Identification of Autoregressive Signals in Colored Noise Using Damped Sinusoidal Model

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***Abstract*—This brief addresses a new method for autoregressive (AR) parameter estimation from colored noise-corrupted observations using a *damped sinusoidal* model for autocorrelation function of the noise-free signal. The *damped sinusoidal* model parameters are first estimated using a least-squares based method from the given noisy observations. The AR parameters are then directly obtained from the *damped sinusoidal* model parameters. The performance of the proposed scheme is evaluated using numerical examples.**

***Index Terms*—Autoregressive (AR) process, colored noise, damped sinusoidal modeling, parameter estimation.**

I. Introduction

Parameter estimation of the stochastic signal model is an important issue in various fields of science and engineering, e.g., econometrics, geophysics, speech processing, image processing, biomedical signal processing, and communication [1], [2]. The most popular stochastic signal model is the Gaussian, minimum phase, AR model. In time-series analysis and signal modeling, both noise-free and noisy autoregressive (AR) systems have been extensively studied by many researchers [3]–[6]. In the latter case, except very few exceptions for colored noise, research results reported so far mostly considered white additive noise.

Zhang and Takeda [7] have proposed a method for parameter estimation of AR moving average (ARMA) systems corrupted by colored noise. In that work, a generalized least-squares (GLS) method has been suggested for estimating the AR parameters using short and noisy data. Although it is claimed in [7] that the estimates converge to the true values within a few iterations, it is shown in [8] that they actually remain unchanged after the first iteration. Furthermore, the GLS method has limitations for certain AR systems and cannot be used to estimate the AR parameters, especially when the poles of the AR system lie relatively near the unit circle and the noise is relatively strong, i.e., signal-to-noise ratio (SNR) is low. To alleviate this problem, a maximum likelihood method for identifying AR systems has been reported in [8]. The algorithm however utilizes a bootstrap technique where the initial values are obtained from the GLS method. The method is highly dependent on initial values and may fail to converge at a low SNR.

Recently, Zheng [9] has extended the improved LS (ILS) type of method to the parameter estimation of AR processes corrupted by colored noise (CN), which is called the ILS-CN method for short. Poor estimate of the initial values by the LS method impedes convergence of the iterative scheme particularly at low SNRs.

In this brief, we introduce a new AR parameter-estimation scheme via *damped sinusoidal* modeling of the autocorrelation function of the noise-free AR signal. The model parameters, leading to AR parameters, are estimated from colored-noise corrupted observations by using a LS type algorithm.

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II. Problem Formulation

The input–output relationship of a pth-order AR process can be expressed as

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + u(n) \tag{1}$$

where the unknown input $u(n)$ is a sequence of zero-mean white Gaussian noise with unknown variance σ_u^2 and $x(n)$ denotes the output signal. Here, a_k $(k = 1, 2, \cdots, p)$ are the unknown AR parameters. The order p of the AR system is assumed to be known.

In many practical situations, observation noise corrupts the data samples. In this brief, we assume that the output signal $x(n)$ contains additive colored noise. Then, the observed process $y(n)$ can be expressed as

$$y(n) = x(n) + w(n). \tag{2}$$

The additive colored noise $w(n)$ originates from an MA process given by

$$w(n) = B(z)v(n) \tag{3}$$

where $v(n)$ is a zero-mean white Gaussian noise with unknown variance σ_v^2 and $B(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{q_b} z^{-q_b}$. We consider that the colored noise $w(n)$ is finitely autocorrelated, i.e.,

$$R_{ww}(\lambda) \equiv E[w(n)w(n-\lambda)] = 0, \quad \text{for } |\lambda| \geq L \tag{4}$$

where $E[\cdot]$ represents the expectation operator and L is a given positive integer. Moreover, $v(n)$ is statistically independent of $u(n)$, i.e., $E[v(n)u(n-t)] = 0$ for all t.

The objective of this brief is to propose a novel method using a *damped sinusoidal* model for autocorrelation function of the noise-free signal to estimate the AR parameters. The *damped sinusoidal* model parameters are estimated using $R_{yy}(m)$, calculated from a finite set of noisy observations. The desired AR parameters $\{a_k\}$ are then directly obtained from this model parameter.

