Assignment 2 for Advanced Macroeconomics III (Fall 2009)

SUFE

Consider a variation of the Aghion-Howitt (1992, Econometrica) model. Time is continuous and is denoted as m. There are three sectors: a final (consumption) goods sector, an intermediate good sector, and a research sector. The final goods sector and the research sector are perfectly competitive, but the intermediate good sector is monopolized by the firm who has the most advanced innovation. As in the Aghion-Howitt paper, the production function is $AF(x)$ (assume that $\frac{d^2xF'(x)}{dx^2} < 0$) for the final goods, $x = l$ for the intermediate good, and a Poission process for the research success.

The main modification of the model is that public knowledge is useful in individual firms' research. To describe public knowledge, let t be the index of the innovation and m_t be the time when the t^{th} innovation is made. Let $A_t = \gamma^t$ and $\gamma > 1$. Consider the time interval $[m_t, m_{t+1}]$. At time $m_t + s$, let $K_t(s)$ be the public knowledge and $N_t(s)$ be the average input of skilled labour by a research firm. Assume:

$$K_t(s) = \begin{cases} A_t \int_0^s h(N_t(\tau))d\tau & \text{if } s \in [0,\ m_{t+1} - m_t) \\ \gamma A_t & \text{if } \ s = m_{t+1} - m_t \end{cases}$$

Here, the function h satisfies $h(0) = 0$, $h' > 0$ and $h'' \le 0$. Denote $k_t(s) = K_t(s)/A_t$ and refer to it as the relative public knowledge. Notice that the long-run growth rate of public knowledge is the same as the growth rate of A.

For an individual research firm, let $n_t(s)$ be the input of skilled hours. Then, the sucess of the research (which results in a new innovation) arrives according to a Poisson process with the following rate:

$$\lambda g(k_t(s))\phi(n_t(s)).$$

Here $\lambda > 0$ is a constant. The function g satisfies $g(0) = 0$, $g' > 0$ and $g'' < 0$. The function ϕ has similar properties. Individual research firms take (k, N) as given.

To simplify the analysis, assume that the representative household has the following utility function:

$$\int_0^\infty \{c(\tau) - A(\tau)w[l(\tau) + n(\tau)]\}e^{-r\tau}d\tau.$$

Here $r > 0$ is the rate of time preference, $w > 0$ a constant, c consumption of final goods, l the hours of skilled labour in producing intermediate goods, and n the hours of skilled labour in research. $A(m) = A(m_t) = A_t$ if the t^{th} innovation is the most advanced one up to m. Notice that the total supply of skilled labour is elastic here, rather than fixed as in the Aghion-Howitt model.

(1) Formulate the decision problem at $m_t + s$ of the producer of the intermediate good using the t^{th} innovation. Denote the optimal input of skilled hours as $x_t(s)$ and the maximized profit as $A_t\pi_t(s)$. Show that $x_t(s) = x$ and $\pi_t(s) = \pi$ for all $s \in [0,\ m_{t+1} - m_t)$, where x and π are both constant.

(2) Let the present value to the innovator of the t^{th} innovation at time $m_t + s$ be $A_t v_t(s)$. Impose the transversality condition and establish the following condition:

$$v_t(0) = \pi \int_0^\infty e^{-\alpha(\tau)} d\tau$$

where $\alpha(s)$ is defined as

$$\alpha(s) \equiv \int_0^s [r + \lambda g(k_t(\tau))\phi(n_t(\tau))] d\tau.$$

(3) Let the input of specialized labour in research be fixed at 1 and let $A_t w_t^R(s)$ be the wage rate of such labour at time $m_t + s$. This wage rate ensures that the net profit of research be zero. Formulate the decision problem of a research firm. Show that, for all $s \in [0,\ m_{t+1} - m_t)$, the optimal choice of $n_t(s)$ satisfies the following condition:

$$\phi'(n_t(s)) = \frac{w}{\gamma\lambda v_{t+1}(0) g(k_t(s))}.$$

Also prove that $\dot{n}_t(s) > 0$, where the dot indicates the derivative with respect to s.

(4) Now consider the social planner who maximizes the households' intertemporal utility. Notice that the welfare function takes into account both consumption and the disutility of skilled hours. However, since the inputs of specialized labour in research and unskilled labour in the production of the final goods are both fixed, they do not enter the welfare function. Let the welfare function at $m_t + s$ be $A_t u_t(s)$. Define

$$Q(s) \equiv F(x_t(s)) - w x_t(s) - w n_t(s) + \gamma\lambda g(k_t(s))\phi(n_t(s)) u_{t+1}(0).$$

Impose the transversality condition and establish the following condition:

$$u_t(s) = \int_s^\infty Q(\tau) e^{-[\alpha(\tau) - \alpha(s)]} d\tau,\ \ s \in [0,\ m_{t+1} - m_t).$$

(5) At any time $m_t + s$, treat $k_t(s)$ and $\alpha(s)$ as the predetermined variables. Use the Hamiltonian to formulate the social planner's problem for the choices of $x_t(s)$, $n_t(s)$ and $k_t(s)$. Show that the socially optimal amount of input of skilled hours in the production of the intermediate good is $x_t(s) = x^*$, where x^* is a constant. Moreover, show that $x^* > x$, where x is defined in part (1).

(6) Derive the optimality conditions for $n_t(s)$, $k_t(s)$ and $\alpha(s)$. Denote the socially optimal choice of $n_t(s)$ as $n_t^*(s)$. Assume that $h(n) = h_0 n$, where $h_0 > 0$, and that $\phi > n\phi'$. Show that $\dot{n}_t^*(s)$ is not necessarily positive.

(7) Explain intuititvely why the equilibrium solution $n_t(s)$ is increasing in s but the socially optimal solution $n_t^*(s)$ is not necessarily so.