SUPPLEMENT TO "INTEGRATED-QUANTILE-BASED ESTIMATION FOR FIRST-PRICE AUCTION MODELS"

YAO LUO AND YUANYUAN WAN

UNIVERSITY OF TORONTO

ABSTRACT. This supplement provides the details of the proofs which were omitted from the main text. Section 1 provides auxiliary lemmas for the proof of Theorem 1. Section 2 provides auxiliary lemmas for the proof of Theorem 2. The proofs to corollaries are collected in Section 3.

We first re-introduce the assumptions made in the main text.

Assumption 1. *There are $L \to \infty$ identical auctions, and for each auction, there are I symmetric and risk neutral bidders. Their private values are i.i.d. draws from a common distribution $F(\cdot)$.*

Assumption 2. *$F(\cdot)$ is continuously differentiable over its compact support $[\underline{v}, \overline{v}]$. There exists $\lambda > 0$ such that $\inf_{v \in [\underline{v}, \overline{v}]} f(v) \geq \lambda > 0$.*

Assumption 3. *The valuation density f is continuously differentiable.*

Assumption 4. *Let K' be the first order derivative of K. Then K satisfies (1) K has compact support and take value zero on the boundary, (2) $\int K'(u)du = \int u^2 K'(u)du = 0$, (3) $\int uK'(u)du = -1$, (4) $\int u^3 K'(u)du \neq 0$.*

1. AUXILIARY LEMMAS FOR THEOREM 1

Lemma 1. *Suppose that Assumptions 1 and 2 hold, then for any $\alpha_0 \in (0,1)$ and uniformly over $t \in \mathscr{T}$, where $\mathscr{T}$ is a compact subset of $\mathbb{R}$,*

$$n^{2/3} \left\{ \int_{\alpha_0}^{\alpha_0 + t/n^{1/3}} Q_{b,n}(\tau) d\tau - \int_{\alpha_0}^{\alpha_0 + t/n^{1/3}} Q_b(\tau) d\tau \right\} \xrightarrow{p} 0.$$

Proof. Assumption 2 implies that Q_b is twice continuously differentiable (see Guerre, Perrigne, and Vuong, 2000, Proposition 1-(iv)). By the Bahadur representation for quantile functions (see, e.g. Bahadur, 1966; Kiefer, 1967), we know that uniform in $\tau \in [\delta, 1-\delta]$,

$$Q_{b,n}(\tau) - Q_b(\tau) = \frac{\tau - \frac{1}{n}\sum_i \mathbf{1}[b_i \leq Q_b(\tau)]}{g(Q_b(\tau))} + O_{a.s.}\left(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}\right).$$

Since $\alpha_0 \in (0,1)$, we have

$$\begin{aligned}
n^{2/3}\int_{\alpha_0}^{\alpha_0+t/n^{1/3}} \left(Q_{b,n}(\tau) - Q_b(\tau)\right) d\tau &= n^{2/3}\int_{\alpha_0}^{\alpha_0+t/n^{1/3}} \left(\frac{\tau - \frac{1}{n}\sum_i \mathbf{1}[b_i \leq Q_b(\tau)]}{g(Q_b(\tau))}\right) d\tau + o_p(1) \\
&= n^{2/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \left(F(u) - \frac{1}{n}\sum_i \mathbf{1}[b_i \leq u]\right) du + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_i n^{1/6}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \left(F(u) - \mathbf{1}[b_i \leq u]\right) du + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_i \xi_n(b_i, t) + o_p(1),
\end{aligned}$$

where $\xi_n(b_i, t) = n^{1/6}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \left(F(u) - \mathbf{1}[b_i \leq u]\right) du$. It is sufficient to show that $\frac{1}{\sqrt{n}}\sum_i \xi_n(b_i, t)$ converges uniformly to zero in probability.

Note that $\mathbb{E}[\xi_n(b_i, t)] = 0$ and the summand are i.i.d.. For each n, define a class of functions indexed by t: $\Xi_n \equiv \{\xi_n(\cdot, t) : t \in \mathscr{T}\}$. Then we can have the following observations.

(i) Let $t^* = \text{argmax}_{t\in\mathscr{T}} |Q_b(\alpha_0 + t/n^{1/3}) - Q_b(\alpha_0)|$ and let $\bar{\xi}(b) \equiv n^{1/6}|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)|(F(u) - \mathbf{1}[b \leq u])$. Then $\bar{\xi}(b)$ is an envelope function for Ξ_n. We also have $\mathbb{E}\bar{\xi}^2(b) = O(1)$ since $|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)| = O(n^{-1/3})$.

(ii) For any $\epsilon > 0$, we have $\mathbb{E}[\bar{\xi}^2(b)\mathbf{1}[\bar{\xi}(b) > \epsilon\sqrt{n}]] = o(1)$. This is because $\bar{\xi}(b) > \epsilon\sqrt{n}$ if and only if $|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)|(F(u) - \mathbf{1}[b \leq u]) > \epsilon n^{1/3}$ and the latter is a probability event with probability approaches zero, whereas $\mathbb{E}[\bar{\xi}^2]$ is bounded.

