Advanced Macroeconomics I

Lecture 2 (1)

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Dynamic optimization

- two common approaches to modelling real-life individuals:
  - they live a finite number of periods
  - they live forever
- two alternative ways of solving dynamic optimization problems:
  - sequential methods: a sequence of choices are determined at once
  - recursive methods: choices in one period are determined in that period

Sequential methods - A finite horizon

- Decide on a consumption stream for T periods
  Additive separable utility function:

$$U(c_1, c_2, ..c_T) = \sum_{t=0}^{T} \beta^t u(c_t)$$

- The standard assumption is $0 < \beta < 1$, which corresponds to the observations that human beings seem to deem consumption at an early time more valuable than consumption further off in the future

A consumption-savings problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\begin{aligned} s.t.\ c_t + k_{t+1} &\leq f(k_t) \equiv F(k_t, L) + (1-\delta) k_t \\ c_t &\geq 0,\ k_{t+1} \geq 0,\ \text{and } k_0 \text{ is given} \end{aligned}$$

It is, in this case, a "planning problem": there is no market where the individual might obtain an interest income from his savings, but rather savings yield production following the transformation rule $f(k_t)$

Assumptions on utility function

- Assume that u is strictly increasing and $\lim_{c\to 0} u'(c) \to \infty$
- Resource constraint will bind
  - resource constraint $c_t + k_{t+1} \leq f(k_t)$ allows for throwing goods away, since strict inequality is allowed
  - but the assumption that u is strictly increasing will imply that goods will not actually be thrown away, because they are valuable
- We can ignore the non-negative constraint of c_t
  - $\lim_{c\to 0} u'(c) \to \infty$, This implies that $c_t = 0$ at any t cannot be optimal

Lagrangian function

- Two decision variables: c_t and k_{t+1}

$$L = \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t (f(k_t) - k_{t+1} - c_t) + \mu_t k_{t+1} \right]$$

- Or one decision variable: k_{t+1}

$$L = \sum_{t=0}^{T} \beta^t \left[u(f(k_t) - k_{t+1}) + \mu_t k_{t+1} \right]$$

We have made use of our knowledge of the fact that the resource constraint will be binding in our solution to get rid of the multiplier $\beta^t \lambda_t$

Sovle the problem - Kuhn-Tucker conditions

- First order conditions for c_t and k_{t+1}

$$
\begin{aligned}
\frac{\partial L}{\partial c_t} &: \quad \beta^t \left[u(c_t) - \lambda_t \right] = 0, \quad t = 0, ..T \\
\frac{\partial L}{\partial k_{t+1}} &: \quad -\beta^t \lambda_t + \beta^t \mu_t + \beta^{t+1} \lambda_{t+1} f'(k_{t+1}) = 0, \quad t = 0, ..., T-1 \\
\frac{\partial L}{\partial k_T} &: \quad -\beta^T \lambda_T + \beta^T \mu_T = 0, \ t = T
\end{aligned}
$$

- Or first order conditions for k_{t+1}

$$
\frac{\partial L}{\partial k_{t+1}} : -\beta^t u'(c_t) + \beta^t \mu_t + \beta^{t+1} u'(c_{t+1}) f'(k_{t+1}) = 0,
$$

$$
t = 0, ..., T-1
$$

$$
\frac{\partial L}{\partial k_T} : -\beta^T u'(c_T) + \beta^T \mu_T = 0, \ t = T
$$

Sovle the problem - Kuhn-Tucker conditions

- the complementary slackness condition

$$\mu_t k_{t+1} = 0, \;\; t = 0, ..., T$$

- Non-negative conditions

$$k_{t+1} \geq 0, \lambda_t \geq 0, \mu_t \geq 0$$

- Derive that $k_{T+1} = 0$: consumers leave no capital for after the last period
$\left({}^{\forall c,\; u'(c)>0}_{-\beta^T u'(c_T)+\beta^T \mu_T = 0} \right) ==> \;\; \mu_T > 0 ==> k_{T+1} = 0$

Sovle the problem - Euler equation

- The summary statement of the first-order conditions is then the "Euler equation":

$$\begin{aligned} u'(f(k_t) - k_{t+1}) &= \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \\ t &= 0, ..., T-1, \ k_0 \text{ given}, \ k_{T+1} = 0 \end{aligned}$$

- Variational conditions: given to boundary conditions k_t and k_{t+2}, it represents the idea of varying the intermediate value k_{t+1} so as to achieve the best outcome
- A difference equation in the capital sequence: there are a total of T + 1 equations and T + 1 unknowns - the unknowns are a sequence of capital stocks with an initial and a terminal condition
    - It is a second-order difference equation because there are two lags of capital in the equation.

