parameter-free and robust in the sense it is consistent under weaker smoothness assumptions than typically imposed in the literature (details later). Whenever the typical smoothness assumptions are satisfied, we can construct a trimming-free and asymptotically normal second step estimator that achieves the optimal rate of GPV. Furthermore, our estimator explicitly incorporates the restriction of the monotone bidding strategy and is monotone in finite samples, which is important for empirical work but not ensured by most of the existing estimators.

To better illustrate the features of our estimator, we begin by reviewing existing approaches in the literature. We focus on the baseline case of homogeneous auctions and will show it can be extended to incorporate auction specific characteristics in Section 2.3. We consider the standard GPV setup of independent private value (IPV) first price auction. Their novel approach is to transform the first-order condition for optimal bids and express a bidder's value as an explicit function of the submitted bid, the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of bids:

$$v = s^{-1}(b) \equiv b + \frac{1}{I-1}\frac{G(b)}{g(b)}, \tag{1}$$

where b is the bid, I is the number of bidders, and $G(\cdot)$ and $g(\cdot)$ are the distribution and density of bids, respectively. A two-step estimation method follows from this observation: first construct a pseudo value for each bid and then apply kernel density estimation to the sample of pseudo values. GPV establish the consistency of their estimator and the optimal rate.

Based on the insight of Haile, Hong, and Shum (2003), who considered a quantile-based-test for the existence of common values, Marmer and Shneyerov (2012, MS hereafter) first proposed to estimate the valuation distribution based on the quantile representation of the first-order condition, that is, when the equilibrium bidding strategy is strictly monotone, valuation quantile function $Q_v(\cdot)$ can be expressed as

$$Q_v(\alpha) = Q_b(\alpha) + \frac{1}{I-1}\frac{\alpha}{g(Q_b(\alpha))}, \quad 0 \leq \alpha \leq 1, \tag{2}$$

where $Q_b(\cdot)$ is the bid quantile function. Note that the right-hand side must be strictly increasing in α, too. MS proposed to first estimate $Q_v(\cdot)$ using plug-in estimators for $g(\cdot)$ and $Q_b(\cdot)$ and subsequently estimate the valuation density using the relationship $f(v) = 1/Q_v'(Q_v^{-1}(v))$. MS show that their estimator is asymptotically normal and achieves the optimal rate of GPV. Guerre and Sabbah (2012, GS) observed that the second term on the right hand side of Equation (2) is a known linear function of α multiplied by the quantile derivative and proposed an optimal local polynomial quantile estimator. Gimenes and Guerre (2013) further explored the observation made by GS and proposed an augmented-quantile regression to overcome the difficulty of incorporating many covariates.

In both estimators of GPV and MS, the bid density $g(\cdot)$ appears in the denominator of the first step estimation; in MS, the derivative of the bid quantile also appears in the denominator of the second step. In practice, trimming near the boundaries is needed but can be troublesome as it is well known that there is no generic guidance. GS does not require trimming but a choice of a bandwidth is needed. In addition, all quantile estimators discussed above may not satisfy the monotonicity restriction imposed by the model.

In this paper, we propose to use the integrated quantile representation of the first order condition to construct our estimator. We use $V(\cdot)$ to denote the integrated quantile function of valuation. When the biding strategy is strictly monotone, there is

$$V(\beta) \equiv \int_0^\beta Q_v(\alpha)d\alpha = \frac{I-2}{I-1}\int_0^\beta Q_b(\alpha)d\alpha + \frac{1}{I-1}Q_b(\beta)\beta, \quad 0 \leq \beta \leq 1. \tag{3}$$

Note that since Q_v is strictly increasing, V is necessarily strictly convex and identical to its greatest convex minorant (g.c.m.). The idea of using the integrated bid quantile and g.c.m. (or least concave majorant, l.c.m.) in auctions first appeared in Liu and Vuong (2013). They focus on testing the monotonicity of the inverse bidding strategy, which is equivalent to the concavity of a known function of the quantile of winning bids. Based on this novel reformation, they propose a test statistic that measures the distance between the sample analog of this function and its l.c.m.. Liu and Luo (2015) also used the integrated quantile function to test the equivalence of valuation distributions. In this paper, we further explore the merits of using integrated quantile function and its g.c.m. for the estimation. First, the sample analog of $V(\cdot)$, denoted by $V_n(\cdot)$, is easy to compute; it essentially requires little more than sorting the observed bids. Neither bandwidth choice nor trimming is needed. Second, we can naturally impose the monotonicity constraint in our estimation procedure by using the g.c.m. of $V_n(\cdot)$ as an estimator for $V(\cdot)$, which we denote as $\widehat{V}$. Since $V_n(\cdot)$ is a piece-wise linear function of β, $\widehat{V}(\cdot)$ can be very easily calculated and is also piece-wise linear. Then we can estimate $Q_v(\cdot)$ by taking the piece-wise derivatives of $\widehat{V}(\cdot)$. As we will formally prove later, this estimator is cube-root-n consistent and requires weaker smoothness on model primitive, that is, it only requires that $F(\cdot)$ be continuously differentiable as opposed to twice continuous differentiability in GPV and MS. We refer it as our first step estimator $\widehat{Q}_v(\cdot)$. Note that $\widehat{Q}_v(\cdot)$ is tuning-parameter-free. If indeed the model admits enough smoothness, we can improve the convergence rate by considering a kernel smoothed version $\hat{q}_v(\cdot)$ of $\widehat{Q}_v(\cdot)$. We show that $\hat{q}_v(\cdot)$ is asymptotically normal and achieves GPV's optimal rate. Note that despite one needs to choose a bandwidth for $\hat{q}_v(\cdot)$ (for which we propose an optimal bandwidth), there is no need for trimming.[1]

Another appealing feature of our estimator is that the monotonicity of bidding strategy is imposed in a simple way through the calculation of g.c.m.. As a result, the estimates $\widehat{Q}_v(\cdot)$ and $\hat{q}_v(\cdot)$ are always increasing by construction. To the best of our knowledge, Henderson,

[1] See Hickman and Hubbard (2014) for a modified version of the GPV estimator which replaces trimming with boundary correction.

List, Millimet, Parmeter, and Price (2012, HLMPP hereafter) were the first to address the imposition of monotonicity in first price auctions. They argued that nonparametric estimators which naturally impose existing economic restrictions have empirical virtue. HLMPP's estimator achieves the desired monotonicity constraint by tilting the empirical distribution of the data and requires repeated re-weighting of the sample. Bierens and Song (2012)'s sieve approach implicitly imposes the monotonicity constraint, but it can be computationally expensive. Our estimator imposes the monotonicity by taking the g.c.m. of the integrated valuation quantile function. The g.c.m. of $V_n(\cdot)$ is easy to compute since it is piece-wise linear. Indeed, satisfying monotonicity in finite samples is a desirable feature of a quantile function estimator in general; see discussions in Chernozhukov, Fernandez-Val, and Galichon (2010). Chernozhukov, Fernandez-Val, and Galichon (2010) proposed a "rearrangement" approach to achieve the monotonicity. We take the g.c.m. approach on the integrate-quantile function in our context because it not only delivers the monotonicity, but also circumvents the necessity of estimating the bid density function in the denominator.