(iii) For any $\epsilon_n \downarrow 0$, there is $\sup_{(t,s)\in\mathscr{T}^2:|t-s|\leq\epsilon_n} \mathbb{E}\{\xi_n(b,t) - \xi_n(b,s)\}^2 = o(1)$. To verify this claim, assume without loss of generality that $t > 0$ and $s < 0$. Then $\xi_n(b,t) -$

$\xi_n(b,s) = n^{1/6}\int_{Q_b(\alpha_0+s/n^{1/3})}^{Q_b(\alpha_0+t/n^{1/3})} (F(u) - \mathbf{1}[b_i \le u])\,du$ and almost surely

$$
\begin{aligned}
\{\xi_n(b,t) - \xi_n(b,s)\}^2 &= n^{1/3}\left\{\int_{Q_b(\alpha_0+s/n^{1/3})}^{Q_b(\alpha_0+t/n^{1/3})} (F(u) - \mathbf{1}[b_i \le u])\,du\right\}^2 \\
&\le n^{1/3}\left\{\int_{Q_b(\alpha_0+s/n^{1/3})}^{Q_b(\alpha_0+t/n^{1/3})} |F(u) - \mathbf{1}[b_i \le u]|\,du\right\}^2 \\
&\le 4n^{1/3}\left\{Q_b(\alpha_0 + t/n^{1/3}) - Q_b(\alpha_0 + s/n^{1/3})\right\}^2 = n^{-1/3}O(|t-s|),
\end{aligned}
$$

where the second inequity holds because $|F(u) - \mathbf{1}[b_i \le u]| \le \sup_u F(u) + 1 \le 2$. The claim is therefore verified.

(iv) Let $\mathscr{N}(\epsilon, \Xi_n, \mathbb{L}^2(\mathscr{P}))$ be the $\mathbb{L}^2$-covering number for Ξ_n with respect to probability measure $\mathscr{P}$, then for every $\epsilon_n \downarrow 0$, we have $\sup_{\mathscr{P}^*}\int_0^{\epsilon_n}\sqrt{\log\mathscr{N}(\epsilon\|\bar{\xi}(b)\|_{\mathscr{P}^*,2}, \Xi_n, \mathbb{L}^2(\mathscr{P}^*))}d\epsilon = o(1)$. This claim holds by observing that $\xi_{b,t}$ is continuously differentiable with respect to t and hence Ξ_n belongs to the parametric class (see Van der Vaart, 2000, Example 19.7), which implies the convergences of the integral.

(v) We derive the limit of the covariance function. Take $t, s \in \mathscr{T}$,

$$
\begin{aligned}
\mathbb{E}[\xi_n(b_i,t)\xi_n(b_i,s)] &= \mathbb{E}\left[n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})} \mathbf{1}[b_i \le u]du \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})} \mathbf{1}[b_i \le u]du\right] + o(1) \\
&= n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})} \mathbb{E}\left\{\mathbf{1}[\min\{u,v\} \ge b_i]\right\} dudv + o(1) \\
&= n^{1/3}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+t/n^{1/3})}\int_{Q_b(\alpha_0)}^{Q_b(\alpha_0+s/n^{1/3})} G(\min\{u,v\})dudv \to 0,
\end{aligned}
$$

where G is the c.d.f. of the bid distribution. Therefore, $H(t,s) \equiv \lim_{n\to\infty}\mathbb{E}[\xi_n(b_i,t)\xi_n(b_i,s)] = 0$ for any $t, s \in \mathscr{T}$.

Based on (i)-(v) and Van Der Vaart and Wellner (1996, Theorem 2.11.22), $\frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t)$ converges weakly to a zero mean Gaussian process $\mathbb{G}$ with sample path define on $\mathscr{T}$ and with covariance function $H(t,s)$. By the property of Gaussian process, $H(t,s) = 0$ implies that the limit process $\mathbb{G}(t) = 0$ for all t almost surely. Because the mapping

$\sup_{t\in\mathscr{T}} f(\cdot) : \mathscr{C} \to \mathbb{R}$ (from the set of continuous functions defined on compact set to $\mathbb{R}$) is continuous with respect to the sup-norm, we can further apply the continuous mapping theorem and have

$$\frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,\cdot) \xrightarrow{w} \mathbb{G} \Rightarrow \sup_{t\in\mathscr{T}} \frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t) \xrightarrow{d} \sup_{t\in\mathscr{T}} \mathbb{G}(t) = 0 \Rightarrow \sup_{t\in\mathscr{T}} \frac{1}{\sqrt{n}}\sum_i \xi_n(b_i,t) \xrightarrow{p} 0.$$

The conclusion of the Lemma holds.

Lemma 2. *Suppose that Assumptions 1 and 2 hold, then*

$$n^{2/3}\alpha_0 \left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0) \right\} \xrightarrow{w} \frac{\alpha_0}{g(Q_b(\alpha_0))}\mathbb{B}(t),$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. By Van Der Vaart and Wellner (1996, Theorem 1.6.1), it is sufficient to show the result holds for a sequence of compact sets $\mathscr{T}_1 \subseteq \mathscr{T}_2 \subseteq \cdots \subseteq \mathscr{T}_k \subseteq \cdots$ such that $0 \in \mathscr{T}_1$ and $\cup_{k=1}^{\infty} \mathscr{T}_k = \mathbb{R}$. Denote $\mathscr{T}_k^+ = \mathscr{T}_k \cap \mathbb{R}^+$ and $\mathscr{T}_k^- = \mathscr{T}_k \cap \mathbb{R}^-$. Given Assumption 2, we can apply Bahadur representation again (see Lemma 1) and know that uniform in τ,

$$Q_{b,n}(\tau) - Q_b(\tau) = \frac{\tau - \frac{1}{n}\sum_i \mathbf{1}[b_i \leq Q_b(\tau)]}{g(Q_b(\tau))} + O_{a.s.}(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}.$$

