Advanced Macroeconomics I

Lecture 2 (2)

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Spring 2011

Sequential Method - Infinite horizon

- Why should macroeconomists study the case of an infinite time horizon?
- Altruism: care about their children
- Let $u(c_t)$ denote the utility flow to generation t, β as the weight an individual attaches to the utility enjoyed by his descendants t generations down the family tree
- His total joy is

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

$\beta < 1$ thus implies that the individual cares more about himself than about his descendants

Care kids' utility or just being happy with giving

- The time horizon for an individual only becomes truly infinite if the altruism takes the form of caring about the utility of the descendants
- If, instead, utility is derived from the act of giving itself, without reference to how the gift influences others' welfare, the individual's problem again becomes finite
  - If I live for one period and care about how much I give, my utility function might be $u(c) + v(b)$, where v measures how much I enjoy giving bequests, b. Although b subsequently shows up in another agent's budget and influences his choices and welfare, those effects are irrelevant for the decision of the present agent, and we have a simple static framework
  - This model is usually referred to as the "warm glow" model

Sudden death model

an overlapping-generations model

- Think of an individual that, if still alive, each period dies with probability $1 - \pi$. Its expected lifetime utility from a consumption stream $\{c_t\}$ is then given by

$$\sum_{t=0}^{\infty} \beta^t \pi^t u(c_t)$$

Looks like infinite horizon

Reason 2 to use infinite horizon - simplicity

- Many macroeconomic models with a long time horizon tend to show very similar results to infinite-horizon models if the horizon is long enough
- Infinite-horizon models are stationary in nature
  - the remaining time horizon does not change as we move forward in time
  - and their characterization can therefore often be obtained more easily than when the time horizon changes over time

When solutions exist

- Finite horizon: Euclidean space R^n
- Infinite horizon: choosing infinite sequence of consumption
- Infinite horizon: it is possible that utility is unbounded
  - If two consumption streams yield infinite utility, it is not clear how to compare them

Preference requirement

consider a constantly increasing consumption stream: $\{c_0(1+\gamma)^t\}_{t=0}^{\infty}$

$$
\begin{aligned}
U &= \sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t u\left[c_0(1+\gamma)^t\right] \\
& \sum_{t=0}^{\infty} \beta^t \frac{\left[c_0(1+\gamma)^t\right]^{1-\sigma} - 1}{1-\sigma}
\end{aligned}
$$

$$\beta(1+\gamma)^{1-\sigma} < 1$$

Technology considerations

- Productivity A

$$\begin{aligned} c_t + k_{t+1} &= Ak_t \\ k_t &\geq 0 \end{aligned}$$

c_t could grow at rate A, if the given A is too large, it is possible that c_t grows explosively

No Ponzi games are allowed

- Consumption: $\{c_t^*\}_{t=0}^{\infty}$, $c_t \leq \bar{c}\ \forall t$

- endow a consumer with a given initial amount of net assets, a_0

$$c_t + a_{t+1} = Ra_t$$

Here, $a_{t+1} < 0$ means borrowing

No Ponzi games are allowed

- Absent no-Ponzi-game condition, the agent could improve on $\{c_t^*\}_{t=0}^{\infty}$ as follows:
  - Put $\tilde{c}_0 = c_0^* + 1$, thus making $\tilde{a}_1 = a_1^* - 1$
  - For every $t \geq 1$ leave $\tilde{c}_t = c_t^*$ by setting $\tilde{a}_{t+1} = a_{t+1}^* - R^t$
- With strictly monotone utility function, the agent will be strictly better off under this alternative consumption allocation, and it also satisfies budget constraint period-by-period
- Because this sort of improvement is possible for any candidate solution, the maximum of the lifetime utility will not exist

No Ponzi games are allowed

- as the agent's debt is growing without bound at rate R, it is never repaid
- nPg condition:

$$\lim_{t \to \infty} \frac{a_t}{R^t} \geq 0$$

- Intuitively, this means that in present-value terms, the agent cannot engage in borrowing and lending so that his "terminal asset holdings" are negative, since this means that he would borrow and not pay back

Transversality condition and the end of period condition

- In the case of finite time horizon it did not make sense for the agent to invest in the final period T, since no utility would be enjoyed from consuming goods at time $T + 1$
- This final condition is the key to determining the optimal path of capital: it provided us with a terminal condition for a difference equation system
- In the case of infinite time horizon there is no such final T: the economy will continue forever. Therefore, the difference equation that characterizes the first-order condition may have an infinite number of solutions
- the missing condition is analogous to the requirement that the capital stock be zero at $T + 1$, for else the consumer could increase his utility

transversality condition

- $$\lim_{t\to\infty} \beta^t u_1(k_t^*, k_{t+1}^*) k_t = 0$$

- Interpretation:
  - $u_1(k_t; k_{t+1})$ is the marginal addition of utils in period t from increasing capital in that period
  - transversality condition simply says that the value (discounted into present-value utils) of each additional unit of capital at infinity times the actual amount of capital has to be zero

- If this requirement were not met, it would pay for the consumer to modify such a capital path and increase consumption for an overall increase in utility without violating feasibility

Transversality condition and nPg

- The no-Ponzi-game and the transversality conditions play very similar roles in dynamic optimization in a purely mechanical sense (at least if the nPg condition is interpreted with equality)
- In fact, they can typically be shown to be the same condition, if one also assumes that the first-order condition is satisfied
- However, the two conditions are conceptually very different
  - The nPg condition is a restriction on the choices of the agent
  - In contrast, the transversality condition is a prescription how to behave optimally, given a choice set