Our estimator is constructed using order statistics of the bids. Indeed, using order statistics is not uncommon in the literature of nonparametric estimation of auction models (see, e.g. Athey and Haile, 2007). Recently, Menzel and Morganti (2013) discussed estimation of value distribution based on the distributions of order statistics. They show that the mapping between distribution of order statistics and valuation distribution is in general non-Lipschitz continuous and established optimal rate for varies of parameters of interest. Our main motivation is to provide computationally-easy estimators for the classical IPV setup of GPV with all bids being observed, which is, as mentioned by Menzel and Morganti (2013), a scenario for which the irregularity of inverting order statistics distribution does not rise. The reason that our first estimator converges at an irregular cube-root-n rate is that we impose weaker smoothness assumption on the valuation distribution, rather than the non-Lipschitz continuous feature of the mapping.

We illustrate our method using the California Highway Procurement auction data set. In practice, it is common that researchers observe auction-specific characteristics.[2] It is worth noting that our method applies naturally if the observed auction-specific characteristics are discrete-valued (or discretization of continuous variables) by conditioning on realizations. The estimate will then be interpreted as conditional valuation quantiles on observed auction characteristics. When the observed auction-specific characteristics are continuous, GPV and MS propose to estimate the conditional valuation density by Kernel method, which suffers the "curse of dimensionality" when the covariates are high dimensional. In Section 2.3, we show our estimation procedure can also take GS's estimator for bids quantile function as an input and deliver a consistent and monotone conditional valuation quantile estimator for each realization of the continuous covariates. Lastly, we can also use the homogenization method proposed by Haile, Hong, and Shum (2003) and apply our estimation methods to the homogenized bids. The homogenization approach requires additional additive separability structure on how valuation depends on observed characteristics. As a result, it is easier to compute and has a faster convergence rate.

The rest of the paper is organized as follows. We lay out the model and propose our estimator in Section 2. We examine the performance of our estimator in Section 3. Section 4 is the empirical illustration. We conclude the paper in Section 5.

2. MODEL AND MAIN RESULTS

We consider the first-price sealed-bid auction model with independent private values. A single and indivisible object is auctioned. We make the following assumptions.

Assumption 1. *There are $L \to \infty$ identical auctions, and for each auction, there are I symmetric and risk neutral bidders. Their private values are i.i.d. draws from a common distribution $F(\cdot)$.*

[2] In general, the first price auction model is not identified if there is unobserved heterogeneity across auctions, see Armstrong (2013b).

Let the total number of bids be $n = LI$. Asymptotics are on the number of auctions, that is, $L \to \infty$. The assumption that number of bidders I is constant across auctions is just for simplifying notation; our analysis can be easily extended to conditional on I.

Assumption 2. *$F(\cdot)$ is continuously differentiable over its compact support $[\underline{v}, \overline{v}]$. There exists $\lambda > 0$ such that $\inf_{v \in [\underline{v}, \overline{v}]} f(v) \geq \lambda > 0$.*

Assumption 2 only requires $F(\cdot)$ is continuously differentiable, which is weaker than the twice continuously differentiability, as assumed in the literature, e.g., GPV and MS. It is well known that the equilibrium strategy is

$$b = s(v|F, I) \equiv v - \frac{1}{F(v)^{I-1}} \int_0^v F(x)^{I-1} dx.$$

GPV show that the first-order condition can be written as Equation (1). Haile, Hong, and Shum (2003) represent this equation in terms of quantiles as in Equation (2). In this paper, we consider the integrated quantile function of the valuation as in Equation (3).

Now let us first propose a tuning-parameter-free estimator for the valuation quantile function. Let $b_{(i)}$ be the i-th order statistic of a sample of bids $\{b_i\}_{i=1}^n$. Employing Equation (3), we construct a raw estimator $V_n(\cdot)$ for $V(\cdot)$ as follows. Let $V_n(0) = 0$. For $\alpha \in \{\frac{1}{n}, \frac{2}{n}, \cdots, 1\}$,

$$V_n(\alpha) = \frac{I-2}{n(I-1)} \sum_{i=1}^{n\alpha} b_{(i)} + \frac{1}{I-1} \alpha b_{(n\alpha)}.$$

For $\alpha \in \left(\frac{j-1}{n}, \frac{j}{n}\right)$, $j = 1, \cdots, n$, define

$$V_n(\alpha) = (j - \alpha n) V_n\left(\frac{j-1}{n}\right) + (\alpha n - j + 1) V_n\left(\frac{j}{n}\right).$$

Note $V_n(\cdot)$ may not be convex in finite samples. Let $\widehat{V}(\cdot)$ be the g.c.m. of $V_n(\cdot)$. To obtain a quantile estimator which respects the monotonicity property, we consider use the left-derivative of $\widehat{V}(\cdot)$. Since $V_n(\cdot)$ is piecewise linear, so is $\widehat{V}(\cdot)$. Define $\widehat{Q}_v(0) = b_{(1)}$

and for $\alpha \in \left(\frac{j-1}{n}, \frac{j}{n}\right]$, $j = 1, \cdots, n$,

$$\widehat{Q}_v(\alpha) = n\left\{\widehat{V}\left(\frac{j}{n}\right) - \widehat{V}\left(\frac{j-1}{n}\right)\right\}.$$

By definition, $\widehat{Q}_v(\cdot)$ is a left-continuous and weakly increasing step function.

Constructing the g.c.m. $\widehat{V}(\cdot)$ for a piecewise linear function $V_n(\cdot)$ is computationally easy. While many algorithms are proposed, the most widely used one is the Pooled Adjacent Violators Algorithm (PAVA, see e.g. Robertson, Wright, Dykstra, and Robertson, 1988; Groeneboom, Jongbloed, and Wellner, 2014). We can envision $\widehat{V}(\cdot)$ as a taut string tied to the leftmost point $(0,0)$ and pulled up and under the graph of $V_n(\cdot)$, ending at the last point $(1, V_n(1))$. See Appendix B for details.

Theorem 1. *Suppose Assumptions 1 and 2 are satisfied at a given $\alpha_0 \in (0,1)$, then*

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \xrightarrow{d} C(\alpha_0)\,\text{argmax}_t\left\{\mathbb{B}(t) - t^2\right\},$$

where $C(\alpha_0)$ is a constant that depends on α_0 and $\mathbb{B}$ is a two-sided Brownian motion process.

Proof. See Appendix A.1. □

We have a few comments on Theorem 1. First, $C(\alpha_0)$ depends on α_0, g and Q_b and is estimable (detailed expression in Appendix A.1). To conduct inference on $Q_v(\alpha_0)$, one can obtain the critical values by estimating $C(\alpha_0)$ and simulating the one-dimensional Brownian motion $\mathbb{B}$, which is straightforward to implement, and an alternative way is subsampling whose validity follows straightforwardly from Theorem 1. Second, the $n^{1/3}$-consistency of our quantile estimator is obtained under weak assumptions on value distribution $F(\cdot)$ and without choosing any tuning parameters. It is slower than the optimal rate of $n^{2/5}$ when $F(\cdot)$ is twice continuously differentiable, as established in GPV. This is similar to the well-known results in the literature on isotonic estimation: without imposing additional smoothness assumptions on the model primitives and without introducing smoothing, one can at most get cube-root-n rate. Third, Theorem 1 shows that our estimator is consistent at quantile level $\alpha \in [\epsilon, 1-\epsilon]$ for any fixed $\epsilon > 0$. In addition, our estimator is super-consistent at $\alpha_0 = 0$ by construction. To see this, note that in equilibrium, the bidder with the lowest valuation bids his/her own valuation, which implies that $\widehat{Q}_v(0) = b_{(1)} \xrightarrow{p} Q_b(0) = Q_v(0)$ at a super-consistent rate under current assumptions. However, our estimator need not be consistent at quantile levels at the right boundary 1. This is because the construction of the g.c.m. starts from the left boundary 0 and tends to use one-sided information as it approaches to the right boundary 1. In the context of isotonic regression, a common way to correct the bias is to use the "penalized cusum diagram", as suggested by Groeneboom, Jongbloed, et al. (2013). Since our main target is imposing the monotonicity constraint and using fewer tuning parameters, we leave this issue for future research.