We consider $t \geq 0$ first. Let $r_{1n} = O_{a.s.}(n^{-1/12}(\log n)^{1/2}(\log\log n)^{1/4}$, we have uniformly in $t \in \mathscr{T}_k^+$,

$$\begin{aligned}
&n^{2/3}\left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0) \right\} \\
&= \frac{n^{1/6}}{\sqrt{n}}\sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 - \mathbf{1}[b_i \leq Q_b(\alpha_0)]}{g(Q_b(\alpha_0))} \right) + r_{1n} \\
&\qquad = \frac{n^{1/6}}{\sqrt{n}}\sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_0) < b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right) + r_{1n} + r_{2n},
\end{aligned}$$

where

$$r_{2n} = \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right)$$

$$= n^{1/6} \left(\frac{1}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{1}{g(Q_b(\alpha_0))} \right) \frac{1}{\sqrt{n}} \sum_i \xi_i = n^{1/6} O(n^{-1/3}) O_p(1) = o_p(1),$$

where $\xi_i = \alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]$. For the leading term, it is can be shown by standard method (e.g. Kim and Pollard, 1990) that

$$\frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_0) < b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right) \overset{w}{\to} \frac{1}{g(Q_b(\alpha_0))} \mathbb{B}(t),$$

where $\mathbb{B}$ is a Brownian motion over a sequence of compact sets $\mathscr{T}_1^+ \subseteq \mathscr{T}_2^+ \subseteq \cdots \subseteq \mathscr{T}_k^+ \subseteq \cdots$.

When $t < 0$, we have uniformly in $t \in \mathscr{T}_k^-$,

$$n^{2/3} \left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0) \right\}$$

$$= \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \le Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 - \mathbf{1}[b_i \le Q_b(\alpha_0)]}{g(Q_b(\alpha_0))} \right) + \tilde{r}_{1n}$$

$$= \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} + \mathbf{1}[Q_b(\alpha_0 + tn^{-1/3}) < b_i \le Q_b(\alpha_0)]}{g(Q_b(\alpha_0))} \right) + \tilde{r}_{1n} + \tilde{r}_{2n},$$

where the two asymptotically negligible terms $\tilde{r}_{1n}$ and $\tilde{r}_{2n}$ are analogously defined as r_{1n} and r_{2n} in the proof of the case $t \ge 0$, respectively. The convergence result holds analogously over a sequence of compact sets $\mathscr{T}_1^- \subseteq \mathscr{T}_2^- \subseteq \cdots \subseteq \mathscr{T}_k^- \subseteq \cdots$.

The conclusion follows by combining the results for both $t \ge 0$ and $t < 0$.

Lemma 3. *Suppose that Assumptions 1 and 2 hold, then*

$$n^{\frac{2}{3}} \left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) \right] - n^{\frac{2}{3}} \left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) \right] \overset{w}{\to} \frac{\alpha_0}{(I-1) g(Q_b(\alpha_0))} \mathbb{B}(t)$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. Recall that for any $\tau \in (0, 1)$,

$$V_n(\tau) = \frac{1}{n}\frac{I-2}{I-1}\sum_i b_i \mathbf{1}[b_i \leq Q_{b,n}(\tau)] + \frac{1}{I-1}\tau Q_{b,n}(\tau) + O_p(1/n)$$
$$\equiv \frac{I-2}{I-1}V_{1n}(\tau) + \frac{1}{I-1}V_{2n}(\tau) + O_p(1/n).$$

Likewise,

$$V(\tau) = \frac{I-2}{I-1}\int_0^\tau Q_v(t)dt + \frac{1}{I-1}\tau Q_b(\tau) \equiv \frac{I-2}{I-1}V_1(\tau) + \frac{1}{I-1}V_2(\tau).$$

The part associates with V_{1n}, that is, $n^{\frac{2}{3}}\left[V_{1n}(\alpha_0 + tn^{-\frac{1}{3}}) - V_{1n}(\alpha_0)\right] - n^{\frac{2}{3}}\left[V_1(\alpha_0 + tn^{-\frac{1}{3}}) - V_1(\alpha_0)\right]$ converges in probability to zero by Lemma 1. For the part associated with V_{2n}, note that

$$n^{\frac{2}{3}}(I-1)\left[V_{2n}(\alpha_0 + tn^{-\frac{1}{3}}) - V_{2n}(\alpha_0)\right] - n^{\frac{2}{3}}\left[V_2(\alpha_0 + tn^{-\frac{1}{3}}) - V_2(\alpha_0)\right]$$
$$= n^{2/3}Q_{b,n}(\alpha_0 + tn^{-1/3})(\alpha_0 + tn^{-1/3}) - n^{2/3}Q_{b,n}(\alpha_0)\alpha_0 - n^{2/3}Q_b(\alpha_0$$
$$+ tn^{-1/3})(\alpha_0 + tn^{-1/3}) + n^{2/3}Q_b(\alpha_0)\alpha_0$$
$$= n^{2/3}\alpha_0\left\{Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0)\right\}$$
$$+ n^{1/3}t\left\{Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_b(\alpha_0 + tn^{-1/3})\right\}$$

The second right hand side term, for $|t| < K$, is uniformly bounded by order $n^{1/3} \times n^{-1/2} \times O_p(1) \xrightarrow{p} 0$. The first right hand side term is dealt with by Lemma 2.

2. Auxiliary Lemmas for Theorem 2

Lemmas 4 to 8 shows that the sup distance between V_n and $\widehat{V}$ is small, which we adapted from Pal and Woodroofe (2006). Lemma 9 is an intermediate step for establishing the limiting distribution of the smoothed quantile estimator.

We introduce some notation. Let k_n be a sequence of integers such that $k_n \to \infty$ and $n/k_n \to \infty$. Without loss of generality we assume k_n divides n and let $\ell_n = n/k_n$. We therefore can divide $[0, n]$ into k_n equal size intervals with each interval contains ℓ_n

consecutive integers. Let $\{s_i, i = 1, 2, \cdots, k_n\}$ be the set of upper boundary of those intervals such that $s_i = i\ell_n$.

