Exercise on Overlapping Generation Model
2011 Spring
SHUFE

Question 1: Consider Diamond's (1965) model of overlapping generations. Let the production function be $F(K,L) = K^{\alpha}L^{1-\alpha}$, $\alpha \in (0,1)$, and let the utility function $U(c^1, c^2)$ satisfy:

$$\frac{U_1(c^1,c^2)}{U_2(c^1,c^2)} = H(\frac{c^2}{c^1}),$$

where $H' > 0$, $H(0) = 0$ and $H(\infty) = \infty$. Denote the inverse function of H as h. That is, for any $x \in (0,\infty)$, $h(H(x)) = x$. Assume that $xh'(x)/h(x) < 1$ for all $x \in (0,\infty)$ and that $\lim_{x\to\infty} x/h(x) < \infty$.

In each period t, the government imposes a proportional tax on labour income (of the young) at a constant tax rate τ (<1). This tax revenue is given to the old agents in the same period as a subsidy which is proportional to the old agent's capital income. That is, each old agent in period t receives a subsidy $\sigma_t r_t k_t^s$, where k_t^s is the agent's saving (in period t-1). The rate σ_t is determined by:

$$\sigma_t r_t \bar{k}_t^s L_{t-1} = \tau w_t L_t,$$

where $\bar{k}_t^s$ is the amount of savings by each agent born at $t-1$. The rate τ is not necessarily positive. When $\tau < 0$, it is a subsidy to labour income, financed by a tax on capital income. Individual agents take the rates (τ, σ) as given, although in equilibrium σ depends on (K, r). Denote:

$$\hat{w}_t = (1-\tau)w_t,\ \hat{r}_{t+1} = (1+\sigma_{t+1})r_{t+1},\ \hat{R}_t = 1 + \hat{r}_t.$$

1. Formulate the maximization problem of an individual young agent born in period t. Derive the savings function $s_t = s(\hat{w}_t, \hat{R}_{t+1})$ (use the function h).

2. Show that the following equation holds in equilibrium:

$$\sigma_t = \frac{1-\alpha}{\alpha}\tau,\ \textit{for all } t.$$

3. Use the result from 2 to rewrite the saving function as $s_t = g(w_t, r_{t+1}, \tau)$. Show that $0 < \frac{\partial s_t}{\partial w_t} < 1-\tau$, $\frac{\partial s_t}{\partial r_{T+1}} < 0$ *and* $\frac{\partial s_t}{\partial \tau} < 0$.

4. Derive the dynamic equation for r in equilibrium as $r_t = m(r_{t+1}, \tau)$. Show that a steady state of r exists. Under what condition is this steady state dynamically stable?

5. Assume dynamic stability. Suppose that the government chooses a level of τ, denoted τ^o, to achieve the golden rule $f'(k) = n$, in the steady state. Find the equation that characterizes τ^o. Find the conditions under which $\tau^o > 0$. (There should not be any endogenous variable in the equation for τ^o or the condition for $\tau^o > 0$)

6. Explain the economics behind the condition for $\tau^o > 0$ in 5.