In practice it is often useful to conduct joint inferences on a set of quantile levels. For example, the test for common values in Haile, Hong, and Shum (2003) and test for different models of entry in Marmer, Shneyerov, and Xu (2013) are characterized by stochastic dominance relations between distributions. The following Corollary shows the quantile estimator is independent across a fixed vector of quantile levels asymptotically.

Corollary 1. *Let* $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_J < 1$ *be a vector of fixed quantile levels. Then*

$$\mathbb{P}\left(\cap_{j=1,2,\cdots,J}\{n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_j) - Q_v(\alpha_j)) \leq z_j\}\right)$$
$$= \Pi_{j=1,2,\cdots,J}\mathbb{P}\left(C(\alpha_j)\,\mathrm{argmax}_t\{\mathbb{B}_j(t) - t^2\} \leq z_j\right),$$

where $\mathbb{B}_j$, $j = 1, 2, \cdots, J$, *are independent two-sided Brownian motions.*[3]

The result in Corollary 1 does not hold in general for quantile estimations. It is useful when researchers would like to compare multiple quantile levels simultaneously, which in practice, is a useful approximation for comparing the whole distribution. One possible choice is the multiple testing procedure of Holm (1979), which controls the familywise error

[3]Please see the online supplement materials (Luo and Wan, 2016) for the proof of all corollaries in this paper.

rate of one false rejection. For example, one can calculate the p-vales $\hat{p}_j$, $j = 1, 2, \cdots J$ for each of the J hypothesizes. Rank all the p-values such that $\hat{p}_{(1)} \leq \hat{p}_{(2)} \leq \cdots \leq \hat{p}_{(J)}$. Let α be the significance level. If $\hat{p}_{(1)} \geq \alpha/J$, then no hypothesis is rejected; otherwise the procedure rejects hypothesizes $H_{(1)}, \cdots, H_{(k)}$, where k is the largest integer such that $\hat{p}_{(j)} \leq \alpha/(J - j + 1)$. One important source of conservativeness of Holm's procedure—dependence among p-values—does not rise here.

Theorem 1 also provides a basis for constructing a simple trimming-free smoothed quantile estimator that converges at the optimal rate of GPV under appropriate smoothness conditions as listed in Assumption 3 below. Numerous smooth quantile function estimators have been studied, see, e.g., Nadaraya (1964) for inverting a kernel distribution function estimator, Harrell and Davis (1982) for using generalized order statistics and Cheng (1995) for a Bernstein polynomial estimator. We adopt the kernel estimator used in Yang (1985), which dates back to Parzen (1979). Specifically, for any $0 < \alpha < 1$, let

$$\hat{q}_v(\alpha) = \int_0^1 \frac{1}{h} K\left(\frac{\alpha - u}{h}\right) \widehat{Q}_v(u) du, \tag{4}$$

where h is a bandwidth and $K(\cdot)$ is a kernel with a compact support. Note that by construction, $\hat{q}_v(\cdot)$ is necessarily increasing since $\widehat{Q}_v(\cdot)$ is increasing.

Assumption 3. *The valuation density f is continuously differentiable.*

Assumption 4. *Let K' be the first order derivative of K. Then K satisfies (1) K has compact support and take value zero on the boundary, (2) $\int K'(u)du = \int u^2 K'(u)du = 0$, (3)$\int uK'(u)du = -1$, (4) $\int u^3 K'(u)du \neq 0$.*

Assumption 3 requires same smoothness as in GPV and MS. Assumption 4 is satisfied by commonly used kernel functions such as second order Epanechnikov or Triweight Kernels.

Theorem 2. *Suppose Assumptions 1 to 4 are satisfied, and let $\alpha \in (0, 1)$,* *(i) if* $nh^5 \to c \in (0, \infty)$*, then* $\sqrt{nh}(\hat{q}_v(\alpha) - Q_v(\alpha)) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$*, where*

$$\mathscr{B} = -\left(\frac{\sqrt{c}Q_b''(\alpha)}{3} + \frac{c^2\alpha}{6(I-1)}Q_b'''(\alpha)\right)\int u^3 K'(u)du, \quad \mathscr{V} = \frac{\alpha^2}{c(I-1)^2}(Q_b'(\alpha))^2 \int K^2(u)du.$$

(ii) if $nh^r \to c$ *for some* $5 < r < 2$*, then*

$$\sqrt{nh}(\hat{q}_v(\alpha) - Q_v(\alpha)) \xrightarrow{d} N(0, \mathscr{V}).$$

Proof. See Appendix A.2. □

We have some remarks on Theorem 2. First, the variance and bias depend on c analytically. One can estimate the optimal choice of c that minimizes the asymptotic mean squared error, provided that the model has enough smoothness for consistent estimation of $Q_b'''(\cdot)$. Part (ii) of the theorem suggests that we can use under-smoothing to eliminate the asymptotical bias. Second, note from the quantile representation Equation (2) that the valuation quantile function is the sum of the bid quantile and a term containing quantile derivative (also known as quantile density function). It is the latter which determines the convergence rate of our estimator since the bid quantile can be estimated at root-n rate. When there is sufficient amount of smoothness, the bid quantile derivative can be estimated at a faster rate, which results in a faster convergence rate than what we obtained in Theorem 1. Third, another way of estimating the quantile derivative is to use the relationship $Q_b'(\alpha) = 1/g(Q_b(\alpha))$ and plug in a kernel density estimator for g, which is the method adopted in Marmer, Shneyerov, and Xu (2013). It is quite interesting to note that by choosing the two bandwidths proportionally, the asymptotic variances of our estimator and Marmer, Shneyerov, and Xu (2013) are identical in the current context.[4]

In the rest of the section, we discuss several interesting extensions of our method.

[4]Marmer, Shneyerov, and Xu (2013) consider entry. In our model all the potential bidders enter with probability one. Let h_{MSX} be the bandwidth of Marmer, Shneyerov, and Xu (2013) and h_{LW} be ours. Then the equality of variances holds by letting $h_{MSX} = h_{LW}Q'(\alpha_0)$ and observing that $Lh^r \to 1 \Leftrightarrow nh^r \to I$ since $n = LI$ and I is fixed. As a matter of fact, taking derivative of a smoothed quantile estimator and taking reciprocal of a kernel density estimator are two general approaches to estimate quantile derivatives. For general comparison of these two approaches, see Jones (1992).

2.1. **Generalization to procurement auctions.** Our method can be easily adapted to first price procurement auction settings. Suppose that there are I bidders competing for a contract in a first-price sealed bid auction. For each auction, every bidder i simultaneously draws an $i.i.d.$ cost c_i from a common distribution $F(\cdot)$ and submits a bid to maximize his/her expected profit $E[(b_i - c_i)\mathbb{1}(b_i \leq s(\min_{j\neq i} c_j))]$. The lowest bid wins the contract, and the bidder is paid the amount he/she bid.

