Example 3.5 ***Solving a parametric dynamic programming problem.*** *In this example we will illustrate how to solve dynamic programming problem by finding a corresponding value function. Consider the following functional equation:*

$$\begin{aligned} V(k) = & \max_{c,\, k'} \{\log c + \beta V(k')\} \\ & s.t.\ c = Ak^{\alpha} - k'. \end{aligned}$$

The budget constraint is written as an equality constraint because we know that preferences represented by the logarithmic utility function exhibit strict monotonicity - goods are always valuable, so they will not be thrown away by an optimizing decision maker. The production technology is represented by a Cobb-Douglass function, and there is full depreciation of the capital stock in every period:

$$\underbrace{F(k,1)}_{Ak^{\alpha}1^{1-\alpha}} + \underbrace{(1-\delta)}_{0}k.$$

A more compact expression can be derived by substitutions into the Bellman equation:

$$V(k) = \max_{k' \geq 0} \{\log[Ak^{\alpha} - k'] + \beta V(k')\}.$$

We will solve the problem by iterating on the value function. The procedure will be similar to that of solving a T-problem backwards. We begin with an initial "guess" $V_0(k) = 0$, that is, a function that is zero-valued everywhere.

$$\begin{aligned} V_1(k) &= \max_{k' \geq 0} \{\log[Ak^{\alpha} - k'] + \beta V_0(k')\} \\ &= \max_{k' \geq 0} \{\log[Ak^{\alpha} - k'] + \beta \cdot 0\} \\ & \max_{k' \geq 0} \{\log[Ak^{\alpha} - k']\}. \end{aligned}$$

This is maximized by taking $k' = 0$. Then

$$V_1(k) = \log A + \alpha \log k.$$

Going to the next step in the iteration,

$$\begin{aligned} V_2(k) &= \max_{k' \geq 0} \{\log[Ak^{\alpha} - k'] + \beta V_1(k')\} \\ &= \max_{k' \geq 0} \{\log[Ak^{\alpha} - k'] + \beta[\log A + \alpha \log k']\}. \end{aligned}$$

The first-order condition now reads

$$\frac{1}{Ak^{\alpha} - k'} = \frac{\beta\alpha}{k'} \Rightarrow k' = \frac{\alpha\beta Ak^{\alpha}}{1+\alpha\beta}.$$

We can interpret the resulting expression for k' as the rule that determines how much it would be optimal to save if we were at period $T-1$ in the finite horizon model. Substitution implies

$$\begin{aligned} V_2(k) &= \log\left[Ak^{\alpha} - \frac{\alpha\beta Ak^{\alpha}}{1+\alpha\beta}\right] + \beta\left[\log A + \alpha \log \frac{\alpha\beta Ak^{\alpha}}{1+\alpha\beta}\right] \\ &= \left(\alpha + \alpha^2\beta\right)\log k + \log\left(A - \frac{\alpha\beta A}{1+\alpha\beta}\right) + \beta \log A + \alpha\beta \log \frac{\alpha\beta A}{1+\alpha\beta}. \end{aligned}$$

We could now use $V_2(k)$ again in the algorithm to obtain a $V_3(k)$, and so on. We know by the characterizations above that this procedure would make the sequence of value functions converge to some $V^*(k)$. However, there is a more direct approach, using a pattern that appeared already in our iteration.

Let

$$a \equiv \log\left(A - \frac{\alpha\beta A}{1+\alpha\beta}\right) + \beta \log A + \alpha\beta \log \frac{\alpha\beta A}{1+\alpha\beta}$$

and

$$b \equiv \left(\alpha + \alpha^2\beta\right).$$

Then $V_2(k) = a + b\log k$. Recall that $V_1(k) = \log A + \alpha \log k$, i.e., in the second step what we did was plug in a function $V_1(k) = a_1 + b_1 \log k$, and out came a function $V_2(k) = a_2 + b_2 \log k$. This clearly suggests that if we continue using our iterative procedure, the outcomes $V_3(k)$, $V_4(k)$, ..., $V_n(k)$, will be of the form $V_n(k) = a_n + b_n \log k$ for all n. Therefore, we may already guess that the function to which this sequence is converging has to be of the form:

$$V(k) = a + b\log k.$$

So let us guess that the value function solving the Bellman has this form, and determine the corresponding parameters a, b :

$$V(k) = a + b\log k = \max_{k'\geq 0}\left\{\log\left(Ak^{\alpha} - k'\right) + \beta\left(a + b\log k'\right)\right\} \;\; \forall k.$$

Our task is to find the values of a and b such that this equality holds for all possible values of k. If we obtain these values, the functional equation will be solved.