For $(i-1)\ell_n \le s < i\ell_n$, $i = 1, 2, \cdots, k_n$, define

$$L(s) = \frac{s-(i-1)\ell_n}{\ell_n} V\left(\frac{i}{n}\right) + \frac{i\ell_n - s}{\ell_n} V\left(\frac{i-1}{n}\right),$$

and

$$L_n(s) = \frac{s-(i-1)\ell_n}{\ell_n} V_n\left(\frac{i}{n}\right) + \frac{i\ell_n - s}{\ell_n} V_n\left(\frac{i-1}{n}\right),$$

That is, L and L_n are the linear interpolation of V and V_n on k_n knots $\{s_1/n, s_2/n, \cdots, s_{k_n}/n\}$, respectively. Note that since V is convex, L is necessarily convex. However L_n may not be convex since V_n is not necessarily convex. Let A_n be the event such that L_n is convex. Since L_n is convex if and only if each segment is convex, the complement of A_n can be written as

$$\begin{aligned} A_n^c = \bigcup_{i=2}^{k_n-1} & \left\{ V_n\left(\frac{(i-1)\ell_n}{n}\right) + V_n\left(\frac{(i+1)\ell_n}{n}\right) < 2V_n\left(\frac{i\ell_n}{N}\right) \right\} \\ &= \bigcup_{i=2}^{k_n} \left\{ V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right) \right. \\ &\qquad \left. + \Delta_n\left(\frac{(i-1)\ell_n}{n}\right) + \Delta_n\left(\frac{(i+1)\ell_n}{n}\right) - 2\Delta_n\left(\frac{i\ell_n}{n}\right) < 0 \right\}, \end{aligned}$$

where $\Delta_n \equiv V_n - V$. Pal and Woodroofe (2006, Proposition 2) shows that A_n^c has probability approaching zero, thus it is sufficient to consider derive the bounds of the distance conditional on A_n (see also Kiefer and Wolfowitz, 1976, Lemma 4).

Lemma 4. *Suppose that Assumption 3 is satisfied, then there exists a positive c_1 such that* $\min_{i=2,\cdots,k_n-1} \left|V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right)\right| \ge \frac{c_1}{k_n^2}$.

Proof. By Assumption 3, there exists $c_1 > 0$ such that $Q_v'(\alpha) \geq c_1 > 0$ for all $\alpha \in [0,1]$. Then we have

$$
\begin{aligned}
&V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right) \\
&\quad = \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} Q_v(\alpha)d\alpha - \int_{\frac{(i-1)\ell_n}{n}}^{\frac{i\ell_n}{n}} Q_v(\alpha)d\alpha \geq \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} \left[Q_v(\alpha) - Q_v\left(\frac{i\ell_n}{n}\right)\right] d\alpha \\
&\quad = \frac{\ell_n}{n}\left[Q_v(\alpha_n^*) - Q_v\left(\frac{i\ell_n}{n}\right)\right] \geq c_1\frac{\ell_n^2}{n^2} = \frac{c_1}{k_n^2}.
\end{aligned}
$$

□

Lemma 5. *Let $\|\cdot\|$ denote the sup norm. Conditional on A_n, there is*

$$\|V_n - \widehat{V}\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|.$$

Proof. By Marshall's Lemma (see Kiefer and Wolfowitz, 1976, Lemma 3), for any convex function m, $\|\widehat{V} - m\| \leq \|V_n - m\|$. Therefore,

$$\|V_n - \widehat{V}\| \leq \|V_n - L_n\| + \|L_n - \widehat{V}\| \leq 2\|V_n - L_n\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|,$$

where the first and third inequalities holds by triangular inequality, the second one holds by Marshall's Lemma. □

Lemma 6. *Suppose that Assumption 3 is satisfied, then there exists $c_3 > 0$ such that for all $s \in [0, n]$,*

$$0 \leq L(s) - V(s) \leq \frac{c_3}{k_n^2}.$$

Proof. $L(s) > V(s)$ follows immediately by the convexity of V. The other inequality holds follows from a similar argument as in Lemma 4 and the fact that $Q_v'(\alpha)$ is bounded from above uniformly.

Lemma 7. *Suppose that Assumptions 1 and 3 is satisfied, then*

$$\|V_n - L_n - V + L\| = O_p\left(\sqrt{\frac{\log k_n}{nk_n}}\right) + O_p\left(\frac{\log n}{n}\right).$$

Proof. Define function V_P such that $V_P(j/n) = V(j/n)$ for each j/n and otherwise equals to its own interpolation. It is obvious that $\|V_P - V\| = O(1/n)$. It is then sufficient to focus on $V_n - L_n - V_P + L$. Note that all four functions are piece-wise linear, and so does there linear combinations. Therefore, the sup must be achieved at some knot(s). Based on this observations, we can write

$$\begin{aligned}
&\|V_n - L_n - V_P + L\| \\
&= \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_n(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_n((i-1)/n) \right|,
\end{aligned}$$

where for $t \in [0,1]$,

$$\begin{aligned}
\Delta_n(t) &= V_n(t) - V_P(t) = V_n(t) - V(t) + O(1/n) \\
&= \frac{I-2}{I-1} \underbrace{\left\{ \sum_{i=1}^{[tn]} \frac{b_{(i)}}{n} - \int_0^t Q_b(\alpha)d\alpha \right\}}_{\Delta_A(t)} + \frac{1}{I-1} \underbrace{\left\{ \frac{[tn]}{n} b_{(j)} - tQ_b(t) \right\}}_{\Delta_B(t)} + O(1/n)
\end{aligned}$$