III. Proposed Identification Method

A. Motivation

It is known that the AR system parameters, i.e. $\{a_k\}$, satisfy the recursive equation of the clean-signal autocorrelation sequence $R_{xx}(m)$ given by [2]

$$R_{xx}(m) = -\sum_{k=1}^{p} a_k R_{xx}(m-k), \quad m \geq 0. \tag{5}$$

In general, $R_{xx}(m)$ is estimated as

$$\widehat{R}_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-1-|m|} x(n)x(n-m) \tag{6}$$

where N is the number of data points. The AR parameters can be estimated solving any p equations given by (5). Usually, $R_{xx}(m)$ for $m = 0, 1, 2, \cdots, p$ is used to form symmetric Toeplitz equations. However, when observation noise satisfying the model in (3) corrupts the data samples $x(n)$, $\{a_k\}$ may be estimated directly using $R_{yy}(m)$ for $m \geq L$, where $R_{yy}(m)$ is the autocorrelation function of the noisy signal $y(n)$, and is usually estimated as

$$\widehat{R}_{yy}(m) = \frac{1}{N} \sum_{n=0}^{N-1-|m|} y(n)y(n-m). \tag{7}$$

Because, it can be shown using (4) that $R_{yy}(m) = R_{xx}(m)$ for $m \geq L$. It is worth mentioning that at a low SNR, the autocorrelation function of the noisy signal $(R_{yy}(m))$ includes significant error at all lags for $m \geq L$ resulting mainly from the nonideal nature of the autocorrelation sequence of the additive noise. As such, the correlation-based methods, e.g. the high-order Yule–Walker (HOYW) method, for computing $\{a_k\}$ using $\widehat{R}_{yy}(m), m \geq L$ fail to estimate the AR parameters with acceptable level of accuracy. In this brief, instead of using the recursive equation for $R_{xx}(m)$ as in (5), we present a new nonrecursive model for calculating $R_{xx}(m)$ based on the roots of the AR systems.

B. AR Parameter Estimation Using Damped Sinusoidal Model

The transfer function of a pth-order AR system in the z domain can be expressed as

$$H(z) = \frac{1}{A(z)} = \sum_{k=1}^{p} \frac{C_k}{1 - z_k z^{-1}} \tag{8}$$

where $A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p}$, z_k denotes the kth pole of the AR system and C_k is the partial fraction coefficient corresponding to the kth pole. The unit impulse response $h(n)$ of the causal AR system described in (8) can be expressed as

$$h(n) = \sum_{k=1}^{p} C_k (z_k)^n. \tag{9}$$

If this relaxed AR system is excited by a sequence of white noise $u(n)$ with distribution $\mathcal{N}(0, \sigma_u^2)$, the response $x^M(n)$ is given by

$$x^M(n) = u(n) * h(n) = \sum_{l=0}^{n} u(l) h(n-l). \tag{10}$$

Using (9), (10) can be written as

$$x^M(n) = \sum_{k=1}^{p} \sum_{l=0}^{n} C_k u(l) (z_k)^{n-l}. \tag{11}$$

Clearly, $x(n)$ and $x^M(n)$ are the same because (1) is the difference equation implementation of input–output using the system parameters and (11) is the convolution sum implementation of the same using the system roots. Using (11), the autocorrelation of the noise-free signal $x^M(n)$ can be obtained as

$$R_{xx}^M(m) = R_{xx}(m) = \sum_{k=1}^{p} \beta_k (z_k)^m \tag{12}$$

where

$$\beta_k = \sigma_u^2 \left[\frac{C_k^2}{1 - z_k^2} + \sum_{q=1, q \neq k}^{p} \frac{C_k C_q}{1 - z_k z_q} \right]. \tag{13}$$

The coefficient β_k may be real or complex depending on whether the pole is real or complex. Since $x(n)$ is real, in the latter case, a complex pole will always be accompanied by its complex conjugate pole. Considering the effect of complex and real poles, (12) can be simplified as

$$R_{xx}(m) = \sum_{j=1}^{g} (r_j)^m \left[P_j \cos(\omega_j m) + Q_j \sin(\omega_j m) \right], \quad \text{for } m \geq 0 \tag{14}$$

where g = {the number of complex conjugate pair of poles +the number of real poles}, r_j and ω_j denote, respectively, the magnitude and angle of the jth pole and, P_j and Q_j are constants. In general, r_j governs the decay rate of the AR system response and ω_j determines the angular position of the pole of the AR system in the z plane.