Differentiating the expected profit with respect to b_i gives the following system of first-order differential equations that define the equilibrium strategy $s(\cdot)$:

$$(b_i - c_i)(I-1)\frac{f[s^{-1}(b_i)]}{[1-F(s^{-1}(b_i))]s'[s^{-1}(b_i)]} = 1,$$

which can be rewritten as

$$c_i = b_i - \frac{1}{I-1}\frac{1-G(b_i)}{g(b_i)}.$$

Therefore, the quantile relationship becomes

$$Q_c(\alpha) = Q_b(\alpha) - (1-\alpha)/[(I-1)g(Q_b(\alpha))],$$

where $Q_c(\cdot)$ represents the cost quantile function. The integrated quantile function becomes

$$C(\beta) \equiv \int_0^\beta Q_c(\alpha)d\alpha = \frac{I-2}{I-1}\int_0^\beta Q_b(\alpha)d\alpha - \frac{1}{I-1}Q_b(\beta)(1-\beta) + \frac{1}{I-1}Q_b(0). \quad (3')$$

To impose the monotonicity constraint, we consider the g.c.m. of the empirical counterpart of the following function:

$$\widetilde{C}(\beta) \equiv C(1-\beta),$$

which is the reflection of the integrated quantile function over the line $\beta = 1/2$. The idea is to utilize the prior information that the maximum possible bid equals the maximum cost in procurement auctions, i.e. $Q_b(1) = Q_c(1)$. As the pseudo values are constructed sequentially, consider the g.c.m. of $\widetilde{C}(\cdot)$ is preferable to $C(\cdot)$. To see this, note that $[\widehat{C}(1) - \widehat{C}(\frac{n-j}{n})]/(j/n) = \frac{I-2}{I-1}\sum_{k=n-j+1}^{N} b_{(k)}/j + \frac{1}{I-1}b_{(N-j)}$ and $[\widehat{C}(1/n) - \widehat{C}(0)]/(1/n) = b_{(1)}$. By definition, the preferred method starts with the largest pseudo valuation $\widehat{c}_{(n)} =$

$\frac{I-2}{I-1}b_{(n)} + \frac{1}{I-1}b_{(n-1)}$. Note the right-hand side converges to $Q_b(1) = Q_c(1)$ at a fast rate. On the other hand, considering the g.c.m. of $C(\cdot)$, we would start with an estimate of the smallest pseudo valuation $\widehat{c}_{(1)} \leq b_{(1)}$. Although $b_{(1)}$ converges to $Q_b(0)$ at a fast rate, it does not guarantee that $\widehat{c}_{(1)}$ converges to $Q_c(0)$ at a fast rate.

For estimation, we construct a raw estimator $\widetilde{C}_n(\cdot)$ for $\widetilde{C}(\cdot)$ by plugging in the bid quantile estimator. We then take the g.c.m. of $\widetilde{C}_n(\cdot)$. The pseudo cost of the bidder whose bid is the jth highest is constructed as the negative of the right-derivative of the g.c.m. at $\beta = (j-1)/n$, where $j = 1, \ldots, n$. A smooth estimator for the cost quantile function follows naturally: $\hat{q}_c(\alpha) = \int_0^1 \frac{1}{h} K\left(\frac{\alpha - u}{h}\right) \widehat{Q}_c(u)du$. Moreover, we can also apply a kernel density estimator on the sample of pseudo costs: $\hat{f}(c) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{\hat{c}_j - c}{h}\right)$.

2.2. **Estimating the valuation distribution function.** Sometimes an analyst might be more interested in the valuation distribution function than the quantile function. Estimators of valuation distribution function can be obtained by inverting $\widehat{Q}$ and $\hat{q}$, respectively. Note $\hat{q}$ is strictly monotone and continuous. The following corollary establishes their limiting distributions.

Corollary 2. *Let $v_0 \in (\underline{v}, \overline{v})$ and $\alpha_0 = F(v_0)$.*

(i) Suppose the conditions of Theorem 1 are satisfied. Define $\widehat{F}(v_0) = \sup\{\alpha : \widehat{Q}_v(\alpha) \leq v_0\}$. Then,

$$n^{1/3}(\widehat{F}(v_0) - F(v_0)) \xrightarrow{d} f(v_0)C(\alpha_0)\,\mathrm{argmax}_t \left\{\mathbb{B}(t) - t^2\right\},$$

where $C(\alpha_0)$ and $\mathbb{B}$ are as defined in Theorem 1.[5]

(ii) Suppose that the conditions of Theorem 2-(ii) are satisfied. Define $\widehat{F}^S(v_0) = \hat{q}^{-1}(v_0)$. Then,

$$\sqrt{nh}(\widehat{F}^S(v_0) - F(v_0)) \xrightarrow{d} N(0, (f(v_0))^2\mathscr{V}).$$

[5]Under a similar set of smoothness assumptions to ours, Armstrong (2013a) proposes to estimate the bidding strategy by maximizing the sample analog of the bidder's objective function and subsequently estimates the valuation distribution function at cube-root-n rate. Our approach is based on the integrated-quantile representation of the first order condition and imposes monotonicity restriction. Both estimators are tuning-parameter-free and robust to the degree of smoothness in the model.

To construct an estimator for valuation density, we can first construct a sample of pseudo valuations employing $\widehat{Q}_v(\cdot)$. Let $\hat{v}_j = \widehat{Q}_v(j/n)$, where $j = 1, \ldots, n$. Second, we apply a kernel density estimator on the sample of pseudo values $\{\hat{v}_j\}_{j=1}^N$: for $v \in (\underline{v}, \overline{v})$

$$\hat{f}(v) = \frac{1}{nh} \sum_{j=1}^{n} K\left(\frac{\hat{v}_j - v}{h}\right).$$

Since our first step estimator $\widehat{Q}_v(\cdot)$ is tuning-parameter-free, our estimator of the valuation density function requires no trimming and only one tuning parameter h.

2.3. **Incorporating auction level heterogeneity.** In practice, it is of empirical interest to incorporate auction-level observed heterogeneity. In this subsection, we discuss several ways of incorporating such heterogeneity.

In many applications, researchers are interested in the valuation quantile/distribution conditional on discrete or discretization of continuous variables. For example, in a procurement auction, researchers may be interested in the valuation (or cost) quantiles of "large" projects, which is defined by whether engineers' estimates (a continuous variable) exceed certain cutoff values. In these cases, our estimation procedure can be directly applied to corresponding subsamples, and the estimates can be interpreted as the conditional valuation quantile.

Our method can also be applied together with the homogenization method (see Haile, Hong, and Shum, 2003)–one common method of controlling both continuous and discrete observed heterogeneity in the empirical auction literature. The homogenization approach assumes valuations depend on auction-level characteristics in an additively (or multiplicatively) separable form, which implies the bids depend on characteristics in the same separable way. Under such an assumption, the effect of auction level characteristics can be controlled by focus on the regression residuals (called homogenized bids) of the original bids (or log of bids in the multiplicative case) on those covariates. Since our estimators converge at rates which are slower than root-n, their asymptotic properties are not affected by the homogenizing step.

If researchers would like to be agnostic about how valuations depend on the continuous covariates X, and if they are interested in the valuations quantiles conditional on a particular realization x, our method can also be applied with modification. In particular, since the quantile representation holds conditional on X, we can define the (sample) conditional integrated-quantile functions as

$$V_n(\beta|X=x) = \begin{cases} \frac{I-2}{n(I-1)} \sum_{i=1}^{n\beta} Q_{n,b}(i/n|X=x) + \frac{1}{I-1}\beta Q_{n,b}(n\beta|X=x), & \beta \in \left\{\frac{1}{n}, \cdots, 1\right\} \\ \\ (j-\beta n) V_n\left(\frac{j-1}{n}|X=x\right) + (\beta n - j + 1) V_n\left(\frac{j}{n}|X=x\right), & \beta \in \left(\frac{j-1}{n}, \frac{j}{n}\right) \end{cases}$$

where $Q_{n,b}(\alpha|X=x)$ is a suitable estimator for the conditional quantile function of bids given $X=x$. Note that by definition, $V_n(\cdot|X=x)$ is still a piecewise linear function and its g.c.m., denoted by $\widehat{V}(\cdot|X=x)$ is as easy to compute as in the unconditional case. Our estimator for conditional valuation quantile function, denoted by $\widehat{Q}_v(\cdot|X=x)$, is then defined as the piecewise derivative of $\widehat{V}(\cdot|X=x)$.