where $[x]$ denotes the integer part of x. Note that Δ_A is an integrated quantile process. By Tse (2009, Theorem 2.1), there exists a Gaussian process $\mathbb{G}_n$ and Brownian bridge $\mathbb{B}_n^A$ defined on proper measurable space such that for any $\tau < 1/6$,

$$\|\sqrt{n}\Delta_A - \psi_n\| \overset{a.s.}{=} O(n^{-\tau}),$$

where $\psi_n(t) = \mathbb{G}_n(t) + \int_0^t \mathbb{B}_n^A(u)dQ_b(u)$. On the other hand, by Csorgo and Revesz (1978, Theorem 6), there exists a sequence of Brownian bridge B_n such that $\sup_{\delta_n \le t \le 1-\delta_n} |g(Q_b(t))\sqrt{n}\Delta_B(t) -$

$B_n(t)| \overset{a.s.}{=} O_p(n^{-1/2}\log n)$. We can then conclude

$$
\begin{aligned}
&\|V_n - L_n - V_P + L\| \\
&\le \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_A(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_A(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_A((i-1)/n) \right| \\
&+ \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \Delta_B(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\Delta_B(i/n) - \frac{i\ell_n - j}{\ell_n}\Delta_B((i-1)/n) \right| + O_p(1/n) \\
&\overset{d}{=} \frac{1}{\sqrt{n}} \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| \psi_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}\psi_n(i/n) - \frac{i\ell_n - j}{\ell_n}\psi_n((i-1)/n) \right| + O_p(n^{-\tau-1/2}) \\
&+ \frac{1}{\sqrt{n}} \max_{i=1,\cdots K_n} \max_{(i-1)\ell_n \le j \le i\ell_n} \left| B_n(j/n) - \frac{j-(i-1)\ell_n}{\ell_n}B_n(i/n) - \frac{i\ell_n - j}{\ell_n}B_n((i-1)/n) \right| + O_p(\log n/n) \\
&\le \frac{1}{\sqrt{n}} \sup_{0 \le t-s \le \frac{1}{k_n}} |\psi_n(t) - \psi_n(s)| + \frac{1}{\sqrt{n}} \sup_{0 \le t-s \le \frac{1}{k_n}} |B_n(t) - B_n(s)| + O_p(\log n/n) + O_p(n^{-\tau-1/2}) \\
&\qquad\qquad \le \frac{\sqrt{2\log\log n}}{\sqrt{n}} \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{n}} \frac{\sqrt{\log\log K_n}}{\sqrt{k_n}} + O_p(\log n/n) + O_p(n^{-\tau-1/2})
\end{aligned}
$$

where the last two inequalities result from the continuity module of Gaussian processes and the fact that $g(b) \ge \underline{b} > 0$ for all b (GPV Proposition 1). Recall that $k_n \propto \frac{n}{\log n}$, we can conclude that the right hand side is of order $O_p((n/\log n)^{-2/3})$.

Lemma 8. *Suppose Assumptions 3 and 4 are satisfied, the* $\|\widehat{V} - V_n\| = O_p((n/\log n)^{-2/3})$.

Proof. The conclusion holds by Lemmas 5 to 7. □

Lemma 9. *Let* $z_{(i)} = n(b_{(i)} - b_{(i-1)})$ *and* $w_i = ((i-1)/n - \alpha)\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u-\alpha)du$. *Suppose Assumption 3 is satisfied, then* $\sum_i z_{(i)} w_i = o_p(1/\sqrt{nh})$.

Proof. Since b_i has bounded support, it is without loss of generality to prove the case when b_i follows the uniform distribution. Pyke (1965, Section 2.1) shows that $z_{(i)}$ are identically distributed across i. Furthermore, $\mathbb{E}[z_{(i)}] = n(n+1)^{-1}$, $V(z_{(i)}) = n^3(n+1)^{-2}(n+2)^{-1}$ and $\mathrm{Cov}(z_{(i)}z_{(j)}) = -n^2(n+1)^{-2}(n+2)^{-1}$. Let ρ_{ij} be the correlation coefficient, so $\rho_{ij} = 1$ if $i = j$, and $\rho_{ij} = -1/n$ otherwise.

Note first that $\mathbb{E}[\sum_i z_{(i)} w_i] = n(n+1)^{-1}\sum_i w_i = (1/h)\left(\int_0^1 (u-\alpha)K(u-\alpha/h)du + O(1/n)\right) = O(1/nh) = o_p(1/\sqrt{nh})$ since $\int uK(u) = 0$ by assumption. Next consider

$$V(\sum_i z_{(i)} w_i) = \sum_i w_i^2 V(z_{(i)}) + 2\sum_{i\neq j} w_i w_j \mathrm{Cov}(z_{(i)}, z_{(j)}) = V(z_{(i)})\left(\sum_i w_i^2 + 2\sum_{i\neq j} w_i w_j \rho_{ij}\right).$$

Consider w_i, there exists a $u_i^* \in ((i-1)/n, i/n)$ such that

$$w_i = \left(\frac{i-1}{n} - \alpha\right)\int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u-\alpha)du = \frac{1}{nh}\left(\frac{i-1}{n} - \alpha\right)K\left(\frac{u_i^* - \alpha}{h}\right),$$

Since the kernel function has bounded support, that is, $K(u) = 0$ if $|u| > \bar{K}$. Then $w_i \neq 0$ only if $|u_i^* - \alpha| \leq \bar{K}h$. Therefore the quantity $i - 1/n$ for nonzero w_i is around h neighborhood of α, which implies that each of the nonzero $|w_i|$ is of order $\frac{1}{nh} \times h = \frac{1}{n}$. Let i_α be the nearest integer to $n\alpha$, then we know $w_i \neq 0$ only if $|i - i_\alpha| \leq Cnh$ for some constant C, which implies that in the expression of $V(\sum_i z_{(i)} w_i)$, there are of order nh nonzero summands. Since each w_i is of order $1/n$, $\rho_{ij} = -1/n$ when $i \neq j$, $V(z_{(i)}) = O(1)$, the order of $V(\sum_i z_{(i)} w_i)$ is $O(nh \times (1/n)^2 + (nh)^2 \times (1/n)^3) = O(h/n)$, which is of smaller order than $1/nh$.