We estimate each of the damped sinusoidal functions of the alternative representation of $R_{xx}(m)$ described in (14) in an iterative fashion. At first, from the given set of noisy data points $y(n)$, the autocorrelation function of the noisy signal $R_{yy}(m)$ is calculated using (7). It is sufficient to consider only a few nonzero positive lags of $R_{yy}(m)$, where $m = L, L+1, \cdots, L+M-1$. The component function $\{(r_j)^m [P_j \cos(\omega_j m) + Q_j \sin(\omega_j m)]\}$ in (14) is then estimated by best fitting a finite sequence of this function with $R_{yy}(m)$ for $L \leq m \leq L + M - 1$. The fitted parameters at the first step will give an estimate of r_j and ω_j, $j = 1$. The corresponding fitted function is then subtracted from $R_{yy}(m)$ to obtain the first residue function $\Re_1(m)$. In the second step, another function of the proposed model is fitted to this residue function to get the second set of r_j and ω_j, $j = 2$. Then, a second residue function $\Re_2(m)$ is calculated by subtracting the second fitted function from the first residue function. The kth residue function is thus defined as

$$\Re_k(m) = \begin{cases} R_{yy}(m), & \text{for } k = 0 \\ \Re_{k-1}(m) - (r_k)^m & \\ \quad \times [P_k \cos(\omega_k m) & \\ \qquad + Q_k \sin(\omega_k m)], & \text{for } k = 1, 2, \cdots, g-1 \end{cases}. \tag{15}$$

For $0 < \omega_k < \pi$, we obtain $r_k \exp[(\pm j \omega_k)]$ as one pair of complex conjugate poles of the AR system. However, $\omega_k = 0$ or π represent a real pole given by r_k or $-r_k$, respectively. Proceeding this way when all the p poles are identified, no further steps are required. As for example, in case of a fourth-order system with two real poles and a pair of complex-conjugate poles, we need three steps. Once the poles are estimated, the AR system parameters can be obtained from their unique relationship [6].

In the proposed method, the parameters ω_k, r_k, P_k, and Q_k of the kth component function are chosen such that the sum-squared error, between the $(k-1)$th residue function and the kth component function, defined by

$$\begin{aligned} J_k^{(i)} = \sum_m \Big| \Re_{k-1}(m) - \left(r_k^{(i)}\right)^m & \\ \times \left[P_k^{(i)} \cos\left(\omega_k^{(i)} m\right) + Q_k^{(i)} \sin\left(\omega_k^{(i)} m\right) \right] \Big|^2, & \\ k = 1, 2, \cdots, g-1; \quad m = L, L+1, \cdots, L+M-1 & \end{aligned} \tag{16}$$

is minimized. Since the proposed method is iterative, the superscript "(i)" denotes the iteration index, i.e., $\omega_k^{(i)}$ denotes the angle of the kth pole at iteration i. The optimum parameters are found as $P_k = P_k^{(i)}$, $Q_k = Q_k^{(i)}$, $r_k = r_k^{(i)}$, and $\omega_k = \omega_k^{(i)}$ for the value of i at which $J_k^{(i)}$ is minimum. For arbitrary values of $r_k^{(i)}$ and $\omega_k^{(i)}$, $P_k^{(i)}$ and $Q_k^{(i)}$ can be obtained by minimizing $J_k^{(i)}$ in the LS sense as