For $Q_{n,b}(\alpha|X=x)$, we adopt the local polynomial estimator proposed by GS and establish the convergence rate of our estimator based on GS's uniform Bahadur representation for the conditional quantile function and its derivatives.

Corollary 3. *Suppose that (i) X has a continuously differentiable density function which is bounded away from zero over its compact support $\mathscr{X} \subset \mathbb{R}^d$; (ii) for every x, the conditional valuation distribution $F(\cdot|x)$ is continuously differentiable over its compact support $[\underline{v}, \overline{v}]$. There exists $\lambda(x) > 0$ such that $\inf_{v \in [\underline{v}, \overline{v}]} f(v|x) \geq \lambda(x) > 0$; (iii) the bandwidth for the local polynomial regression is chosen such that $h = cn^{-r}$ for some r satisfying $\frac{1}{d+3} < r < \frac{1}{d}$. The kernel function is chosen to satisfy Assumption 4. Then for a given x in the interior of $\mathscr{X}$ and $\alpha_0 \in (0,1)$,*

$$\sqrt[3]{nh^d}\left(\widehat{Q}_v(\alpha_0|X=x) - Q_v(\alpha_0|X=x)\right) \xrightarrow{d} \tilde{C}(\alpha_0, x) \operatorname{argmax}_t \left\{\mathbb{B}(t) - t^2\right\},$$

where $\tilde{C}(\alpha_0, x)$ is a constant that depends on α_0, x and the kernel function.

In the presence of covariates, the rate of convergence of our first stage estimator is $\sqrt[3]{nh^d}$, which is slower than the optimal $n^{2/(5+d)}$ rate obtained by GPV and MS under stronger smoothness assumption on $F(\cdot|x)$; when there is no covariates (i.e. $d = 0$), this reduces to the $\sqrt[3]{n}$ versus $n^{2/5}$ comparison in the homogeneous auction case.

3. SIMULATION

To study the finite sample performance of our estimation method, we conduct Monte Carlo experiments. We adopt the setup of the Monte Carlo simulations from MS. The true valuation distribution is

$$F(v) = \begin{cases} 0 & \text{if } v < 0, \\ v^\gamma & \text{if } 0 \le v \le 1, \\ 1 & \text{if } v > 1, \end{cases}$$

where $\gamma > 0$. Such a choice of private value distributions is convenient since the distributions correspond to linear bidding strategies as:

$$s(v) = \big(1 - \frac{1}{\gamma(I-1)+1}\big) \cdot v. \tag{5}$$

We consider $I = 7$ bidders, $n = 4200$ and $\gamma \in \{0.5, 1, 2\}$. The number of Monte Carlo replications is 1000. For each replication, we first generate randomly n private values from $F(\cdot)$. Second, we obtain the corresponding bids b_i employing the linear bidding strategy (5). Third, we construct a raw estimator $V_n(\cdot)$ for $V(\cdot)$. Let $\widehat{V}(\cdot)$ be the g.c.m. of $V_n(\cdot)$. Fourth, we obtain a sample of pseudo values $\widehat{v}_j$ as the left-derivative of $\widehat{V}(\cdot)$ at j/N and estimate the valuation density function using a kernel estimator.

We compare our method with MS and GPV. For the MS and GPV methods, we use the same setups as in MS: the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidths in estimation of densities. For our method, we also use the tri-weight kernel function for the kernel estimators and the normal rule-of-thumb bandwidth in estimation of f: $h = 1.06\widehat{\sigma}_v n^{-1/7}$, where $\widehat{\sigma}_v$ is the estimated standard deviation of the constructed pseudo valuations $\{\widehat{v}_j\}_{j=1}^N$.

Table 1 shows the simulation results for density estimation. When the distribution is skewed to the left ($\gamma = 0.5$), our method improves MSE and MAD but seems to produce larger biases near the boundaries. While the MS and GPV methods behave similarly in terms of MSE and MAD, the former seems to produce larger biases. When the distribution is uniform or skewed to the right ($\gamma = 1$ or 2), our method performs similarly to the GPV method, both of which seem to perform slightly better than the MS method.

4. Empirical Illustration

In this section, we implement our method using the California highway procurement data. In particular, we analyze the data used in Krasnokutskaya and Seim (2011). It covers highway and street maintenance projects auctioned by the California Department of Transportation (Caltrans) between January 2002 and December 2005. We focus on the procurement auctions with 2 to 7 bidders. For each auction, the data contains the engineer's estimate of the project's total cost, the type of work involved, the number of days allocated to complete the project, and the identity of the bidders and their bids.

Following Haile, Hong, and Shum (2003), we homogenize the bids before implementing our method to control for observable heterogeneity for each sample (with the same number of bidders). In particular, we regress the logarithm of the bid ($logb$) on the logarithm of the engineer's estimate ($logX$), the logarithm of the number of days ($logDays$), and the project type dummies. Table 2 displays the results. The homogenized bids (bid_new) are calculated as the exponential of the differences between the logarithm of the original bids and the demeaned fitted values of the regression. Table 3 displays the mean and standard deviation of the original and homogenized bids.

We estimate a first price auction model with each sample. Figure 1 displays the estimated inverse bidding strategies and the estimated valuation quantile functions without and with smoothing, respectively. The curves represented are: from the sample with 2 bidders (yellow

TABLE 1. Simulation Results for Density Estimation

	v		0.2	0.3	0.4	0.5	0.6	0.7	0.8
$\gamma = 0.5$	MSE	MS	0.0068	0.0073	0.0103	0.0131	0.0132	0.0171	0.0202
		GPV	0.0056	0.0072	0.0101	0.0132	0.0139	0.0188	0.0218
		Ours	0.0044	0.0057	0.0080	0.0100	0.0109	0.0140	0.0163
	Bias	MS	-0.0041	-0.0019	-0.0086	-0.0029	-0.0159	-0.0156	-0.0185
		GPV	0.0038	0.0018	-0.0034	0.0037	-0.0019	0.0025	0.0072
		Ours	0.0120	0.0043	-0.0016	0.0037	-0.0022	0.0038	0.0056
	MAD	MS	0.0672	0.0689	0.0806	0.0907	0.0908	0.1027	0.1094
		GPV	0.0611	0.0688	0.0800	0.0925	0.0940	0.1106	0.1186
		Ours	0.0543	0.0608	0.0711	0.0806	0.0825	0.0952	0.1030
$\gamma = 1$	MSE	MS	0.0036	0.0050	0.0066	0.0076	0.0102	0.0122	0.0148
		GPV	0.0025	0.0035	0.0050	0.0060	0.0082	0.0102	0.0127
		Ours	0.0023	0.0033	0.0049	0.0061	0.0083	0.0102	0.0129
	Bias	MS	0.0003	0.0000	-0.0047	-0.0035	0.0014	-0.0060	-0.0113
		GPV	0.0000	0.0015	-0.0023	-0.0011	0.0053	0.0007	-0.0021
		Ours	0.0000	0.0016	-0.0027	-0.0020	0.0056	0.0007	-0.0026
	MAD	MS	0.0479	0.0557	0.0647	0.0688	0.0800	0.0892	0.0961
		GPV	0.0402	0.0470	0.0563	0.0610	0.0724	0.0806	0.0904
		Ours	0.0389	0.0459	0.0557	0.0615	0.0730	0.0812	0.0901
$\gamma = 2$	MSE	MS	0.0016	0.0025	0.0037	0.0063	0.0085	0.0108	0.0154
		GPV	0.0011	0.0016	0.0025	0.0044	0.0060	0.0078	0.0112
		Ours	0.0011	0.0017	0.0028	0.0049	0.0069	0.0091	0.0130
	Bias	MS	-0.0006	-0.0031	-0.0008	-0.0013	-0.0033	-0.0085	-0.0001
		GPV	0.0005	-0.0020	0.0007	0.0002	-0.0004	-0.0044	0.0021
		Ours	0.0006	-0.0019	0.0013	0.0002	-0.0006	-0.0048	0.0020
	MAD	MS	0.0320	0.0394	0.0481	0.0637	0.0739	0.0830	0.1008
		GPV	0.0263	0.0321	0.0396	0.0528	0.0624	0.0707	0.0864
		Ours	0.0266	0.0329	0.0415	0.0555	0.0668	0.0767	0.0929