The above argument shows that $\mathbb{E}[\sum_i z_{(i)} w_i] = o_p(1/\sqrt{nh})$ and $V(\sum_i z_{(i)} w_i) = o_p(1/nh)$, therefore we can conclude that $\sum_i z_{(i)} w_i = o_p(1/\sqrt{nh})$.

3. Proofs to Corollaries

3.1. **Proof of Corollary 1.** Consider a $J \times 1$ vector of mutually different quantile levels $(\alpha_1, \alpha_2, \cdots, \alpha_J)$, following the arguments in the proof of Theorem 1, the following events are equivalent:

$$\cap_{j=1,2,\cdots,J}\{n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_j) - Q_v(\alpha_j)) \leq z_j\} \Leftrightarrow \cap_{j=1,2,\cdots,J}\left\{\underset{t\in[-\alpha_j n^{\frac{1}{3}}, (1-\alpha_j)n^{\frac{1}{3}}]}{\operatorname{argmin}}\{W_{jn}(t) - z_j t\} \geq 0\right\},$$

where for $j = 1, 2, \cdots, J$

$$W_{jn}(t) = n^{\frac{2}{3}} \left[V_n(\alpha_j + tn^{-\frac{1}{3}}) - V_n(\alpha_j) \right] - n^{\frac{2}{3}} \left[V(\alpha_j + tn^{-\frac{1}{3}}) - V(\alpha_j) \right] + n^{\frac{2}{3}} \left[V(\alpha_j + tn^{-\frac{1}{3}}) - V(\alpha_j) - Q_v(\alpha_j) tn^{-\frac{1}{3}} \right].$$

Following the same arguments, we have the convergence of each single component:

$$W_{jn}(t) \xrightarrow{w} \frac{\alpha_j}{(I-1)g(Q_b(\alpha_j))} \mathbb{B}_j(t) + \frac{1}{2} Q_v'(\alpha_j) t^2.$$

where $\mathbb{B}_j$ is a two sided Brownian motion. Since $\mathbb{B}_j$ is Gaussian, it remains to find their covariance. Following the arguments in Lemmas 1 to 3 and ignoring the small order terms, we know that for each given t, the joint limiting distribution of W_{jn}, $j = 1, \cdots, J$, is determined by the joint limiting distribution of

$$n^{2/3} \left\{ Q_{b,n}(\alpha_j + tn^{-1/3}) - Q_{b,n}(\alpha_j) - Q_b(\alpha_j + tn^{-1/3}) + Q_b(\alpha_j) \right\}, \quad j = 1, \cdots, J,$$

or alternatively, the joint limiting distribution of (for $t > 0$, the case of $t < 0$ is similar)

$$\frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_j) < b_i \le Q_b(\alpha_j + tn^{-1/3})]}{g(Q_b(\alpha_j))} \right), \quad j = 1, \cdots, J,$$

To calculate the limit of covariance of above expression at different quantile levels, it is sufficient to focus on same observation index i since bids are i.i.d.. Since all the α_j are mutually different, the we have for $j \neq j'$, there is $Q_b(\alpha_j) \neq Q_b(\alpha_{j'})$ by the strict monotonicity of Q_b. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} n^{1/3} \mathbb{E} \Big[\Big(tn^{-1/3} - \mathbf{1}[Q_b(\alpha_j) < b_i \le Q_b(\alpha_j + tn^{-1/3})] \Big) \\
&\qquad \times \Big(tn^{-1/3} - \mathbf{1}[Q_b(\alpha_{j'}) < b_i \le Q_b(\alpha_{j'} + tn^{-1/3})] \Big) \Big] \\
&= \lim_{n\to\infty} \mathbb{E} \Big[-t^2 n^{-1/3} + n^{1/3} \mathbf{1}[Q_b(\alpha_j) < b_i \le Q_b(\alpha_j + tn^{-1/3})] \\
&\qquad \times \mathbf{1}[Q_b(\alpha_{j'}) < b_i \le Q_b(\alpha_{j'} + tn^{-1/3})] \Big] = 0.
\end{aligned}$$

Therefore, we can conclude that $\mathbb{B}_j$ are asymptotically uncorrelated and hence independent.