$$\mathbf{DU} = \mathbf{V} \tag{17}$$

where the elements of (2 × 2) matrix $\mathbf{D}$ are defined by

$$\begin{aligned} D_{11} &= \sum_m \left(r_k^{(i)}\right)^{2m} \cos^2(\omega_k^{(i)} m) \\ D_{22} &= \sum_m \left(r_k^{(i)}\right)^{2m} \sin^2(\omega_k^{(i)} m) \\ D_{12} &= D_{21} = \sum_m \left(r_k^{(i)}\right)^{2m} \cos(\omega_k^{(i)} m) \sin(\omega_k^{(i)} m) \\ \mathbf{U}^T &= [P_k^{(i)} \; Q_k^{(i)}] \end{aligned}$$

and

$$\mathbf{V}^T = [V_1 \; V_2]$$

with

$$\begin{aligned} V_1 &= \sum_m \Re_{k-1}(m) \left(r_k^{(i)}\right)^m \cos(\omega_k^{(i)} m) \\ V_2 &= \sum_m \Re_{k-1}(m) \left(r_k^{(i)}\right)^m \sin(\omega_k^{(i)} m). \end{aligned}$$

C. Efficient Implementation of Damped Sinusoidal Method

To estimate the kth component function of the damped sinusoidal model, if ω_k and r_k are searched in their entire domain, e.g., [0, π] and [0,1], respectively with an acceptable resolution, the computational cost will be extremely high. As such, we look for an alternative approach. It is known that the noisy process $y(n)$ can be more accurately characterized by a higher order AR process containing both noise and system poles [10]. Then, these mixture of noise plus system poles can be used as candidate solutions for (16). Therefore, to derive a computationally efficient method, at first we estimate poles of a higher order AR model fitted to the observed noisy process by using the stable noise-uncompensated lattice filter (NULF) [11]. The order of the overfitted AR model may be determined using a standard technique [12]. Now, instead of scanning the entire domains of ω_k and r_k, the angular positions ω_l, $l = 1, 2, \cdots, \tilde{p}$, and magnitude r_l, $l = 1, 2, \cdots, \tilde{p}$, of these poles are used to minimize (16). Here, $\tilde{p}$ denotes the order of the higher order AR model. Note that unlike conventional noise-compensation techniques [9], [13], the proposed method is inherently stable.

IV. Numerical Results

In this section, we illustrate the performance of the proposed method using three numerical examples. First, the noisy sequence $y(n) = x(n) + w(n)$ is generated using the AR(3) process and noise model expressed by

$$x(n) = 2.299x(n-1) - 2.1262x(n-2) + 0.7604x(n-3) + u(n) \quad (18)$$

$$w(n) = v(n) - v(n-1) + 0.2v(n-2). \quad (19)$$

The roots of the third-order AR process are located at $0.7245 \pm j0.6080$ and 0.8501. The roots of the noise model are located at 0.7236 and 0.2764. The variance of the input signal is fixed at $\sigma_u^2 = 1$ and the variance σ_v^2 of the noise process $v(n)$ is selected to give different SNR's defined as

$$\mathrm{SNR} = 10 \log_{10} \frac{\sum_{n=1}^{N} x^2(n)}{\sum_{n=1}^{N} w^2(n)} \mathrm{dB} \quad (20)$$

In all the simulations $\tilde{p} = 30$ and $N = 4000$ data samples from noisy observations were used. For determining the *damped sinusoidal* model parameters we have used $R_{yy}(m)$ for $m = L, L+1, L+2, \cdots, L+M-1$. In simulations $M = 10p$ was used, where p is the AR system order and L is chosen to be equal to p as also assumed in [9].

The estimated AR parameters using the proposed method and the ILS-CN method reported in [9] are presented in Table I for different SNR's. The entries denote arithmetic means and standard deviations of the estimated a_1, a_2, and a_3 based on 10 independent runs. As can be seen, the accuracy of estimation of both the methods are comparable at $\mathrm{SNR} = 10$ dB and $\mathrm{SNR} = 5$ dB. But at $\mathrm{SNR} = 0$ dB, the ILS-CN method completely fails to estimate the AR parameters while no significant deterioration in performance of the proposed method is observed. The deteriorating performance of the ILS-CN method with decreasing SNRs is due to increasingly poor estimation of the noise autocovariance function in the bias correction term using the recursive technique described in [9]. However, in our model such an estimation of the noise autocovariance function is not required. It can also be seen from Table I that the standard deviations of estimation using the proposed method are noticeably lower than the ILS-CN method demonstrating better consistency of the proposed scheme.