solid line); 3 bidders (magenta dash-dot line); 4 bidders (cyan solid line); 5 bidders (red dash-dot line); 6 bidders (green solid line); 7 bidders (blue dash-dot line), and the 45-degree line (black dash line).

All inverse bidding strategies are increasing. The valuation quantile functions seem to be close except for $I = 2$. Table 3 displays some summary statistics of the estimated pseudo costs. The auctions with two bidders tend to be less costly to finish in percentage terms. In

TABLE 2. Regression Results

	2	3	4	5	6	7
logX	0.978***	0.966***	1.015***	0.957***	0.932***	0.938***
	(34.11)	(56.68)	(50.59)	(51.81)	(49.91)	(56.58)
logDays	0.00650	0.00473	-0.00271	0.0901***	0.138***	0.00430
	(0.15)	(0.25)	(-0.13)	(4.76)	(6.31)	(0.18)
type	Yes	Yes	Yes	Yes	Yes	Yes
n	206	474	564	470	402	252
adj. R^2	0.871	0.906	0.857	0.929	0.930	0.947

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

TABLE 3. Summary Statistics

	2	3	4	5	6	7	Total
bid	993.8	967.6	757.7	1136.9	990.9	1769.7	1042.8
	(1644.5)	(1935.9)	(843.7)	(4584.7)	(3350.3)	(7288.0)	(3595.9)
bid_new	652.5	587.7	566.3	508.9	464.4	478.5	540.0
	(208.4)	(190.6)	(178.6)	(129.0)	(135.0)	(137.4)	(174.0)
cost	402.1	468.0	477.8	453.8	423.6	441.7	451.5
	(259.6)	(223.6)	(218.9)	(164.4)	(156.3)	(159.5)	(200.0)
profit	250.4	119.7	88.46	55.09	40.79	36.81	88.59
	(75.81)	(77.65)	(60.62)	(49.56)	(42.83)	(46.68)	(83.51)
profit rate	0.439	0.244	0.197	0.136	0.109	0.0978	0.190
	(0.213)	(0.208)	(0.194)	(0.167)	(0.153)	(0.158)	(0.206)

Std. Dev. in parentheses. $profit = bid\_new - cost$. *Profit rate=profit / bid.*

fact, the generated profit rate is almost twice that of the sample with three bidders. As the auction becomes more competitive when the number of bidders increases from two to seven, the profit rate decreases from 44% to about 10%.

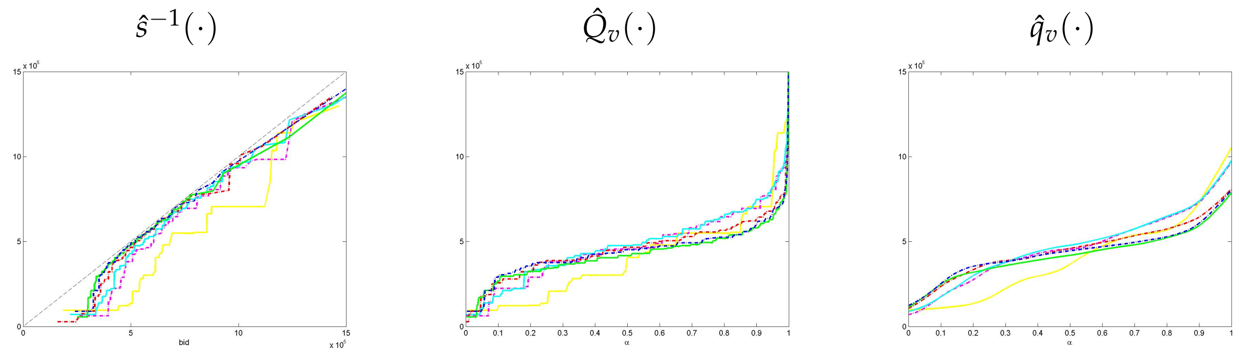

FIGURE 1. Estimation results

5. CONCLUSION

This paper considers nonparametric estimation of first-price auction models based on an integrated-quantile representation of the first-order condition. The monotonicity of bidding strategy is imposed in a natural way. We propose two estimators for the valuation quantile function and derive their asymptotics: a non-smoothed estimator that is tuning-parameter-free and a smoothed one that is trimming-free. We show the former is cube-root consistent under weaker smoothness assumptions, and the latter achieves the optimal rate of GPV under standard ones. Monte Carlo simulations show our method works well in finite samples. We apply our method to data from the California highway procurement auctions.

REFERENCES

ARMSTRONG, T. (2013a): "Notes on Revealed Preference Estimation of Auction Models," *Working paper*.

ARMSTRONG, T. B. (2013b): "Bounds in auctions with unobserved heterogeneity," *Quantitative Economics*, 4(3), 377–415.

ATHEY, S., AND P. A. HAILE (2007): "Chapter 60 Nonparametric Approaches to Auctions," vol. 6, Part A of *Handbook of Econometrics*, pp. 3847 – 3965. Elsevier.

BIERENS, H. J., AND H. SONG (2012): "Semi-nonparametric estimation of independently and identically repeated first-price auctions via an integrated simulated moments method," *Journal of Econometrics*, 168(1), 108–119.

CHENG, C. (1995): "The Bernstein polynomial estimator of a smooth quantile function," *Statistics & probability letters*, 24(4), 321–330.

CHERNOZHUKOV, V., I. FERNANDEZ-VAL, AND A. GALICHON (2010): "Quantile and probability curves without crossing," *Econometrica*, pp. 1093–1125.

GIMENES, N., AND E. GUERRE (2013): "Augmented quantile regression methods for first price auction," Discussion paper.

GROENEBOOM, P., G. JONGBLOED, ET AL. (2013): "Smooth and non-smooth estimates of a monotone hazard," in *From Probability to Statistics and Back: High-Dimensional Models and Processes–A Festschrift in Honor of Jon A. Wellner*, pp. 174–196. Institute of Mathematical Statistics.

GROENEBOOM, P., G. JONGBLOED, AND J. A. WELLNER (2014): *Nonparametric estimation under shape constraints*, vol. 38. Cambridge University Press.

GUERRE, E., I. PERRIGNE, AND Q. VUONG (2000): "Optimal Nonparametric Estimation of First-price Auctions," *Econometrica*, 68(3), 525–574.