Unique solution

- Assumption: u is concave
- $$U = \sum_{t=0}^{T} \beta^t u(c_t) \text{ is concave in } \{c_t\}$$

  since the sum of concave functions is concave
- the constraint set is convex in $\{c_t, k_{t+1} | c_t + k_{t+1} \leq f(k_t)\}$, provided that we assume concavity of f
- concavity of the functions u and f makes the overall objective concave and the choice set convex, and thus the first-order conditions are suffient

Interpret the Euler equation

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1})$$

$$\begin{aligned}
u'(f(k_t) - k_{t+1}) &: \text{ utility lost if you invest one more unit} \\
&\quad \text{ i.e. marginal cost of saving} \\
\beta u'(f(k_{t+1}) - k_{t+2}) &: \text{ utility increase next period} \\
&\quad \text{ per unit of increase in } c_{t+1} \\
f'(k_{t+1}) &: \text{ return on the invested unit:} \\
&\quad \text{ by how many units } c_{t+1} \text{ can increase}
\end{aligned}$$

because of the concavity of u, equalizing the marginal cost of saving to the marginal benefit of saving is a condition for an optimum

How do the premitives affect savings behavior

the concavity of utility, the discounting, and the return to saving

1. Consumption "smoothing": if the utility function is strictly concave, the individual prefers a smooth consumption stream

Example

Suppose that technology is linear, i.e. $f(k) = Rk$, and that $R\beta = 1$. Then

$$\beta f'(k_{t+1}) = \beta R = 1$$

$$u'(c_t) = u'(c_{t+1})$$

if u is strictly concave, u' is monotonically increasing, $c_t = c_{t+1}$

How do the premitives affect savings behavior

2 Impatience: via β we see that a low β (a low discount factor, or a high discount rate $\frac{1}{\beta} - 1$) will tend to be associated with low c_{t+1}'s and high c_t's.

3 The return to savings: $f'(k_{t+1})$ clearly also affcts behavior

- but its effect on consumption cannot be signed unless we make more specific assumptions
- Moreover, k_{t+1} is endogenous, so when f' nontrivially depends on it, we cannot vary the return independently
- The case when f' is a constant, such as in the Ak growth model, is more convenient

Example 1 - logarithmic utility function

- $u(c) = \log(c)$, $f(k) = Ak$
- Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1})$$

-

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} A$$

- Optimal consumption growth rule

$$c_{t+1} = \beta A c_t$$

Intertemporal budget constraint

- Resource constraints

$$\begin{aligned} c_0 + k_1 &= Ak_0 \\ c_1 + k_2 &= Ak_1 \\ &\ldots \\ c_T + k_{T+1} &= Ak_T \\ k_{T+1} &= 0 \end{aligned}$$

- Intertemporal budget constraint

$$c_0 + \frac{1}{A}c_1 + \frac{1}{A^2}c_2 + \ldots + \frac{1}{A^T}c_T = Ak_0$$

present value of consumption stream = present value of income

Optimal consumption

Using the optimal consumption growth rule $c_{t+1} = \beta A c_t$,

$$c_0 + \frac{1}{A}\beta A c_0 + \frac{1}{A^2}(\beta A)^2 c_0 + ... + \frac{1}{A^T}(\beta A)^T c_0 = Ak_0$$

$$c_0 \left[\beta + \beta^2 + ... + \beta^T\right] = Ak_0$$

$$c_0 = \frac{Ak_0}{\beta + \beta^2 + ... + \beta^T}$$

Share of consumption c_t is $\frac{\beta^t}{\beta+\beta^2+...+\beta^T}$

The Effects of productivity A

- An increase in A will cause a rise in consumption in all periods
  - Crucial to this result is the chosen form for the utility function: Logarithmic utility has the property that income and substitution effcts, when they go in opposite directions, exactly offset each other
- Changes in A have two components: a change in relative prices (of consumption in different periods) and a change in present-value income: Ak_0