Second, consider the AR(4) process given by

$$x(n) = 2.595x(n-1) - 3.339x(n-2) + 2.2x(n-3) - 0.731x(n-4) + u(n) \quad (21)$$

TABLE I

Results for AR(3) Process Using the ILS-CN and Proposed Methods [$(\cdot)$ Denotes the Standard Deviation]

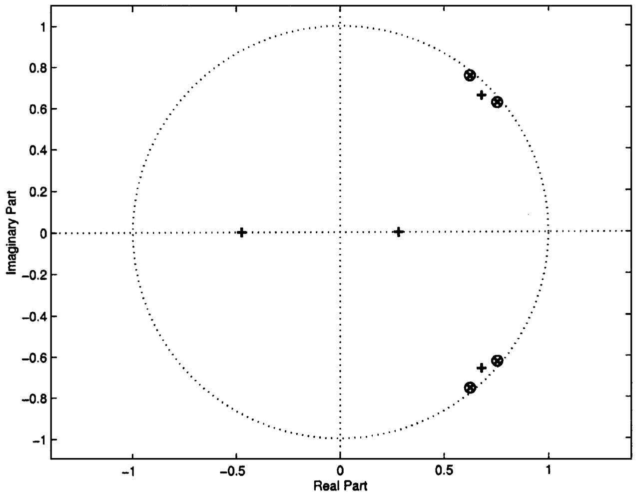
True $\{a_k\}$	Estimated AR parameters SNR (dB)	ILS-CN	Proposed
$a_1 = -2.2990$	10	−2.2963 (±0.1080)	−2.3304 (±0.0300)
	5	−2.2322 (±0.3983)	−2.3471 (±0.0067)
	0	2.1740 (±3.0840)	−2.3286 (±0.0298)
$a_2 = 2.1262$	10	2.1246 (±0.1691)	2.1768 (±0.0418)
	5	2.1804 (±0.6221)	2.1999 (±0.0168)
	0	15.9248 (±22.0143)	2.1716 (±0.0474)
$a_3 = -0.7604$	10	−0.7613 (±0.0848)	−0.7881 (±0.0213)
	5	−0.7985 (±0.3171)	−0.8012 (±0.0081)
	0	106.8487 (±174.9853)	−0.7830 (±0.0266)

TABLE II

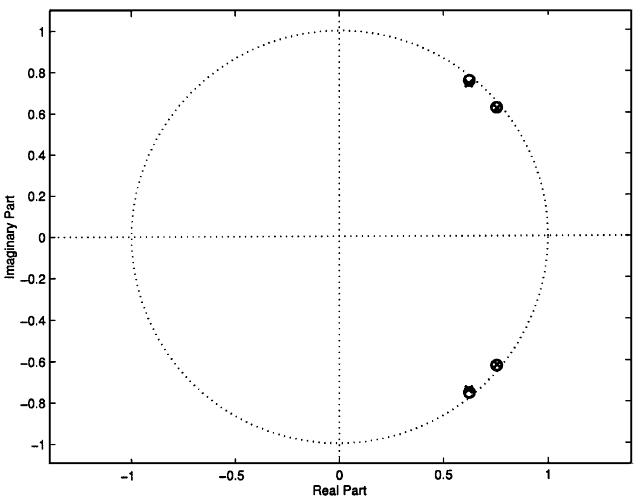
Results for AR(4) Process Using the ILS-CN and Proposed Methods [$(\cdot)$ Denotes the Standard Deviation]

