Advanced Macroeconomics I

Lecture 2 (3)

Zhe Li

SUFE

Spring 2011

Motivate dynamic programming

- The sequencial approach is to look for a sequence of real numbers $\{k_{t+1}\}_{t=0}^{\infty}$ that generates an optimal consumption plan
  - Searching for a solution to an infinite sequence of equations (the Euler equation)
- The search for a sequence is sometimes impractical
- Alternative approach - dynamic programming

Dynamic Programming

- Key to dynamic programming
  - to think of dynamic decisions as being made not once and for all
  - but recursively: time period by time period
  - The savings between t and $t + 1$ are thus decided on at t, and not at 0

Stationary

in terms of decisions

Definition

We will call a problem **stationary** whenever the *structure* of the choice problem that a decision maker faces is *identical* at every point in time

Stationary in decision rule

- A consumer placed at the beginning of time choosing his infinite future consumption stream given an initial capital stock k_0
  - $\{k_{t+1}\}_{t=0}^{\infty}$ indicating the level of capital that the agent will choose to hold in each period
- Stationary means that the choice of k_{t+1} given k_t obeys a single rule, for any k_t

Stationary

he would not change his mind if he could decide all over again

- Once the consumer has chosen a capital path, suppose that we let the consumer keep it for s periods
- At $t = s$ he will find himself with the k_s decided on initially
- If at that moment we told the consumer to forget about his initial plan and asked him to decide on his consumption stream again, from then onwards

Stationary

he would not change his mind if he could decide all over again

- Using as new initial level of capital $k_0 = k_s$, what sequence of capital would he choose?
- If the problem is stationary then for any two periods $i \neq j$, $k_i = k_j$, $k_{i+n} = k_{j+n}$, for all $n > 0$

Decision Rule

- If a problem is stationary, we can think of a function that, for every period t, assigns to each possible initial level of capital k_t an optimal level for next period's capital k_{t+1}:

$$k_{t+1} = g(k_t)$$

(and therefore an optimal level of current period consumption)

Decision Rule

- Stationarity means that the function $g(k_t)$ has no other argument than current capital
  - In particular, the function does not vary with time
- We will refer to $g(k_t)$ as the decision rule

Stationary

in terms of properties of the solution to a dynamic problem

- What types of dynamic problems are stationary?
  - Intuitively, a dynamic problem is stationary if one can capture all relevant information for the decision maker in a way that does not involve time

Stationary - finite horizon

- In our neoclassical growth framework, with a finite horizon, time is important, and the problem is not stationary:
  - it matters how many periods are left - the decision problem changes character as time passes

Stationary - infinite horizon

- With an infinite time horizon, however, the remaining horizon is the same at each point in time
  - The only changing feature of the consumer's problem in the infinite-horizon neoclassical growth economy is his initial capital stock, k_0
  - hence, his decisions will not depend on anything but this capital stock

State variables

- Whatever is the relevant information for a consumer solving a dynamic problem, we will refer to it as his state variable
  - So the state variable for the planner in the one-sector neoclassical growth context is the current capital stock

Value function

- Let us denote by $V(k_t)$ the value of the optimal program from period t for an initial condition k_t:

$$V(k_t) \equiv \max_{\{k_{s+1}\}_{s=t+1}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(k_s, k_{s+}$$

$$\text{s.t. } k_{s+1} \in \Gamma(k_s) \ \forall s \geq t$$

where $\Gamma(k_s)$ represents the feasible choice set for k_{s+1} given k_s

Value function

- V is an indirect utility function, with k_t representing the parameter governing the choices and resulting utility
- Using the maximization-by-steps idea, we can write

$$V(k_t) \equiv \max_{k_{t+1} \in \Gamma(k_t)} \{u(k_t, k_{t+1}) + \beta V(k_{t+1})\}$$

Bellman equation

- A functional equation – the unknown is a function

$$V(k) \equiv \max_{k' \in \Gamma(k)} \{u(k, k') + \beta V(k')\}$$

- The decision rule for $k' : k' = g(k)$

$$g(k) = \arg \max_{k' \in \Gamma(k)} \{u(k, k') + \beta V(k')\}$$

The equivalence

between the sequential formulation of the dynamic optimization and its recursive Bellman formulation

- If a function represents the value of solving the sequential problem (for any initial condition), then this function solves the dynamic programming equation (DPE)
- If a function solves the DPE, then it gives the value of the optimal program in the sequential formulation

The equivalence

between the sequential formulation of the dynamic optimization and its recursive Bellman formulation

- If a sequence solves the sequential program, it can be expressed as a decision rule that solves the maximization problem associated with the DPE
- If we have a decision rule for a DPE, it generates sequences that solve the sequential problem

Existence of a value function

The proofs of all these results can be found in Stokey and Lucas with Prescott (1989)

Assumptions:

1. $u(k, k')$ is continuously diffrentiable in its two arguments, that it is strictly increasing in its first argument (and decreasing in the second), strictly concave, and bounded

Existence of a value function

2. Γ is a nonempty, compact-valued, monotone, and continuous correspondence with a convex graph
3. $\beta \in (0, 1)$

- There exists a function V that solves the Bellman equation. This solution is unique

Find the value function

Contraction Mapping

- It is possible to find V by the following iterative process:
    - Pick any initial V_0 function, for example $V_0(k) = 0$
    - Find V_{n+1}, for any value of k, by evaluating the right-hand side of (*) using V_n

$$V(k) \equiv \max_{k' \in \Gamma(k)} \{u(k, k') + \beta V(k')\} \qquad (*)$$

- The outcome of this process is a sequence of functions $\{V_j\}_{j=0}^{\infty}$ which converges to V

Properties of a value function

- V is strictly concave
- V is strictly increasing
- V is diffrentiable

Optimal decision rule

- Optimal behavior can be characterized by a function g, with $k' = g(k)$, that is increasing so long as u_2 is increasing in k
- First order condition:

$$V(k) \equiv \max_{k' \in \Gamma(k)} \{u(k, k') + \beta V(k')\} \quad (*)$$

$$-u_2(k, k') = \beta V'(k')$$

implied relationship between k and k'