Advanced Macroeconomics I

Lecture 2 (1)

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Dynamic optimization

- two common approaches to modelling real-life individuals:
  - they live a finite number of periods
  - they live forever
- two alternative ways of solving dynamic optimization problems:
  - sequential methods
  - recursive methods

Sequential methods - A finite horizon

- Decide on a consumption stream for T periods
  Additive separable utility function:

$$U(c_1, c_2, ..c_T) = \sum_{t=0}^{T} \beta^t u(c_t)$$

- The standard assumption is $0 < \beta < 1$, which corresponds to the observations that human beings seem to deem consumption at an early time more valuable than consumption further off in the future

A consumption-savings problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\begin{aligned} s.t.\ c_t + k_{t+1} &\leq f(k_t) \equiv F(k_t, L) + (1-\delta) k_t \\ c_t &\geq 0,\ k_{t+1} \geq 0,\ \text{and } k_0 \text{ is given} \end{aligned}$$

It is, in this case, a "planning problem": there is no market where the individual might obtain an interest income from his savings, but rather savings yield production following the transformation rule $f(k_t)$

Assumptions on u

- Assume that u is strictly increasing
- Notice that our resource constraint $c_t + k_{t+1} \leq f(k_t)$ allows for throwing goods away, since strict inequality is allowed
- But the assumption that u is strictly increasing will imply that goods will not actually be thrown away, because they are valuable
- We know in advance that the resource constraint will need to bind at our solution to this problem
- $\lim_{c \to 0} u'(c) \to \infty$, This implies that $c_t = 0$ at any t cannot be optimal, so we can ignore the non-negativity constraint on consumption: we know in advance that it will not bind in our solution to this problem

Lagrangian function

- Two decision variables: c_t and k_{t+1}

$$L = \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t (f(k_t) - k_{t+1} - c_t) + \mu_t k_{t+1} \right]$$

- Or one decision variable: k_{t+1}

$$L = \sum_{t=0}^{T} \beta^t \left[u(f(k_t) - k_{t+1}) + \mu_t k_{t+1} \right]$$

We have made use of our knowledge of the fact that the resource constraint will be binding in our solution to get rid of the multiplier $\beta^t \lambda_t$

Sovle the problem - Kuhn-Tucker conditions

- First order conditions for c_t and k_{t+1}

$$\frac{\partial L}{\partial c_t} \; : \; \beta^t \left[u(c_t) - \lambda_t \right] = 0, \quad t = 0, ..T$$

$$\frac{\partial L}{\partial k_{t+1}} \; : \; -\beta^t \lambda_t + \beta^t \mu_t + \beta^{t+1} \lambda_{t+1} f'(k_{t+1}) = 0, \quad t = 0, ..., T-1$$

$$\frac{\partial L}{\partial k_T} \; : \; -\beta^T \lambda_T + \beta^T \mu_T = 0, \; t = T$$

- Or first order conditions for k_{t+1}

$$\frac{\partial L}{\partial k_{t+1}} : -\beta^t u'(c_t) + \beta^t \mu_t + \beta^{t+1} u'(c_{t+1}) f'(k_{t+1}) = 0,$$

$$t = 0, ..., T-1$$

$$\frac{\partial L}{\partial k_T} : -\beta^T u'(c_T) + \beta^T \mu_T = 0, \; t = T$$

Sovle the problem - Kuhn-Tucker conditions

- the complementary slackness condition

$$\mu_t k_{t+1} = 0, \ \ t = 0, ..., T$$

- Non-negative conditions

$$\lambda_t \geq 0, k_{t+1} \geq 0, \mu_t \geq 0$$

- Derive that $k_{T+1} = 0$: consumers leave no capital for after the last period
$\left(\overset{\forall c, \ u'(c)>0}{-\beta^T u'(c_T)+\beta^T \mu_T = 0} \right) ==> \ \mu_T > 0 ==> k_{T+1} = 0$

Sovle the problem - Euler equation

- The summary statement of the first-order conditions is then the "Euler equation":

$$\begin{aligned} u'(f(k_t) - k_{t+1}) &= \beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) \\ t &= 0, ..., T-1, \ k_0 \text{ given}, \ k_{T+1} = 0 \end{aligned}$$

- Variational conditions: given to boundary conditions k_t and k_{t+2}, it represents the idea of varying the intermediate value k_{t+1} so as to achieve the best outcome
- A difference equation in the capital sequence: there are a total of T + 2 equations and T + 2 unknowns - the unknowns are a sequence of capital stocks with an initial and a terminal condition
  - It is a second-order difference equation because there are two lags of capital in the equation.

Unique solution

- Assumption: u is concave
- $$U = \sum_{t=0}^{T} \beta^t u(c_t) \text{ is concave in } \{c_t\}$$
  since the sum of concave functions is concave
- the constraint set is convex in $\{c_t, k_{t+1} | c_t + k_{t+1} \leq f(k_t)\}$, provided that we assume concavity of f
- concavity of the functions u and f makes the overall objective concave and the choice set convex, and thus the first-order conditions are suffient

Interpret the Euler equation

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1})$$

$$\begin{aligned} u'(f(k_t) - k_{t+1}) &: \text{ utility lost if you invest one more unit,} \\ & \quad \text{i.e. marginal cost of saving} \\ \beta u'(f(k_{t+1}) - k_{t+2}) &: \text{ utility increase next period} \\ & \quad \text{per unit of increase in } c_{t+1} \\ f'(k_{t+1}) &: \text{ return on the invested unit:} \\ & \quad \text{by how many units } c_{t+1} \text{ can increase} \end{aligned}$$

because of the concavity of u, equalizing the marginal cost of saving to the marginal benefit of saving is a condition for an optimum

How do the premitives affect savings behavior

the concavity of utility, the discounting, and the return to saving

1. Consumption "smoothing": if the utility function is strictly concave, the individual prefers a smooth consumption stream

Example

Suppose that technology is linear, i.e. $f(k) = Rk$, and that $R\beta = 1$. Then

$$\beta f'(k_{t+1}) = \beta R = 1$$

$$u'(c_t) = u'(c_{t+1})$$

if u is strictly concave, $c_t = c_{t+1}$

How do the premitives affect savings behavior

2 Impatience: via β we see that a low β(a low discount factor, or a high discount rate $\frac{1}{\beta} - 1$) will tend to be associated with low c_{t+1}'s and high c_t's.

3 The return to savings: $f'(k_{t+1})$ clearly also affcts behavior, but its effect on consumption cannot be signed unless we make more specific assumptions. Moreover, k_{t+1} is endogenous, so when f' nontrivially depends on it, we cannot vary the return independently. The case when f' is a constant, such as in the Ak growth model, is more convenient