True $\{a_k\}$	Estimated AR parameters SNR (dB)	ILS-CN	Proposed
$a_1 = -2.595$	20	−2.5978 (±0.1479)	−2.6104 (±0.0363)
	10	−1.4897 (±1.4057)	−2.6075 (±0.0299)
	0	−0.4344 (±0.1934)	−2.6354 (±0.0394)
$a_2 = 3.339$	20	3.3430 (±0.3732)	3.3769 (±0.0565)
	10	1.4641 (±3.7896)	3.3711 (±0.0572)
	0	−0.0641 (±0.2999)	3.3975 (±0.0812)
$a_3 = -2.2$	20	−2.2038 (±0.3677)	−2.2291 (±0.0394)
	10	−0.8130 (±3.9980)	−2.2302 (±0.0482)
	0	0.1203 (±0.2314)	−2.2298 (±0.0703)
$a_4 = 0.731$	20	0.7297 (±0.1524)	0.7373 (±0.0155)
	10	0.4358 (±1.8517)	0.7411 (±0.0183)
	0	0.3813 (±0.1083)	0.7278 (±0.0237)

The roots of the AR process are located at $0.7681 \pm j0.5587$ and $0.5294 \pm j0.7281$. The noise model is assumed to be the same as in (19). Table II displays the arithmetic means and standard deviations of the estimated AR parameters a_1, a_2, a_3, and a_4 based on 10 independent runs. It is evident that the ILS-CN method can estimate the AR parameters with good accuracy only at $\mathrm{SNR} = 20$ dB while it fails to identify the parameters at 10 and 0 dB SNR's. On the contrary, the estimates of the AR parameters obtained from the proposed method at all these SNRs are close enough to their respective true values with better standard deviations. Also, it was observed that at a relatively low SNR, the ILS-CN method faces nonconvergence problem and there was an average of three flop tests out of ten simulation runs. Nonconvergence within 2500 iterations had been considered a "flop test" and was ignored.



(a)



(b)

Fig. 1. Estimated poles of the AR(4) system at different SNR's (o: true, +: ILS-CN, x: proposed).

Next, we present in Fig. 1(a), the estimated poles by the two methods at $\mathrm{SNR} = 20$ dB of an AR(4) system given by

$$x(n) = 2.7606x(n-1) - 3.8106x(n-2) + 2.6535x(n-3) - 0.9238x(n-4) + u(n). \quad (22)$$

Here also, the noise model is assumed to be the same as in (19). As shown, for this system having poles very close to the unit circle the estimates obtained by the proposed method match the true ones with high accuracy in contrast to the complete failure of the ILS-CN method. The results in Fig. 1(b) show that the accuracy of estimation of the proposed method is consistently good even at $\mathrm{SNR} = 0$ dB for such a system. Note that in both the plots, the results indicate the mean of ten independent runs.

To compare the computational efficiency of the two methods, we present in Table III the number of MATLAB FLOPS (floating point operations) associated with each one. The comparison is made for the results shown in Table I, Table II, and Fig. 1. The performance results in terms of number of FLOPS shown in the table indicate that the proposed method is computationally more expensive than the ILS-CN method.

TABLE III
COMPARISON OF COMPUTATIONAL COMPLEXITY OF THE TWO ESTIMATORS

Simulated Example	Computational Complexity FLOPS ($\times 10^6$)	
	ILS-CN	Proposed
AR(3) system (TABLE I, SNR=10 dB)	0.33	1.78
AR(4) system (TABLE II, SNR=20 dB)	1.05	1.79
AR(4) system (Fig. 1, SNR=20 dB)	0.57	1.78

V. Conclusion

In this brief, a new method has been presented for estimating the parameters of AR signals corrupted by colored noise. In the proposed method, the AR parameters are computed from the *damped sinusoidal* model parameters introduced here for the autocorrelation sequence of the noise-free AR signal. A LS type algorithm is used for estimating the *damped sinusoidal* model parameters iteratively from the noisy data. Compared with the extended improved LS technique reported in [9], the proposed one consistently gives more accurate results particularly at low SNRs with the cost paid in computational complexity. Moreover, stability using the proposed method is always guaranteed, a feature seldom seen in noise-compensation techniques.

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