$$c_0 + \frac{1}{A}c_1 + \frac{1}{A^2}c_2 + ... + \frac{1}{A^T}c_T = Ak_0$$

The Effects of productivity A - logarithmic utility

- With logarithmic utility, a relative price change between two goods will make the consumption of the favored good go up whereas the consumption of other good will remain at the same level
- The unfavored good will not be consumed in a lower amount since there is a positive income effct of the other good being cheaper, and that effect will be spread over both goods
- Thus, the period 0 good will be unfavored in our example (since all other goods have lower price relative to good 0 if A goes up), and its consumption level will not decrease
- The consumption of good 0 will in fact increase because total present-value income is multiplicative in A

Varying productivity A

Productivity stream $\{A_t\}$

$$c_0 + \frac{1}{A_1}c_1 + \frac{1}{A_1 A_2}c_2 + ... + \frac{1}{A_1 A_2 ... A_T}c_T = A_0 k_0$$

Plugging in the optimal path $c_{t+1} = \beta A_{t+1} c_t$,

$$c_0 \left[\beta + \beta^2 + ... + \beta^T\right] = A_0 k_0$$

$$\begin{aligned} c_0 &= \frac{A_0 k_0}{\beta + \beta^2 + ... + \beta^T} \\ c_1 &= \frac{(A_1 \beta) A_0 k_0}{\beta + \beta^2 + ... + \beta^T} \\ &\cdots \\ c_T &= \frac{A_0 A_1 ... A_T \beta^T k_0}{\beta + \beta^2 + ... + \beta^T} \end{aligned}$$

Comparative Statics: $A_t \uparrow \implies c_0, c_1, ..., c_{t-1}$ are unaffected

Example 2 - CES Utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

$$\begin{aligned} \sigma &= 0: \text{ linear utility} \\ \sigma &> 0: \textit{ concave} \\ \sigma &= 1: \text{ limit: logarithmic utility} \end{aligned}$$

Intertemporal Elasticity of Substitution

Let $R_{t,t+k}$ denote the interest rate: $R_{t,t+k} = A_{t+1}...A_{t+k}$
The intertemporal elasticity of substitution is

$$IES \equiv \frac{\frac{d\frac{c_{t+k}}{c_t}}{\frac{c_{t+k}}{c_t}}}{\frac{dR_{t,t+k}}{R_{t,t+k}}}$$

IES measures the elasticity of the relative share of conumption with respect to interest rate

Intertemporal Elasticity of Substitution

$R_{t+1} = A_{t+1}$

$$u'(c_t) = \beta u'(c_{t+1}) R_{t+1}$$

Replacing repeatedly, we have

$$\left.\begin{array}{c} u'(c_t) = \beta^k u'(c_{t+k}) R_{t+1} \ldots R_{t+k} \\ u'(c) = c^{-\sigma} \end{array}\right\} \Longrightarrow c_t^{-\sigma} = \beta^k c_{t+k}^{-\sigma} R_{t,t+k}$$

$$IES \equiv \frac{\frac{d \frac{c_{t+k}}{c_t}}{\frac{c_{t+k}}{c_t}}}{\frac{dR_{t,t+k}}{R_{t,t+k}}} = \frac{d \log \frac{c_{t+k}}{c_t}}{d \log R_{t,t+k}} = \frac{1}{\sigma}$$

Intertemporal Elasticity of Substitution

- When $\sigma = 1$, expenditure shares do not change: this is the logarithmic case
- When $\sigma > 1$, an increase in $R_{t,t+k}$ would lead c_t to go up and savings to go down: the income effect, leading to smoothing across all goods, is larger than substitution effect
- Finally, $\sigma < 1$, the substitution effect is stronger: savings go up whenever $R_{t,t+k}$ goes up
- When $\sigma = 0$, the elasticity is infinite and savings respond discontinuously to $R_{t,t+k}$