GUERRE, E., AND C. SABBAH (2012): "Uniform bias study and Bahadur representation for local polynomial estimators of the conditional quantile function," *Econometric Theory*, 28(01), 87–129.

HAILE, P. A., H. HONG, AND M. SHUM (2003): "Nonparametric tests for common values at first-price sealed-bid auctions," Discussion paper, National Bureau of Economic Research.

HARRELL, F. E., AND C. DAVIS (1982): "A new distribution-free quantile estimator," *Biometrika*, 69(3), 635–640.

HENDERSON, D. J., J. A. LIST, D. L. MILLIMET, C. F. PARMETER, AND M. K. PRICE (2012): "Empirical implementation of nonparametric first-price auction models," *Journal of Econometrics*, 168(1), 17–28.

HICKMAN, B. R., AND T. P. HUBBARD (2014): "Replacing Sample Trimming with Boundary Correction in Nonparametric Estimation of First-Price Auctions," *Journal of*

Applied Econometrics.

HOLM, S. (1979): "A simple sequentially rejective multiple test procedure," *Scandinavian journal of statistics*, pp. 65–70.

JONES, M. (1992): "Estimating densities, quantiles, quantile densities and density quantiles," *Annals of the Institute of Statistical Mathematics*, 44(4), 721–727.

KRASNOKUTSKAYA, E., AND K. SEIM (2011): "Bid preference programs and participation in highway procurement auctions," *The American Economic Review*, 101(6), 2653–2686.

LIU, N., AND Y. LUO (2015): "Nonparametric Tests for Comparing Valuation Distributions in First-Price Auctions," Working Paper, University of Toronto.

LIU, N., AND Q. VUONG (2013): "Nonparametric Test of Monotonicity of Bidding Strategy in First-price Auctions," Working Paper, New York University.

LUO, Y., AND Y. WAN (2016): "Supplement to "Integrated-quantile-based Estimation for First Price Auction Models"," *Working Paper*.

MARMER, V., AND A. SHNEYEROV (2012): "Quantile-based nonparametric inference for first-price auctions," *Journal of Econometrics*, 167(2), 345–357.

MARMER, V., A. SHNEYEROV, AND P. XU (2013): "What model for entry in first-price auctions? A nonparametric approach," *Journal of Econometrics*, 176(1), 46–58.

MENZEL, K., AND P. MORGANTI (2013): "Large sample properties for estimators based on the order statistics approach in auctions," *Quantitative Economics*, 4(2), 329–375.

NADARAYA, E. A. (1964): "Some new estimates for distribution functions," *Theory of Probability & Its Applications*, 9(3), 497–500.

PARZEN, E. (1979): "Nonparametric statistical data modeling," *Journal of the American statistical association*, 74(365), 105–121.

ROBERTSON, T., F. WRIGHT, R. L. DYKSTRA, AND T. ROBERTSON (1988): *Order restricted statistical inference*, vol. 229. Wiley New York.

VAN DER VAART, A. W., AND J. A. WELLNER (1996): *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer.

VAN ES, B., G. JONGBLOED, AND M. V. ZUIJLEN (1998): “Isotonic Inverse Estimators for Nonparametric Deconvolution,” *The Annals of Statistics*, 26(6), pp. 2395–2406.

WELSH, A. (1988): “Asymptotically efficient estimation of the sparsity function at a point,” *Statistics & probability letters*, 6(6), 427–432.

YANG, S.-S. (1985): “A Smooth Nonparametric Estimator of a Quantile Function,” *Journal of the American Statistical Association*, 80(392), pp. 1004–1011.

Appendix A. Proof of main results

A.1. **Proof of Theorem 1.** For a generic $c > 0$, let $Z_n(c) = \text{argmin}_{t\in[0,1]}\{V_n(t) - ct\}$. If the argmin is a set, then we take the sup of the set. For any $\alpha_0 \in (0,1)$, by van Es, Jongbloed, and Zuijlen (1998, Theorem 2), the two following events are equivalent

$$Z_n(c) \geq \alpha_0 \Leftrightarrow \widehat{Q}_v(\alpha_0) \leq c.$$

Therefore, we have for a fixed $\alpha_0 \in [0,1)$

$$\begin{aligned}
n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \leq z \Leftrightarrow \widehat{Q}_v(\alpha_0) \leq zn^{-\frac{1}{3}} + Q_v(\alpha_0) \Leftrightarrow Z_n(zn^{-\frac{1}{3}} + Q_v(\alpha_0)) \geq \alpha_0 \\
\overset{i}{\Leftrightarrow} \operatorname*{argmin}_{s\in[0,1]}\{V_n(s) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))s\} \geq \alpha_0 \\
\overset{ii}{\Leftrightarrow} \operatorname*{argmin}_{\{t:\alpha_0+tn^{-\frac{1}{3}}\in[0,1]\}}\{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))(\alpha_0 + tn^{-\frac{1}{3}})\} \geq 0 \\
\overset{iii}{\Leftrightarrow} \operatorname*{argmin}_{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}\{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}} - ztn^{-\frac{2}{3}}\} \geq 0 \\
\overset{iv}{\Leftrightarrow} \operatorname*{argmin}_{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}\{n^{\frac{2}{3}}V_n(\alpha_0 + tn^{-\frac{1}{3}}) - n^{\frac{2}{3}}V_n(\alpha_0) - Q_v(\alpha_0)tn^{\frac{1}{3}} - zt\} \geq 0,
\end{aligned}$$

where (i) holds by definition of Z_n, (ii) holds by changing variable $s = \alpha_0 + tn^{-\frac{1}{3}}$, and (iii) and (iv) hold because the argmin stays unchanged when constants are multiplied or added to, or subtracted from the objective function.

Let $W_n(t) = n^{\frac{2}{3}}\left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}}\right]$, then the above displayed equation reduces to

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \leq z \Leftrightarrow \operatorname*{argmin}_{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}\{W_n(t) - zt\} \geq 0$$

It remains to analyze the asymptotic behavior of $W_n(t)$. Decompose W_n as following

$$\begin{aligned}
W_n(t) = n^{\frac{2}{3}}\left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0)\right] - n^{\frac{2}{3}}\left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0)\right] \\
+ n^{\frac{2}{3}}\left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}}\right].
\end{aligned}$$

The second component equals to $\frac{1}{2}Q_v'(\alpha_0)t^2 + o(1)$ by Assumption 2. As shown by Lemma 3 in the Supplement Material (Luo and Wan, 2016), the first right hand side term converges weakly to $\frac{\alpha_0}{(I-1)\sqrt{g(Q_b(\alpha_0))}}\mathbb{B}$, where $\mathbb{B}$ is a two sided Brownian Motion. Therefore, we have

$$W_n(t) \xrightarrow{w} \frac{\alpha_0}{(I-1)g(Q_b(\alpha_0))}\mathbb{B}(t) + \frac{1}{2}Q_v'(\alpha_0)t^2.$$