Let constants (a_j, b_j) be defined as

$$a_j = \frac{\alpha_j}{(I-1)g(Q_b(\alpha_j))}, \quad b_j = \frac{1}{2}Q'_v(\alpha_j).$$

Then we have the joint limiting distribution be given by

$$\begin{aligned}
&\mathbb{P}\left(\cap_{j=1,2,\cdots,J}\{n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_j) - Q_v(\alpha_j)) \le z_j\}\right) \\
&\quad \to \mathbb{P}\left(\cap_{j=1,2,\cdots,J}\left\{\text{argmax}_{t\in\mathbb{R}}\{\mathbb{B}_j(t) - t^2\} \le \frac{z_j}{2b_j}\left(\frac{b_j}{a_j}\right)^{2/3}\right\}\right) \\
&\qquad = \Pi_{j=1,\cdots J}\mathbb{P}\left(\text{argmax}_{t\in\mathbb{R}}\{\mathbb{B}_j(t) - t^2\} \le \frac{z_j}{2b_j}\left(\frac{b_j}{a_j}\right)^{2/3}\right).
\end{aligned}$$

3.2. **Proof to Corollary 2.** Consider inverting $\widehat{Q}(\cdot)$ first. Recall that $\widehat{F}(v_0) = \sup\{\alpha : \widehat{Q}_v(\alpha) \le v_0\}$. Consistency of $\widehat{F}(v_0)$ holds by the consistency of $\widehat{Q}_v$ and the continuity of the sup operator. It remains to work out the convergence rate and limiting distribution. Let $Z = \text{argmax}_{t\in\mathbb{R}}\{\mathbb{B}(t) - t^2\}$. Now,

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0) - F(v_0)) < x\right) = \mathbb{P}\left(\widehat{F}(v_0) < n^{-1/3}x + F(v_0)\right)$$

Note that the event $\{\widehat{F}(v_0) < n^{-1/3}x + F(v_0)\}$ is equivalent to $\{v_0 < \widehat{Q}_v(n^{-1/3}x + F(v_0))\}$. Using the fact that $F(v_0) = \alpha_0$, $Q_v(\alpha_0) = v_0$, and $(Q'_v(\alpha_0))^{-1} = f(v_0)$, we have

$$\begin{aligned}
&\mathbb{P}\left(\widehat{F}(v_0) < n^{-1/3}x + F(v_0)\right) = \mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x + F(v_0)) > v_0)\right) \\
&\quad = \mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x + \alpha_0) - Q_v(n^{-1/3}x + \alpha_0) > v_0 - Q_v(n^{-1/3}x + \alpha_0)\right) \\
&\quad = \mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x + \alpha_0) - Q_v(n^{-1/3}x + \alpha_0) > -n^{-1/3}Q'_v(\alpha_0)x + O(n^{-2/3}))\right) \\
&\quad = \mathbb{P}\left(f(v_0)n^{1/3}(\widehat{Q}_v(n^{-1/3}x + \alpha_0) - Q_v(n^{-1/3}x + \alpha_0)) < x + O(n^{-1/3}))\right)
\end{aligned}$$

Repeat the proof of Theorem 1 shows that for each x, $n^{1/3}(\widehat{Q}_v(n^{-1/3}x+\alpha_0)-Q_v(n^{-1/3}x+\alpha_0))$ has the same limiting distribution as $n^{1/3}(\widehat{Q}_v(\alpha_0)-Q_v(\alpha_0))$. Therefore, we have

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0)-F(v_0))<x\right)\to\mathbb{P}\left(f(v_0)C(\alpha_0)Z<x\right).$$

Next, by the definition of $\widehat{F}$, we have for any positive $\eta_n\downarrow 0$,

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0)-F(v_0))=x\right)\le\mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x+F(v_0))\le v_0<\widehat{Q}_v(n^{-1/3}x+F(v_0)+\eta_n)\right),$$

the right hand side coverages to zero. Observe that Z is continuous, we have

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0)-F(v_0))\le x\right)\to\mathbb{P}\left(f(v_0)C(\alpha_0)Z\le x\right).$$

Lastly, because $\hat{q}(\cdot)$ is continuous and strictly increasing, the result for $\widehat{F}^S$ follows essentially the same (but simpler) argument as above and therefore omitted.

3.3. **Proof to Corollary 3.** We give the sketch of the proof for brevity. Let γ_n be a deterministic diverging sequence whose rate will be determined later. For a given x, define

$$W_n(t|x)=\gamma_n^2\left[V_n(\alpha_0+t\gamma_n^{-1}|x)-V_n(\alpha_0|x)-Q_v(\alpha_0|x)t\gamma_n^{-1}\right].$$

Following the same argument as in Theorem 1, we have

$$\gamma_n^{-1}(\widehat{Q}_v(\alpha_0|x)-Q_v(\alpha_0|x))\le z\Leftrightarrow\underset{t\in[-\alpha_0\gamma_n,(1-\alpha_0)\gamma_n]}{\operatorname{argmin}}\{W_n(t|x)-zt\}\ge 0$$

Then we conduct the same decomposition:

$$\begin{aligned}W_n(t|x)=&\underbrace{\gamma_n^2\left[V_n(\alpha_0+t\gamma_n^{-1}|x)-V_n(\alpha_0|x)\right]-\gamma_n^2\left[V(\alpha_0+t\gamma_n^{-1}|x)-V(\alpha_0|x)\right]}_{\equiv W_n^A(t)}\\&+\underbrace{\gamma_n^2\left[V(\alpha_0+t\gamma_n^{-1}|x)-V(\alpha_0|x)-Q_v(\alpha_0|x)t\gamma_n^{-1}\right]}_{=\frac{1}{2}Q_v'(\alpha_0|x)t^2+o(1)}.\end{aligned}$$

It remains to analyze the asymptotic behavior of W_n^A. It can be observed from the definition of $V_n(\cdot|x)$ that for any $\tau \in (0,1)$,

$$V_n(\tau|x) = \frac{I-2}{I-1}\int_0^{\tau} Q_{n,b}(t|x)dt + \frac{1}{I-1}\tau Q_{n,b}(\tau|x) + O(1/n),$$

where $Q_{n,b}(\tau|x)$ is chosen to be the local polynomial estimator of Guerre and Sabbah (2012), whose Assumptions X, F and K can be verified to hold in our context. In particular, Assumption F is satisfied since the continuous differentiability of $F(\cdot|x)$ implies that $Q_b(\cdot|x)$ is twice continuously differentiable, as shown in Guerre, Perrigne, and Vuong (2000, Proposition 1-iv).