The first-order condition reads:

$$\frac{1}{Ak^{\alpha} - k'} = \frac{\beta b}{k'} \;\Rightarrow k' = \frac{\beta b}{1+\beta b} Ak^{\alpha}.$$

We can interpret $\frac{\beta b}{1+\beta b}$ as a savings rate. Therefore, in this setup the optimal policy will be to save a constant fraction out of each period's income.

Define

$$LHS \equiv a + b\log k$$

and

$$RHS \equiv \max_{k'\geq 0}\left\{\log\left(Ak^{\alpha} - k'\right) + \beta\left(a + b\log k'\right)\right\}.$$

Plugging the expression for k' into the RHS, we obtain:

$$\begin{aligned}
RHS &= \log\left(Ak^{\alpha} - \frac{\beta b}{1+\beta b}Ak^{\alpha}\right) + a\beta + b\beta\log\left(\frac{\beta b}{1+\beta b}Ak^{\alpha}\right) \\
&= \log\left[\left(1 - \frac{\beta b}{1+\beta b}\right)Ak^{\alpha}\right] + a\beta + b\beta\log\left(\frac{\beta b}{1+\beta b}Ak^{\alpha}\right) \\
&= (1+b\beta)\log A + \log\left(\frac{1}{1+b\beta}\right) + a\beta + b\beta\log\left(\frac{\beta b}{1+\beta b}\right) + (\alpha + \alpha\beta b)\log k.
\end{aligned}$$

Setting LHS=RHS, we produce

$$\begin{cases} a = (1+b\beta)\log A + \log\left(\dfrac{1}{1+b\beta}\right) + a\beta + b\beta\log\left(\dfrac{\beta b}{1+\beta b}\right) \\ b = \alpha + \alpha\beta b, \end{cases}$$

which amounts to two equations in two unknowns. The solutions will be

$$b = \frac{\alpha}{1-\alpha\beta}$$

and, using this finding,

$$a = \frac{1}{1-\beta}\left[(1+b\beta)\log A + b\beta\log(b\beta) - (1+b\beta)\log(1+b\beta)\right],$$

so that

$$a = \frac{1}{1-\beta}\frac{1}{1-\alpha\beta}\left[\log A + (1-\alpha\beta)\log(1-\alpha\beta) + \alpha\beta\log(\alpha\beta)\right].$$

so that

$$a = \frac{1}{1-\beta}\frac{1}{1-\alpha\beta}\left[\log A + (1-\alpha\beta)\log(1-\alpha\beta) + \alpha\beta\log(\alpha\beta)\right].$$

Going back to the savings decision rule, we have:

$$\begin{aligned} k' &= \frac{b\beta}{1+b\beta}Ak^{\alpha} \\ k' &= \alpha\beta Ak^{\alpha}. \end{aligned}$$

If we let y denote income, that is, $y \equiv Ak^{\alpha}$, then $k' = \alpha\beta y$. This means that the optimal solution to the path for consumption and capital is to save a constant fraction $\alpha\beta$ of income.

This setting, we have now shown, provides a microeconomic justification to a constant savings rate, like the one assumed by Solow. It is a very special setup however, one that is quite restrictive in terms of functional forms. Solow's assumption cannot be shown to hold generally.

We can visualize the dynamic behavior of capital as is shown in Figure 3.1.

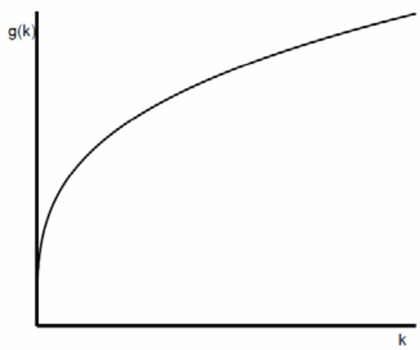

Figure 3.1: The decision rule in our parameterized model