To simplify the notation, denote the constants in front of $\mathbb{B}$ and t^2 with a and b, respectively. Note that $a > 0$ and $b > 0$. By Van Der Vaart and Wellner (1996, Theorem 3.2.2) and the property of Brownian motion,

$$\begin{aligned}
\underset{t\in[-\alpha_0 n^{\frac{1}{3}},(1-\alpha_0)n^{\frac{1}{3}}]}{\operatorname{argmin}} \{W_n(t) - zt\} &\xrightarrow{d} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + bt^2 - zt\} \overset{d}{\sim} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{a\mathbb{B}(t) + b(t - \frac{z}{2b})^2\} \\
&\overset{d}{\sim} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\frac{a}{b}\mathbb{B}(t) + (t - \frac{z}{2b})^2\} \overset{d}{\sim} \left(\frac{a}{b}\right)^{2/3} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} + \frac{z}{2b},
\end{aligned}$$

where $\overset{d}{\sim}$ denote equivalence in distribution. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \le z\right) \to \mathbb{P}\left(\left(\frac{a}{b}\right)^{2/3} \underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} + \frac{z}{2b} \ge 0\right) \\
&= \mathbb{P}\left(\underset{t\in\mathbb{R}}{\operatorname{argmin}}\{\mathbb{B}(t) + t^2\} \ge -\frac{z}{2b}\left(\frac{b}{a}\right)^{2/3}\right) = \mathbb{P}\left(\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t) - t^2\} \le \frac{z}{2b}\left(\frac{b}{a}\right)^{2/3}\right)
\end{aligned}$$

Thus we can conclude that for $C(\alpha_0) = 2a^{2/3}b^{1/3}$,

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \xrightarrow{d} C(\alpha_0)\operatorname{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t) - t^2\}.$$

A.2. **Proof of Theorem 2.** For notation simplicity, let $K_h(\cdot) = (1/h)K(\cdot/h)$. Then

$$\begin{aligned}
\hat{q}_v(\alpha) = \int K_h(\alpha - u)\, d\widehat{V}(u) &= \int K_h(\alpha - u)\, dV_n(u) + \int K_h(\alpha - u)\, d(\widehat{V} - V_n)(u) \\
&= \int K_h(\alpha - u)\, dV_n(u) + \frac{1}{h}\int K_h'(\alpha - u)(\widehat{V}(u) - V_n(u))du \\
&= \int K_h(\alpha - u)\, dV_n(u) + \frac{1}{h}\int K_h'(t)(\widehat{V}(\alpha + ht) - V_n(\alpha + ht))dt \\
&= \int K_h(\alpha - u)\, dV_n(u) + O_p((n/\log n)^{-2/3}/h) \quad (6)
\end{aligned}$$

where the third inequality holds by integration by parts, and the last equality holds because the sup distance from V_n to its g.c.m. $\widehat{V}$ is of order $O_p((n/\log n)^{-2/3})$ by Lemma 8 in the Supplement Material (Luo and Wan, 2016). Therefore it is then sufficient to focus on the first right hand side term. Since $Q_{v,n}$ is piecewise flat and is left-continuous, we have

$$\int K_h(\alpha-u)\,dV_n(u) - Q_v(\alpha) = \int K_h(\alpha-u)\,Q_{v,n}(u)du - Q_v(\alpha)$$
$$= \underbrace{\sum_{i=1}^{n} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du - Q_b(\alpha)}_{A_n(\alpha)} + \frac{1}{I-1}\underbrace{\left(\sum_{i=1}^{n}(i-1)(b_{(i)}-b_{(i-1)})\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du - \frac{\alpha}{g(Q_b(\alpha))}\right)}_{B_n(\alpha)}.$$

$A_n(\alpha)$ is the standard smooth quantile estimator. Yang (1985, Theorem 1) shows that when $nh^5 \to c$, $\sqrt{nh}A_n(\alpha) \xrightarrow{p} \sqrt{c}Q_b''(\alpha)\int u^2K(u)du = -\frac{\sqrt{c}}{3}Q_b''(\alpha)\int u^3K'(u)du$, and when $nh^5 \to 0$, $\sqrt{nh}A_n(\alpha) \xrightarrow{p} 0$.

It remains to consider the B_n part. Define $\tilde{B}_n(\alpha)$ as

$$\tilde{B}_n(\alpha) = \sum_{i=1}^{n} \alpha n(b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du - \frac{\alpha}{g(Q_b(\alpha))}.$$

Note first when n is large,

$$n\sum_{i=1}^{n}(b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du = n\sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du$$
$$- n\sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha-u)\,du + nb_{(n)}\int_{\frac{n-1}{n}}^{1} K_h(\alpha-u)\,du - nb_{(0)}\int_{0}^{1/n} K_h(\alpha-u)\,du$$
$$\approx n\sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du - n\sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha-u)\,du.$$

The last equality holds because under Assumption 4, when n is large, $K_h(t) = 0$ for any $t \neq 0$. Recall that $K_h(\cdot) = (1/h)K(\cdot/h)$, we know that

$$\tilde{B}_n(\alpha) = \alpha n\sum_{i=1}^{n-1} b_{(i)} \left\{\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha-u)\,du - \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha-u)\,du\right\}$$
$$= \frac{\alpha}{h^2}\sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K'\left(\frac{u-\alpha}{h}\right)du + O_p(1/n).$$

By Welsh (1988, main theorem, part (ii)), under Assumptions 3 and 4 and $nh^5 \to c$, $\sqrt{nh}(\tilde{B}_n(\alpha) - \frac{\alpha}{g(Q_b(\alpha))}) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$, where

$$\mathscr{B} = -\frac{c^2\alpha}{6(I-1)} Q_b'''(\alpha) \int u^3 K'(u) du \quad \mathscr{V} = \frac{\alpha^2}{c(I-1)^2} (Q_b'(\alpha))^2 \int K^2(u) du.$$

When $h = cn^{-r}$ for some $\frac{1}{5} < r < \frac{1}{2}$, $\sqrt{n^{1-r}}(\tilde{B}_n(\alpha) - Q_v(\alpha)) \xrightarrow{d} N(0, \mathscr{V})$.

Lastly, let $z_{(i)} = n(b_{(i)} - b_{(i-1)})$ and $w_i = ((i-1)/n - \alpha) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u-\alpha) du$. Observe that $B_n - \tilde{B}_n = \sum_i z_{(i)} w_i$. By Lemma 9 in the Supplement Material (Luo and Wan, 2016), $B_n - \tilde{B}_n$ is of order $o_p(1/\sqrt{nh})$. Therefore we can conclude that $\sqrt{nh} B_n(\alpha) \xrightarrow{d} N(\mathscr{B}, \mathscr{V})$.

Appendix B. Compute the Greatest Convex Minorant of $V_n(\cdot)$

We now describe how to compute the greatest convex minorant of $V_n(\cdot)$.

First, consider the coordinate vectors of the piecewise linear function $V_n(\cdot)$: $\{(0,0),(1/n, V_n(1/n)), \ldots, (1, V_n(1))\}$. We find the smallest slope of each $(j/n, V_n(j/n))$ with respect to the origin, which defines the first partition on the g.c.m.. Let $j_1 = \text{argmin}_{j \in \{1,\ldots,n\}} \frac{V_n(j/n)-0}{j/n-0}$. The first partition is the line segment connecting $(0,0)$ and $(j_1, V_n(j_1/n))$.

Second, we find the next smallest slope, after removing the first partition from further consideration. In particular, consider the coordinate vectors $\{(j_1, V_n(j_1/n)), \ldots, (1, V_n(1))\}$. Let $j_2 = \text{argmin}_{j \in \{j_1+1,\ldots,n\}} \frac{V_n(j/n)-V_n(j_1/n)}{j/n-j_1/n}$. The second partition is the line segment connecting $(j_1, V_n(j_1/n))$ and $(j_2, V_n(j_2/n))$.

We continue in this manner until we reach the end of the points $(1, V_n(1))$. The resulting coordinate vectors $\{(0,0), (j_1, V_n(j_1/n)), (j_2, V_n(j_2/n)), \ldots, (1, V_n(1))\}$ define the greatest convex minorant of $V_n(\cdot)$, which is also piecewise linear.