Since we only need to estimate the quantile function, we choose the order ν of the polynomial as $\nu = 0$. Using Guerre and Sabbah (2012, page 98)'s uniform Bahadur representation, we have for any $\tau \in (0,1)$,

$$Q_{n,b}(\tau|x) - Q_b(\tau|x) = \frac{\beta_n(\tau)}{(nh^d)^{1/2}} + O\left(h^2\right) + O_p\left(\frac{\log n}{nh^d}\right)^{3/4},$$

where the first right hand side (RHS) is the first order approximation, the second RHS term is the bias and its order is determined by the twice continuous differentiability of Q_b, and the third RHS term is the Bahadur representation error, and β_n is defined as

$$\beta_n(\tau) = J_n^{-1}\frac{2}{(nh^d)^{1/2}}\sum_i^n \left\{\mathbf{1}[b_i \le Q_b^*(\tau|x)] - \tau\right\} K\left(\frac{X_i - x}{h}\right),$$

where $J_n \xrightarrow{p} J$ for some constant, $K(\cdot)$ is the kernel function and Q_b^* is the argmin of the population criterion function of the local polynomial regression.

Following similar argument as in Lemmas 1 to 3, we need to make sure that both the bias term and the Bahadur representation error term converges (in probability) to zero faster than γ_n^2. Take $\gamma_n = (nh^d)^{1/3}$, then $O(\gamma_n^2 h^2) = o(1)$ since h is chosen such that $nh^{d+3} \to 0$; $\gamma_n^2\left(\frac{\log n}{nh^d}\right)^{3/4} = o_p(1)$ since $nh^d \to \infty$. As the consequence, the limiting behavior of $W_n^A(t)$ when $t \ge 0$ (the case of $t < 0$ similar) depends on the following dominant term (up to additive asymptotically negligible and some multiplicative constant terms):

$$\frac{1}{\sqrt{nh^d}}\sum_i^n \underbrace{(nh^d)^{1/6}\left\{t(nh^d)^{-1/3} - \mathbf{1}[Q_b^*(\alpha_0|x) < b_i \leq Q_b^*(\alpha_0 + t(nh^d)^{-1/3}|x)]\right\}}_{\equiv \xi_i(t)} K\left(\frac{X_i - x}{h}\right).$$

Guerre and Sabbah (2012, Lemma A.1) shows that $\xi_i(t)$ has zero mean. Furthermore, for arbitrary $t, s > 0$,

$$\begin{aligned}\lim_{n\to\infty} h^{-d}\mathbb{E}[\xi_i(t)\xi_i(s)] &= \lim_{n\to\infty} h^{-d}\mathbb{E}\{\mathbb{E}[\xi_i(t)\xi_i(s)|X_i]\} \\ &= \lim_{n\to\infty} h^{-d}\mathbb{E}\left\{K^2\left(\frac{X_i - x}{h}\right)[\min\{t,s\} + O(\gamma_n^{-1})]\right\} \\ &= f_X(x)\min\{t,s\}\int K^2(u)du.\end{aligned}$$

It follows that $\frac{1}{\sqrt{nh^d}}\sum_i \xi_i(t)$ converges in distribution to normal for each t and given $\xi_i(t)$ is sum of indicator functions, $\frac{1}{\sqrt{nh^d}}\sum_i \xi_i(\cdot)$ weakly converge to a constant multiplied by a Brownian motion process. The rest of the proof follows similarly from Theorem 1.

REFERENCES

BAHADUR, R. R. (1966): "A note on quantiles in large samples," *Annals of Mathematical Statistics*, 37(3), 577–580.

CSORGO, M., AND P. REVESZ (1978): "Strong approximations of the quantile process," *The Annals of Statistics*, pp. 882–894.

GUERRE, E., I. PERRIGNE, AND Q. VUONG (2000): "Optimal Nonparametric Estimation of First-price Auctions," *Econometrica*, 68(3), 525–574.

GUERRE, E., AND C. SABBAH (2012): "Uniform bias study and Bahadur representation for local polynomial estimators of the conditional quantile function," *Econometric Theory*, 28(01), 87–129.

KIEFER, J. (1967): "On Bahadur's representation of sample quantiles," *The Annals of Mathematical Statistics*, pp. 1323–1342.

KIEFER, J., AND J. WOLFOWITZ (1976): "Asymptotically minimax estimation of concave and convex distribution functions," *Probability Theory and Related Fields*, 34(1), 73–85.

KIM, J., AND D. POLLARD (1990): "Cube Root Asymptotics," *The Annals of Statistics*, 18(1), 191–219.

PAL, J. K., AND M. WOODROOFE (2006): "On the distance between cumulative sum diagram and its greatest convex minorant for unequally spaced design points," *Scandinavian journal of statistics*, 33(2), 279–291.

PYKE, R. (1965): "Spacings," *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 395–449.

TSE, S. (2009): "On the Cumulative Quantile Regression Process," *Mathematical Methods of Statistics*, 18(3), 270–279.

VAN DER VAART, A. W. (2000): *Asymptotic statistics*, vol. 3. Cambridge university press.

VAN DER VAART, A. W., AND J. A. WELLNER (1996): *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer.

WELSH, A. (1988): "Asymptotically efficient estimation of the sparsity function at a point," *Statistics & probability letters*, 6(6), 427–432.